

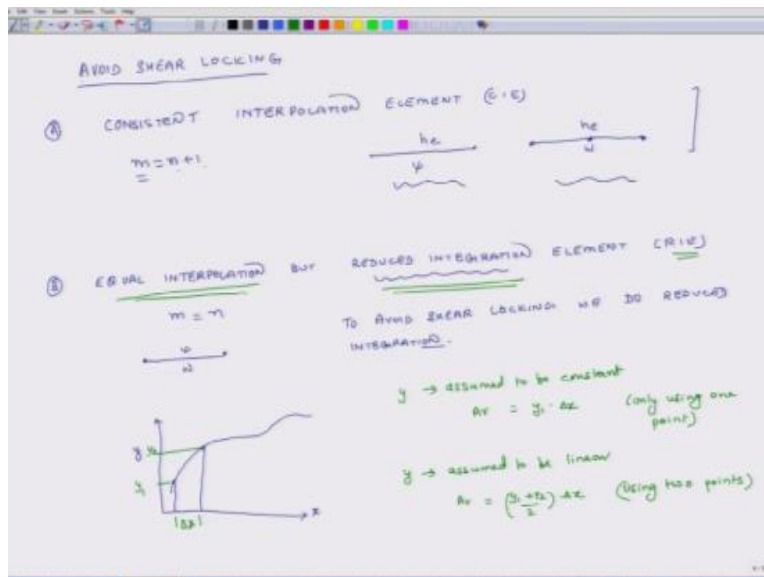
Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 40
Equal interpolation but reduced integration element

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Hello, welcome to basics of finite element analysis, today is the fourth day of this current week and we will continue our discussion on reduced integration element and specifically we will actually conduct an exercise and see how this reduced integration exercise is actually performed, so.

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In one of our earlier classes

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Updated weak form

$$\int_0^{h_e} \frac{dw_1}{dx} GA E_0 \left(\psi + \frac{dw_2}{dx} \right) dx = \int_0^{h_e} w_1 f dx + w_1(0) a_1^e + w_1(h_e) a_2^e \quad (51)$$

$$\int_0^{h_e} \frac{dw_2}{dx} EI \frac{d\psi}{dx} dx + \int_0^{h_e} w_2 GA E_0 \left(\psi + \frac{dw_2}{dx} \right) dx = w_2(0) a_2^e + w_2(h_e) a_3^e \quad (52)$$

we express w and ψ as:

$$w(x) = \sum_{j=1}^m w_j^e \phi_j^e \quad \psi(x) = \sum_{j=1}^n \psi_j^e \beta_j^e \quad (53)$$

we chose $m = n = 2$ //

$$\frac{dw}{dx} = w_1^e \frac{d\phi_1^e}{dx} + w_2^e \frac{d\phi_2^e}{dx} = \frac{w_2^e - w_1^e}{h_e} \quad (54)$$

$$\psi = \psi_1 \beta_1 + \psi_2 \beta_2$$

For two beams

$$\psi + \frac{dw}{dx} = 0 \rightarrow \psi = -\frac{dw}{dx}$$

$$\beta_1 = \frac{x}{h_e} \quad \beta_2 = 1 - \frac{x}{h_e}$$

$$\beta_1 = \frac{x}{h_e} \quad \beta_2 = 1 - \frac{x}{h_e}$$

We had seen that the weak form of the two equilibrium equations are given here okay, and using these weak forms and these interpolation functions we will now develop the stiffness matrices for the entire system.

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Updated wave form

$$\int_0^{\hbar\epsilon} \frac{dw_1}{dz} \text{Grav}_2 \left(\psi + \frac{dw}{dz} \right) dz = \int_0^{\hbar\epsilon} w_1 f dz + w_1(0) a_1^e + w_1(\hbar\epsilon) a_2^e \quad (51)$$

$$\int_0^{\hbar\epsilon} \frac{dw_2}{dz} \text{Grav}_2 \left(\psi + \frac{dw}{dz} \right) dz = w_2(0) a_2^e + w_2(\hbar\epsilon) a_1^e \quad (52)$$

we express w and ψ as:

$$w(x) = \sum_{j=1}^m w_j^e a_j^e \quad \psi(x) = \sum_{j=1}^n s_j^e a_j^e \quad (53)$$

