Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title

Basics of Finite Element Analysis

Lecture – 42
Eigenvalue problem : example

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Hello, welcome to basics of finite element analysis course, today is the last day of this week and in today's lecture we will discuss a particular Eigen value problem and actually develop its finite element formulation and solve it, so this problem relates to vibrations in a bar and the governing differential equation of the bar is.

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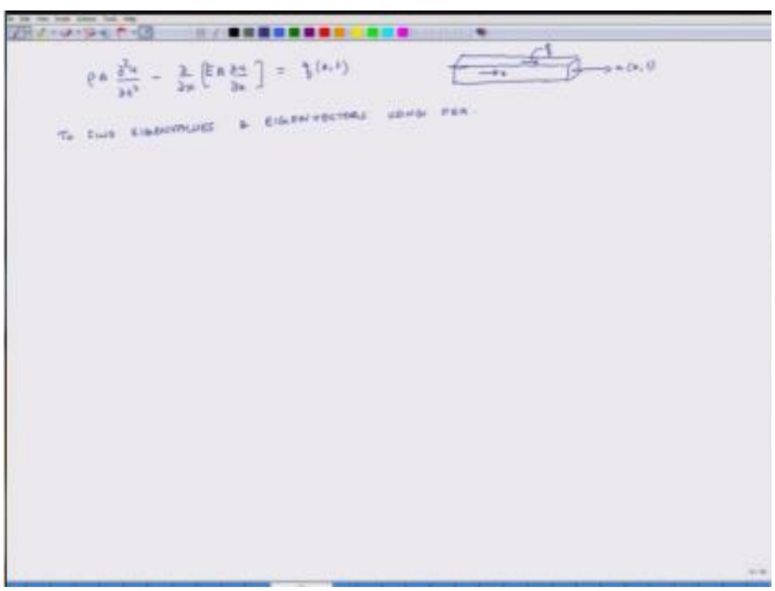

ρ a times ∂u with respect to time - ∂e, a δu δx with respect to x equals Q and to explain suppose this is the point where, which I am considering u is the displacement in next direction and it can vary with time and position this is my x-coordinate okay, ρ is the density of the bar in this case we are assuming that ρ is homogeneous or it is same on that particular small element of the bar.

A is the radius of cross section, e is the Young's modulus and Q is the traction per unit length, it is traction per unit length so it is units of Newton's per meter. Our goal is to find Eigen value problems, Eigen values, and Eigen vectors using FEA okay. Now our first step is so this is our second order partial differential equation it is second-order both in time and in space, so the first thing we do is we try to convert this into a second order ordinary differential equation and we do that by realizing that if the bar is vibrating like this, this motion will be and if it is vibrating on its own it will have some sort of a harmonic motion with some natural frequency Ω right.

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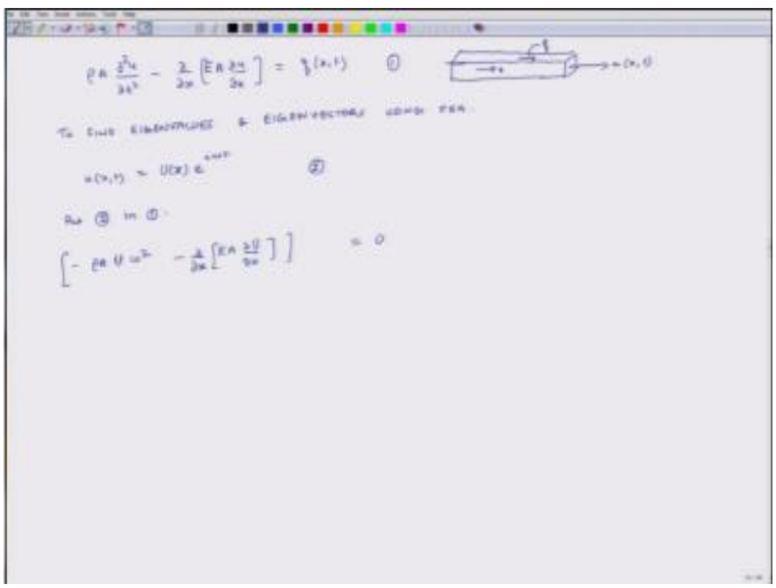

So we say that u(x, t) equals a function u(x) times e $^{j\Omega t}$ oh excuse me it is e $^{i\Omega t}$, so this is equation 1, this is equation 2, so the amplitude of this motion can vary from place to place but at every point the point will vibrate in a harmonic way but each point may have a different amplitude which is u(x) but the motion of each point will be harmonic, that is what it means. So now what I do is put two in one, so what I get is ρA and I am going to just write u rather than u(x) for purposes of privacy so ρAu and because when it is differentiated by time two times.

So I get $-\Omega^2$- δ over δx EA δu δx and this entire thing $e^{i\Omega t}$ right and because we are interested in finding out the Eigen values of the problem the right side of the system will be 0. Now we note that $e^{i\Omega t}$ is common to the entire expression so I can eliminate this right so I will erase this, the other thing I note so this entire equation is now only in x because this U.