we chose $m = n = 2$ //

$$\frac{dw}{dz} = w_1^e \frac{dw_1}{dz} + w_2^e \frac{dw_2}{dz}$$

$$= \frac{w_2^e - w_1^e}{\hbar\epsilon} \quad (54)$$

$$\psi = s_1 a_1 + s_2 a_2$$

For thin beams

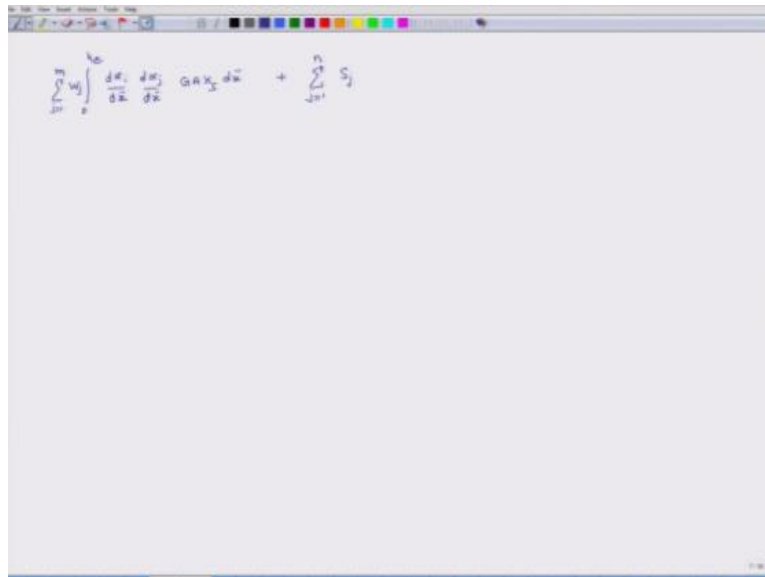
$$\psi + \frac{dw}{dz} = 0 \rightarrow \psi = -\frac{dw}{dz}$$

$$a_2^e = \frac{\hbar\epsilon}{\hbar\epsilon} \quad a_1^e = 1 - \frac{\hbar\epsilon}{\hbar\epsilon}$$

$$a_2^e = \frac{\hbar\epsilon}{\hbar\epsilon} \quad a_1^e = 1 - \frac{\hbar\epsilon}{\hbar\epsilon}$$

So, so that is what we will do, we will start with the first equation so corresponding to the first equation.

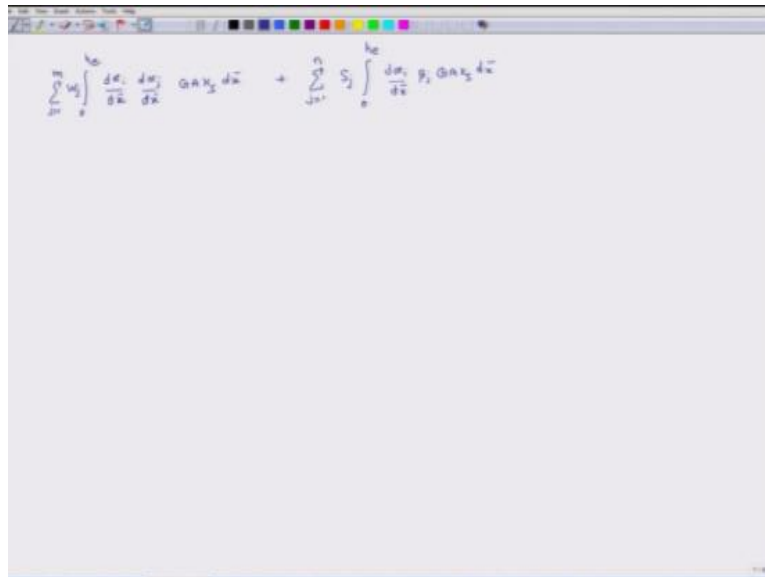
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$$\sum_{j=1}^m w_j \int_0^{h_e} \frac{d\alpha_i}{d\bar{x}} \frac{d\alpha_j}{d\bar{x}} G A K_s d\bar{x} + \sum_{j=1}^n S_j$$

We get integral 0 to h_e $d\alpha_i$ over $d\bar{x}$ $d\alpha_j$ over $d\bar{x}$ $G A K_s d\bar{x}$, and then this entire thing is multiplied by this unknown constant, and I have to sum this up on j is equal to 1 2 m okay. So plus sum on j is equal to one to n S_j so wherever we had ψ 's we are replacing them by S times betas and wherever we had W 's we are replacing that by alpha times W 's right, so that, that is why we are getting.