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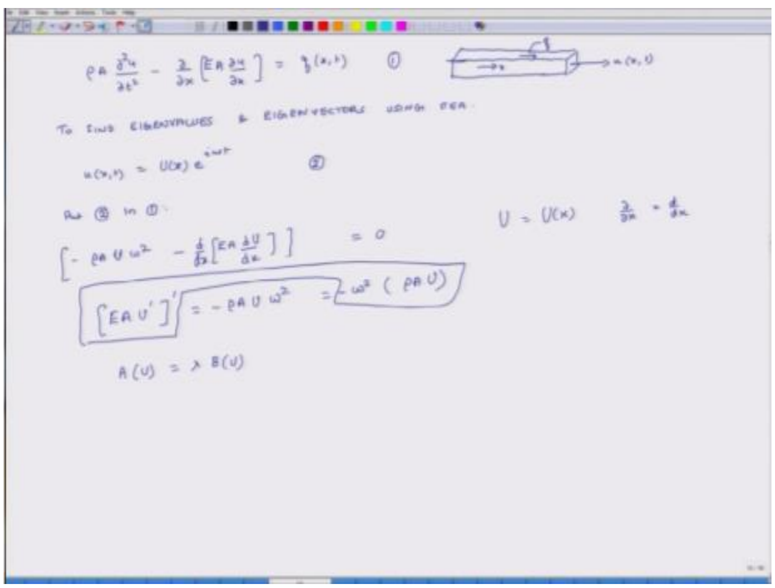

U is nothing but a function of x only which means that δ over δx equals d over dx right so I will replace these partial differential operators with regular differential, operators total derivative operators, so it becomes d over dx. So now I see that I can express this entire expression as EAU´ is equal to ρ AUΩ^2 or I can write it as Ω^2 times ρAU, and there should be a negative sign here okay, so this if I look at this it is in the same form as A(U) equals λB (U) which is the standard form for Eigen value problem.

So I have developed this converted this form into an Eigen value problem statement okay, now at this stage I am going to solve these equations using the finite element method and all of you already know how to solve this problem using the finite element method. The first step in the finite element method if we want to solve this problem is we will break the domain into small elements, the second step will be we will assume interpolation function for U, we will plug those things here, we will develop and find the residue of the overall thing over the entire element multiplied by a weight function, equate that residue weighted into the thing 0, weaken the differential form, weaken the statement.

And develop a system of equations and then we will solve those equations to get our answer, so that is the overall process and that is what we are going to do.

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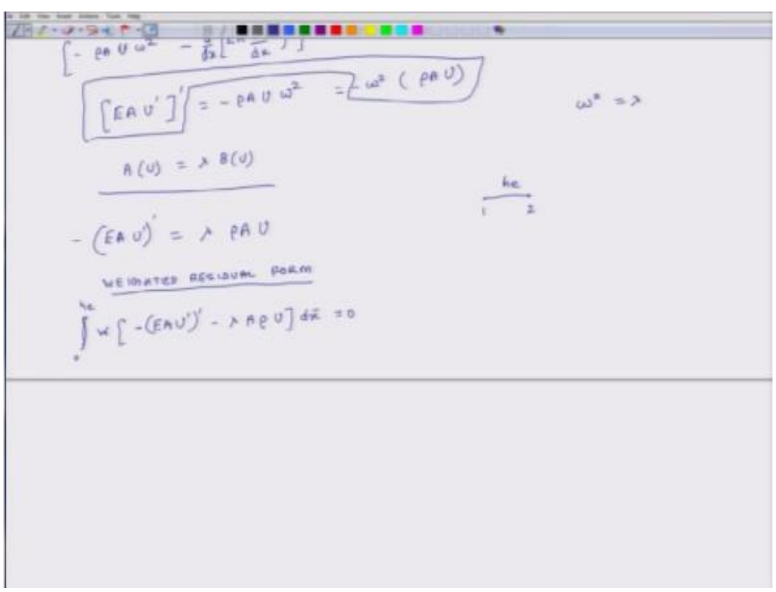

So our overall equation is EAU´ and I remove the negative sign on this side equals Ω^2 and what I will do is just for purposes of convenience I will replace Ω^2 by λ, so it is λ times ρAU okay, so the weak form, so now we are going to get the weak form but before that we will express it as a weighted residual form, so at this stage so suppose my bar is this long and this is one, so I have broken it into several elements, this is node 1, node2 let us say the element is h_e and so for the e^{th} element I have length is h_e and there are only two nodes so I am assuming that this is a linear element but before I get to the element order I will develop a weighted residual form, so I integrate it from zero to $h_{e,}$ I have a weight function w multiplied by x-EAU´ the entire thing differentiated with respect to x- λ **A**ρU dx equals 0 d$\bar{x}$ because I am in local coordinates.

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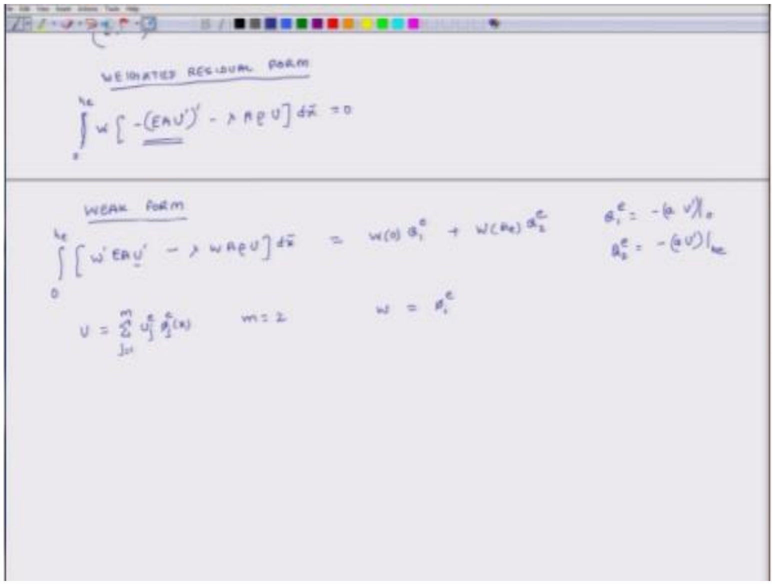

From this I develop a weak form, so my weak form is 0 to h_e and I reduce the differentiability on the first term and shift it to w so I get w´ and when I do that the negative sign before EA goes away right $-\lambda$ wAρUd$\bar{x}$ equals on the right side I have, so I will get some extra boundary terms and I am shifting them to the right side and what I get is w evaluated 0 times Q_1^e + w evaluated1 no h_e times Q_2^e where Q_1^e equals - AU´ evaluated at 0 and Q_2^e equals –A this entire thing is evaluated.

AU´ evaluated at h_e okay. At this stage my next thing is that I assume an interpolation function for u, so u is equal to Σ of U_j and $Ø_j$x and this is for the e[th] element and j equals 1 to m and in this case I am using M is equal to 2 because I assume that it is a linear element, also my weight function equals $Ø_i$ for the e[th] element which is the variation using principles of variational mechanics this is nothing but it represents a variation in U.

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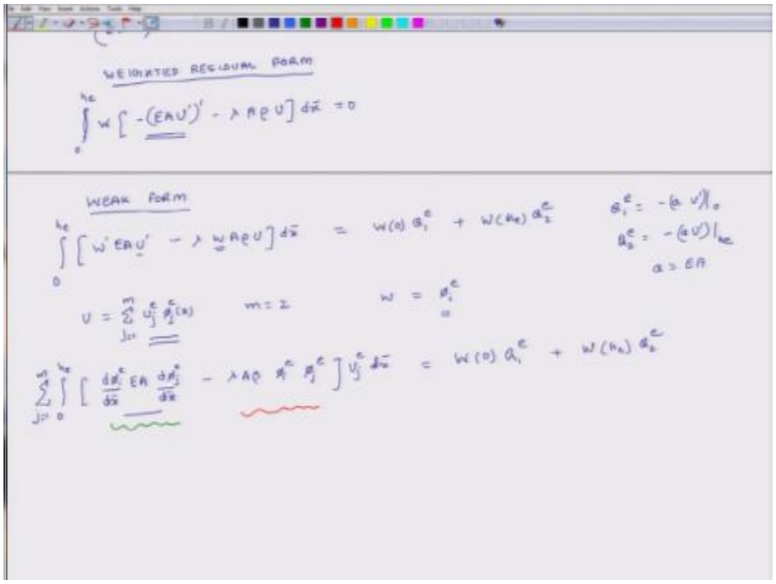

So then I get 0 to h_e $d\text{Ø}_i$ over $d\bar{x}$ EA $d\text{Ø}_j$ over $d\bar{x}$ all for e^{th} equation $-\lambda$ Aρ over here by the way a equals e times A so coming back to my weak form this is my first term $-\lambda$ A ρ and w I have assumed as Ø_i for the e^{th} term and u will be Øj e^{th} term, and because I have to some, you is Σ of u_j Ø_j so I have to multiply it by U_j and sum it up j is equal to 1 to m and integrate it over the entire domain or, because this is may not appear on your screen clearly so I will again j is equal to 1 to m and I am going to multiply it by $U_j{}^e$ $d\bar{x}$ okay, and this equals w evaluate at 0 $Q_1{}^e$+ w evaluated h_e $Q_2{}^{e.}$ Now we look at these two terms, the one in green and the one which is underlined as red okay, the green term has EA embedded in it A is the area of cross section and e is the Young's modulus of the material.

And so it represents kind of the stiffness of the system, if you have a stiffer system e will be high if you have a very compliant system e will be less right so this represents.

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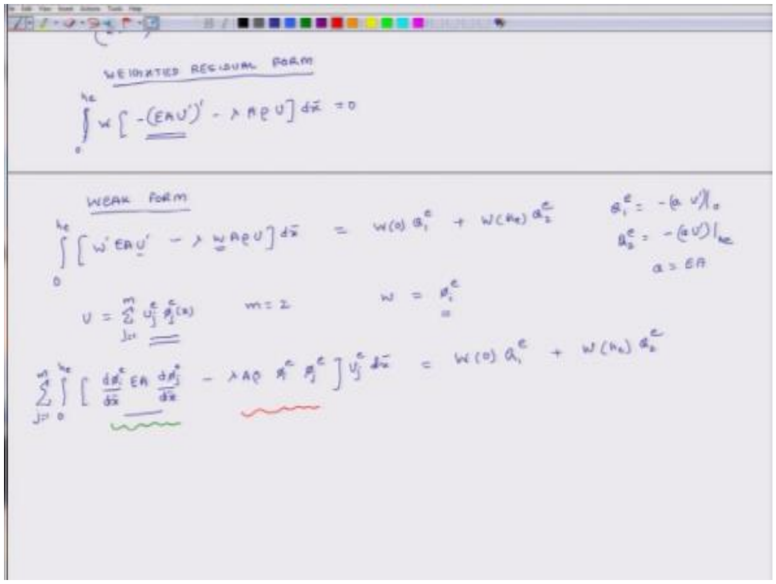

Stiffness of the system, on the other hand the red term the one which is underlined red if we do not worry about λ then it is a times ρ times dx is, so suppose you have a small element which is dx long, cross-sectional area is a times ρ, ρ is the density so it represents mass of the system.