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$$\sum_{j=1}^m w_j \int_0^{h_e} \frac{d\alpha_i}{dx} \frac{d\alpha_j}{dx} GAK_s d\bar{x} + \sum_{j=1}^n S_j \int_0^{h_e} \frac{d\alpha_i}{dx} \beta_j GAK_s d\bar{x}$$

Here the index of u , we are adding from 1 to m , and in the second case we are adding from 1 to n okay, 0 to h_e $d\alpha_i$ over dx β_j GAK_s $d\bar{x}$, the other thing is the first equation equilibrium equation was multiplied by weight function w_1 and here we assume that weight function w_1 is same as α_i . So that is why we are getting this.

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The image shows handwritten mathematical derivations for the finite element method, specifically for a beam element. The derivations are as follows:

$$\sum_{j=1}^n w_j \int_0^{h_e} \left[\frac{dw_i}{dx} \frac{dw_j}{dx} G A K_s \right] dx + \sum_{j=1}^n S_j \int_0^{h_e} \frac{dw_i}{dx} \beta_j G A K_s dx = \int_0^{h_e} \alpha_i [dx + \alpha_i B A_i^e + \alpha_i (h_e) A_3^e] - E I$$

$$\sum_{j=1}^n w_j \int_0^{h_e} \beta_i \frac{dw_j}{dx} G A K_s dx + \sum_{j=1}^n S_j \int_0^{h_e} \left[\frac{d\beta_i}{dx} \frac{d\beta_j}{dx} E I + \beta_i \beta_j G A K_s \right] dx = \beta_i(0) A_3^e + \beta_i(h_e) A_4^e - E I$$

$$\sum_{j=1}^n K_{ij}^w w_j + \sum_{j=1}^n K_{ij}^s S_j = F_i^1 \rightarrow E 1$$

$$\sum_{j=1}^n K_{ij}^w w_j + \sum_{j=1}^n K_{ij}^s S_j = F_i^2 \rightarrow E 2$$

$$K_{ij}^w = \int_0^{h_e} \frac{dw_i}{dx} \frac{dw_j}{dx} G A K_s dx$$

$$K_{ij}^s = \int_0^{h_e} \beta_i G A K_s \frac{d\beta_j}{dx} dx$$

$$K_{ij}^{12} = \int_0^{h_e} \frac{d\beta_i}{dx} \frac{d\beta_j}{dx} E I + \beta_i \beta_j G A K_s dx$$

α_i here okay. So this equals integral of $\alpha_i f dx$ plus the boundary terms, α_i evaluated at 0 Q_1^e as α_i evaluated at h_e times Q_3^e okay, so that is my first equation. The second equation is again j is equal to 1 to m , w_j 0 to h_e and here the whole equation is being multiplied by weight function w_2 which is beta. So I am multiplying the whole equation by $\beta_i d\alpha_j dx G A K_s dx$ plus j is equal to 12 $n S_j$ 0 to $h_e d\beta_i$ over $dx d\beta_j$ over dx and it should be actually $\bar{x}$ times EI and this entire thing is in brackets plus $\beta_i \beta_j G A K_s$ equals β_i evaluated at 0 Q_2^e plus β_i evaluated at h_e times Q_4^e .

Now couple of comments, here w_j is present and S_j is present in the first equation, and also w_j and S_j is also present in the second equation. w_j and S_j are constants of associated with w and ψ and they are unknowns so we have to find them out right, and specifically physically they mean the displacement at node 1, deflection at node 1 will be w_1 , deflection at node 2 will be w_2 right, similarly rotation at node 1 will be S_1 and rotation at node 2 will be S_2 .

So this one, so these are two equations and in w 's and S these are two equations which are linear, they are linear equations because the powers of w 's and s are 1 and these are coupled equations, so I cannot so these are simultaneous equations in w and s . So from here I can compress all this as $K_{ij}^{11} w_j + K_{ij}^{12} S_j = F_i^1$ okay. So this is the equation I get so this is equation 1, this is equation

2, this is the thing which I get from E1 and what is the definition of K^{11} , I will explain that, so please bear with me and here I am adding these $K_{ij}^{21} W_j + K_{ij}^{22} S_j$ equals F_i^2 where, so this is E2, and now I will define K_{ij}^{11} equals this term, then my this is K^{12}_j okay, this is K^{21}_j , K_{ij}^{21} and finally I have K^{22} this entire thing okay yeah. So if I have a node, an element with two nodes node 1 and node 2, let us say the element is h_e long then I will get essentially two equations from the first weak form E1.