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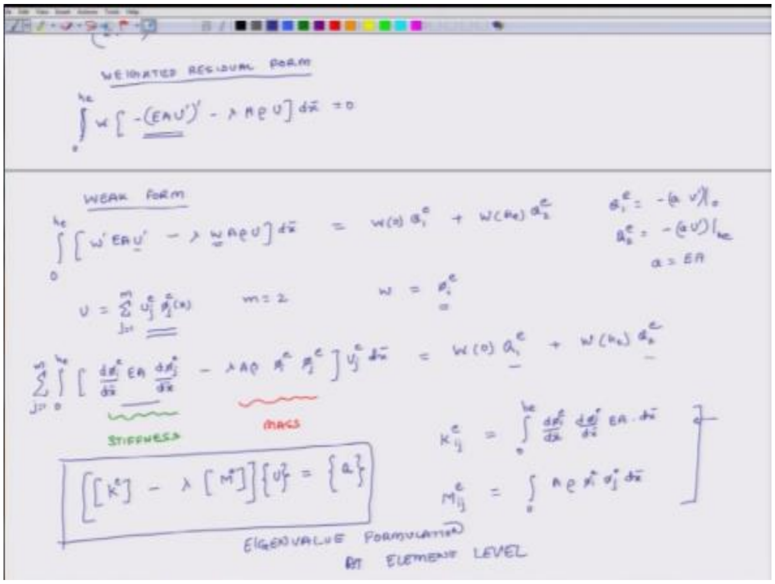

Okay so what we get is two matrixes k – λ m and both these matrices are for the eth element and this entire thing is multiplied by an unknown vector U, and on the right side I get a Q vector a force vector, but these forces are internal to the system okay, these are internal to the system. Here $k_{i\ j}$ for the eth element equals integral of $d\O_i$ over $d\bar{x}$, $d\O_j$ over $d\bar{x}$ times EA $d\bar{x}$, also the mass matrix so the stiffness matrix is defined earlier and then mass matrix is defined as integral of A $\rho\O_i\O_j$ $d\bar{x}$ and of course everything is with the super script d so that is my stiffness matrix and that is my mass matrix.

So in our earlier problems we had only one metrics which was multiplied by the unknown vector u, here we have two matrices one is the stiffness matrix the other one is the mass matrix right and they are and the mass matrix is multiplied by that λ which is the Eigen value of the system, and on the right side you have the Q vector which is basically the internal reaction forces at the two ends of the element okay, couple of things.

So this is stiffness matrix if you look at it, it is symmetric because if you replace jby i and i by j you get the same answer, so k_{ij} is equal to $k_{ji.}$ The mass matrix is also symmetric in nature because when you replace i by j so $M_{i,j}$ is equal to M_{ji} okay so this is the, so these are the two matrices k and M, so now at the element level we have developed the Eigen value formulation for the problem.

So our next step is to assemble all these things okay so.

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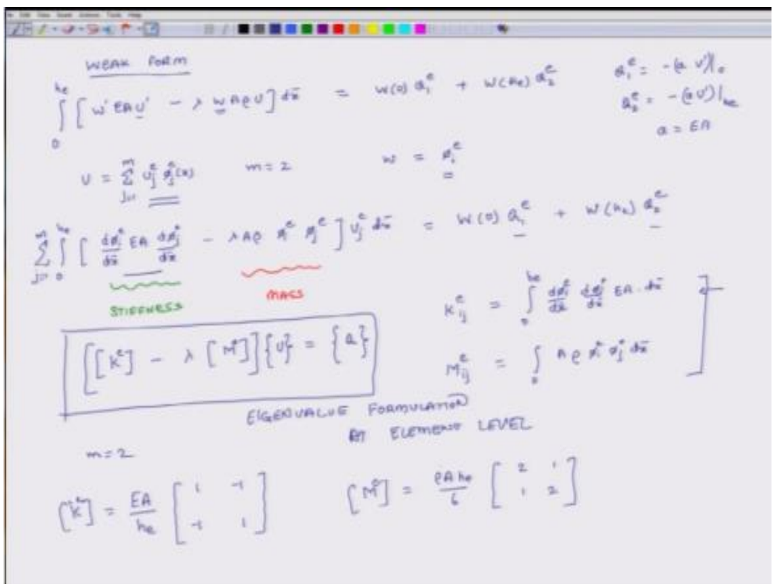

That is what we are going to do, so before I do that actually I will write down the values of these terms so K for the e^{th} element is equal to EA over h_e and this is for the case when m is equal to two, it is for a linear element, for a quadratic element it will be a three-by-three matrix okay for a linear element it will be a two-by-two matrix and the terms are 1 -1 -1 and 1, and then the mass matrix for the e^{th} element is ρ Ah_e over 6 2 1 1 2, there is one thing interesting usually you will note at the mass matrix, the overall mass of the system which is an element h_e long will be what, its cross-sectional area times density times h_e right.

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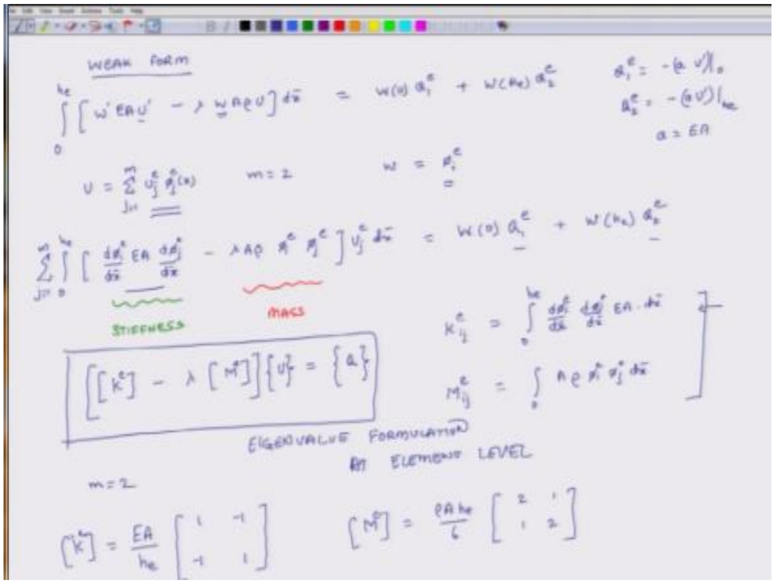

So you have in mass matrix at least in this case the numerator ρ Ah_e is the mass it is divided by 6 and then you look at all the terms in the matrix, you have 2 2 1 and 1 if you add them up two plus, two plus, two plus, four plus one five and one 6 that divided by six is so, if you add up all the elements of the mass matrix they end up with the total mass and the mass matrix can never be negative because mass is never negative it can never be negative okay

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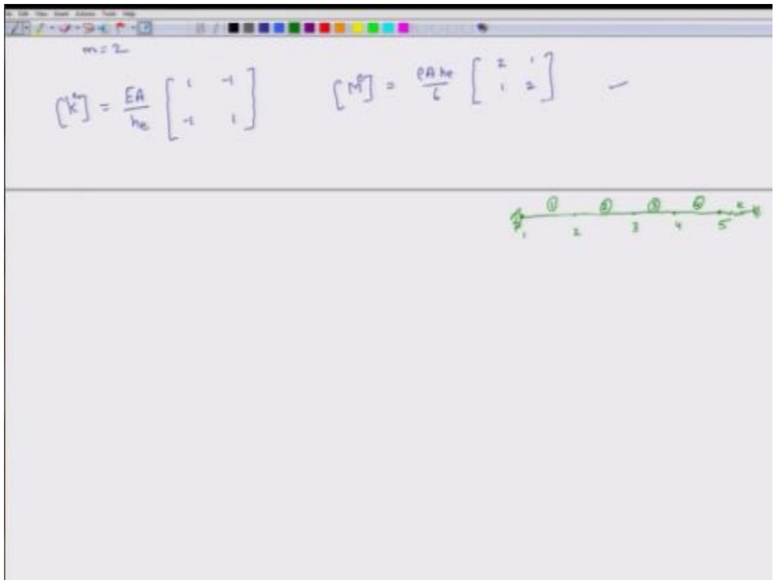

So there is something interesting about your mass matrix, so with this understanding what we will do is we will develop assembly for our system, so our system is like this, so it is a long bar and at this end of the bae it is rigidly fixed, at the other end of the bar I have a spring whose stiffness is K okay, now what I am going to do is I am going to split this bar into four elements.

So this is element number one, element number two, element number three, element number four, so I have a total nodes number of nodes is one, two, three, four and five okay, and I am interested in finding out the Eigen values of this problem okay. One thing you will immediately notice that suppose this spring was not there then the bar would be would have less problem vibrating back and forth right, once the spring is there then the overall stiff system becomes stiffer.

So bar will have a higher angular frequency because its over all stiffness that is k it goes up right so we have to figure out how to incorporate the

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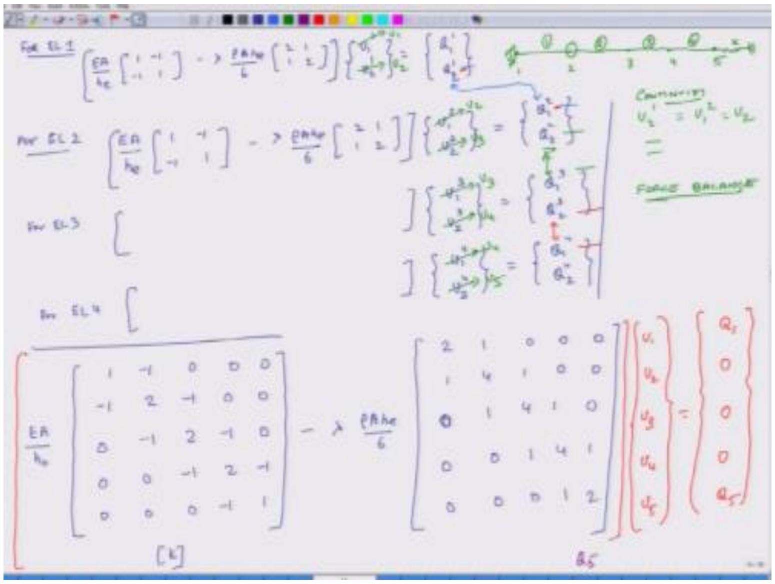

Influence of this k in the overall assembly level equations and to do that first we have to develop the assembly level equations, so let us write down these assembly level equations for element 1. Our assembly level equation is EA over h_e 1 -1 - 1 1 – λ ρAh_e over 6 2 1 1 2, the entire thing is multiplied by U_1 U_2 ^{e} and in this case e is this the first element so e is one and this equals$Q_1^1Q_2^1$ okay, so this is the these are the two equations for the first element. The equations for the second element are identical except that the superscript 1becomes two okay because we have assumed that the length of the element is same in all the cases.

So for element 2 EA over h_e 1- 1 -1 1 –λ ρAh_e over 6 2 1 1 2, this entire thing is multiplied by U_1 U_2 for the second element and that equals Q vector when Q_1 Q_2 for the second element. Likewise I will have another set of equations for element 3 and here it will be the U vector will be U_1 U_2^3 = Q_1 Q_2^3 as the superscript and for element four again similar set of equations, U_1 U_2^{44} equals Q_1 Q_2^{44} okay, so we have developed four sets of element level element equations total number of equations her is eight right and we have five nodes.

1, 2, 3 and 4 and 5, we have five nodes, so the first thing we do is we established the continuity conditions for U, what are those continuity conditions at node 2 U_2^1 is equal to U $_1^2$ is equal to

$U_{2,}$ so this becomes $U_{2,}$ this becomes U1 same using same approach so this is U_2 I am replacing all these local degrees of freedom with their global values, this is $U_{3,}$ this is $U_{4,}$ this is $U_{4,}$ and this is U_5 okay.

The second thing I have to note is that the total force let us say node 2 so this is continuity and they are 3, 3 more relations for that and then the second one is force balance, and for force balance the total force at node 2 is $Q_2^2 + Q_1^3$ right, similarly the total force at node yeah total force at node 4 is $Q_3^2 + Q_1^{4}$, and the total force at node two is the sum of these two correct, so if the total force is that much then I have to add these equations so I have to add this equation with this equation, I have to add these two equations, and I have to add these two equations, and then in that way I will get total of five equations, right now I have eight so I have five equations and I have five degrees of freedom.

So the total number of, so the final set of equation becomes if I do that is EA over h_0 1 -1 0 0 0o -1 2 -1 0 0 0 0 0 -1 2 -1 0 0 -1 2 -1 0 0 -1 1, this is my K matrix global K matrix - λ $\rho A h_e$ over 6 and I get global mass matrix. What is the mass matrix, 21000, 14100, 01410, 0 this is 00141, 00014 okay, and this entire thing would be 2, so this entire thing is multiplied by u_1, u_2, u_3 u_4, $u_{5,}$ and this equals the Q vector and the sum of Q_{12} this guy and this guy is 0 so what I will get is Q_1 000 and Q_5 okay. Now let us look at Q_5, Q_5 is the force which is being applied on the system due to the spring, so if the node 5 it moves by a distance u the spring will apply a force in the opposite direction and its value will be K times u_5.

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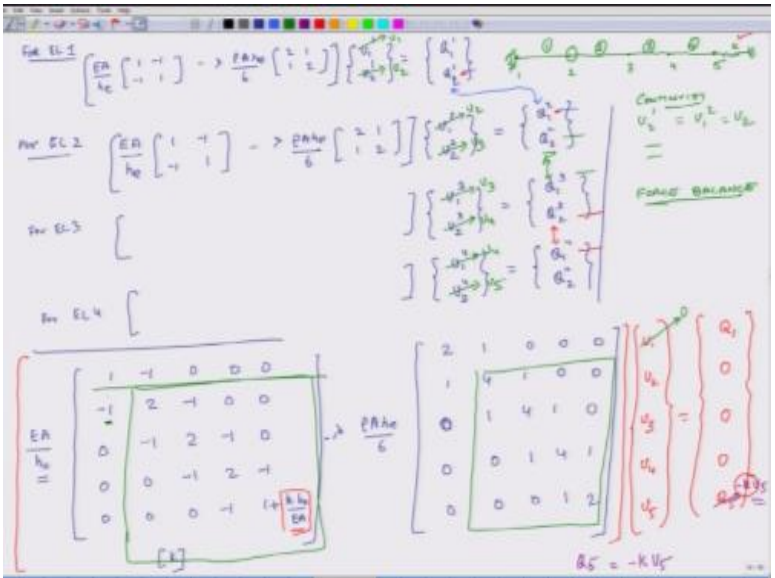

So Q_5 is equal to -K times $u_5$okay, so this guy becomes -K times u_5 and what I, and because this is unknown u_5 I ship this, I move this u_5 on the left-hand side. So here what it gets is I get another term here and this is plus, I do not have space to write here, this is negative and it will be K times h_0 by EA. Because there is a common term out here okay, so this is coming this extra term is coming because of this term, and once I have done this then on the right side this term goes away so this entire the Q_5 the term the constant on the right side in the fifth equation becomes zero okay.

So this is the first boundary condition which is this one, the second boundary condition is that at x is equal to 0 that is at node 1, u is 0 so I specify this to be 0 okay. So now I have to worry only about finding u_2 u_3 u_4 and u_5 okay, which means yeah. So I do not have to worry about my first equation, I do not have to worry about my first equation because u_1 I have to only worry, I have to only solve for u_2, u_3, u_4 and u_5.

So I have to worry about the four, last four equations and also in the last four equations when I multiply this minus 1, oh excuse me by u_1 it will be 0 so contribution of these terms will also be 0 so I will consider only this block in the K matrix, and for the same logic I will consider only this block in the m matrix. So first I have assembled, now I have reduced the number of equations.

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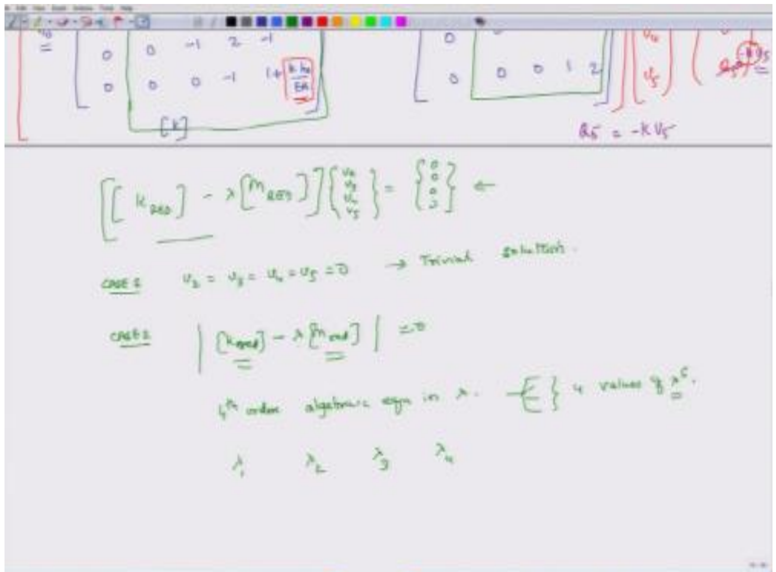

So ultimately what I get is, $[[K_{RED}] - \lambda[M_{RED}]]$ u2 u3 u4 u5 is equal to 0000 okay, now these, these are four equations and four unknowns, these equations will be true in two situations. Situation one case 1, u2 = u3 = u4 = u5 = 0 this is one solution. But we do not like this solution it is a trivial solution for this I did not have to do all this finite element analysis, the second solution will be case two when the determinant of $|[K_{red}]-\lambda\ [M_{red}]|$ equals 0, if this determinant is zero and I know all the values in K reduced, I know all the values in M reduce the only thing which I do not know is λ, that is the only thing I do not know.

So because these are four equations, I will get a fourth-order algebraic equation in λ, fourth order equation, so it will have λ^4 times something plus λ Q times something and so on and so forth. So I will get four values, four values of λs, each of these values is called an Eigen value because it is a characteristic of the system, that is what it literally means okay. Now once I get that λ so, I so I get four λs, λ_1, λ_2, λ_3, λ_4. But what is our goal we have to find u right, we have to find u that was our goal so we have found these λs.

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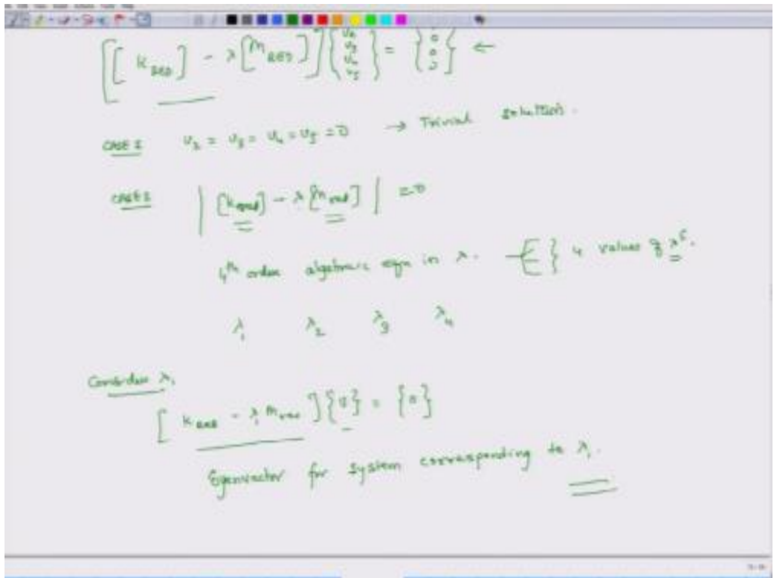

Next what I do is I consider λ_1 and I plug that λ_1 in [Kred - λ_1 Mred] times this reduced vector u_2 u_3 u_4 u_5 equals 0. Now I know everything in this matrix, I also know λ_1, so I can find the relationships between u_2, u_3, u_4 and u_5. When I get those relationships that solution for u is called Eigen vector for system corresponding to λ_1, corresponding to the first Eigen value. Similarly I consider now the second Eigen value so I get a second Eigen vector, then I take the third Eigen value I get the third Eigen vector, and if their system is having n number of equations reduced number of equations then I will get n Eigen values, n Eigen vectors and each Eigen vector will be corresponding to one particular Eigen values okay. So that is the, how I figured out my Eigen vectors okay.

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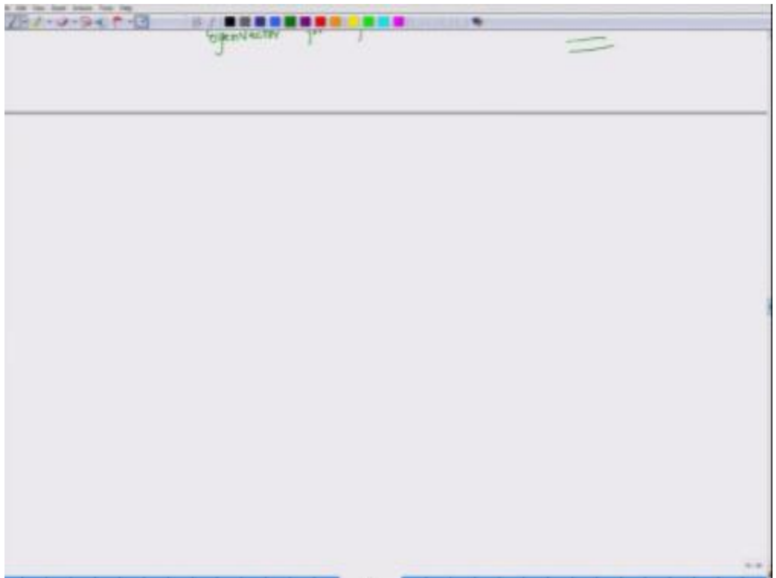

And finally our original goal was what to find u, our original goal was to find u.

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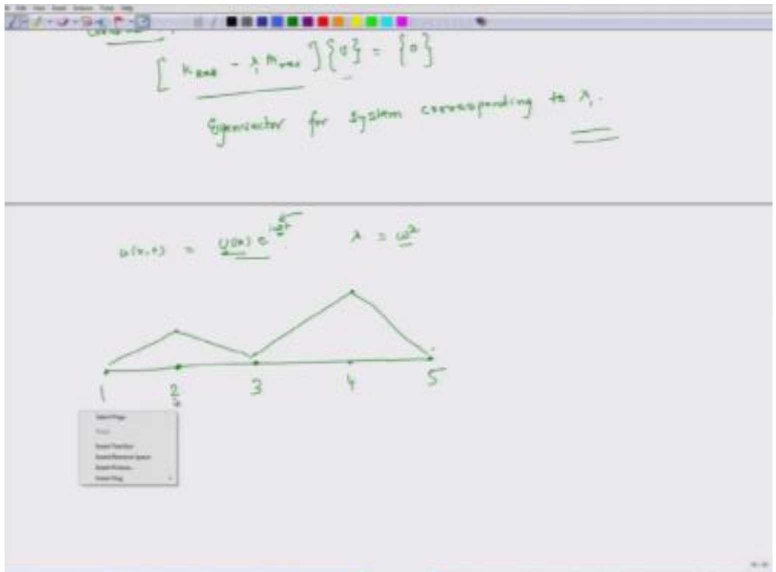

And we had said that u(x, d) is $u(x)e^{i\omega t}$ right, so u(x) so in ω we know that λ equals ω^2 that is what we had assumed right. So I know what is ω, so I take first ω_1 come corresponding to λ_1, I put ω_1 in this equation, and I also put u_1 which is the first Eigen vector. So that is my first solution, then I take the second Eigen value, I find out its Eigen vector, put both these Eigen vectors and Eigen values in this solution, and I get the second solution for the system, that is the that is the second solution and each Eigen vector in this case is also called a mode shape of the system.

Mode shape of the system means so you have four nodes suppose two three four and five, first node does not move because why? Because it is fixed, so what does a mode shape mean that when you are, when the system gets excited by first Eigen frequency maybe u_1 is like this, u_3 is this, this and this, so this is your first mode shape, which means this is the shape or the boundaries between which the system is going to vibrate you know, maybe the second mode shape will be this.

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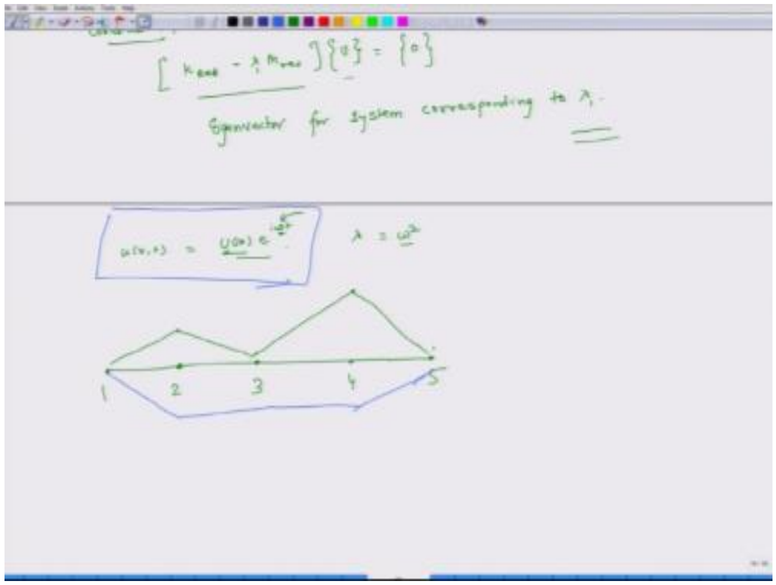

So it all depends on the system you know, so you will have four different shapes corresponding to four different Eigen values and the overall solution for each Eigen value is this, so this completes our treatment for the Eigen value problem. Next week we will, which will be our final week we will extend this discussion into two important topics. The first topic will be time dependent problems, how do we solve time-dependent problems, and the second topic we will cover will be numerical integration that when we develop all these K matrix relations how do we numerically integrate them using our computers. So that is pretty much it and look forward to seeing you tomorrow. Bye.

Acknowledgement

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