How do I get two equations, when I first in first case I put α_i I give i as the value, I give the value of i as one and I get one equation okay, then I give the second value i is equal to two, I get the second equation okay. Similarly I get two equilibrium equations from the second weak form E2, here also β is the weight function so I have I give two betas, in first case I give β_1 and in the second case I give β_2 , so I get two equations, so overall.

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$$\sum_{j=1}^n w_j \int_0^{h_e} \frac{dw_j}{dx} \frac{dw_i}{dx} EA dx + \sum_{j=1}^n S_j \int_0^{h_e} \frac{dw_j}{dx} \beta_i EA dx = \int_0^{h_e} w_i f dx + \alpha_i \theta \Delta T + \alpha_i (h_e) \Delta T - E1$$

$$\sum_{j=1}^n w_j \int_0^{h_e} \beta_j \frac{dw_j}{dx} EA dx + \sum_{j=1}^n S_j \int_0^{h_e} \left[\frac{d\beta_j}{dx} \frac{dw_j}{dx} EI + \beta_j \beta_i EA \right] dx = \beta_i (h_e) \Delta T + \beta_i (h_e) \Delta T - E2$$

$$\sum_{j=1}^n K_{ij}^{11} w_j + \sum_{j=1}^n K_{ij}^{12} S_j = F_i^1 \rightarrow E1$$

$$\sum_{j=1}^n K_{ij}^{21} w_j + \sum_{j=1}^n K_{ij}^{22} S_j = F_i^2 \rightarrow E2$$

$$K_{ij}^{11} = \int_0^{h_e} \frac{dw_i}{dx} \frac{dw_j}{dx} EA dx$$

$$K_{ij}^{12} = \int_0^{h_e} \frac{dw_i}{dx} EA \beta_j dx$$

$$K_{ij}^{21} = \int_0^{h_e} \beta_i EA \frac{dw_j}{dx} dx$$

$$K_{ij}^{22} = \int_0^{h_e} \left[EI \frac{d\beta_i}{dx} \frac{d\beta_j}{dx} + \beta_i \beta_j EA \right] dx$$

These are four equations right, these are overall four equations associated with two nodes of an element which is h_e long. Now we have talked about reduced integration. So what we note is that when we are integrating this function the function which is underlined in green we do not do reduced integration if M . So here we have deliberately specified m and n as different, now if m is

So there will not be any problem. Well now to avoid shear locking problem we will go on each of these terms and we will figure out which term has to be done for on using a reduced integration method and which terms when we integrate we do it without reduced integration.

$$\sum_{j=1}^m w_j \left[\alpha_1 \frac{d\phi_1}{dz} \frac{d\psi_j}{dz} \text{GAX}_2 \frac{d\bar{w}}{dz} + \sum_{i=1}^n s_i \int_0^{h_0} \frac{d\phi_1}{dz} p_i \text{GAX}_2 \frac{d\bar{w}}{dz} \right] = \int_0^{h_0} \alpha_1 f d\bar{w} + \alpha_1 B d_1^E + \alpha_1 (h_0) d_3^E - E I$$

$$\sum_{j=1}^m w_j \left[p_1 \frac{d\phi_1}{dz} \text{GAX}_2 \frac{d\bar{w}}{dz} + \sum_{i=1}^n s_i \int_0^{h_0} \frac{d\phi_1}{dz} \frac{d\phi_i}{dz} \frac{d\psi_j}{dz} E I + p_i p_j \text{GAX}_2 \right] d\bar{w} = p_1(0) d_2^E + p_1(h_0) d_4^E - E I$$

Normal Inter Required Integration

$$\frac{d\bar{w}}{dz} = \sum_{j=1}^m w_j \left[\sum_{i=1}^n k_{ij}^1 w_i + \sum_{i=1}^n s_i s_i \right] = F_1^z \rightarrow E I$$

$$\sum_{j=1}^m k_{ij}^1 w_i + \sum_{i=1}^n s_i s_i = F_i^z \rightarrow E I$$

$$\sum_{j=1}^m k_{ij}^2 w_i + \sum_{i=1}^n s_i s_i = F_i^z$$

$$k_{ij}^2 = \int_0^{h_0} \frac{d\phi_1}{dz} \frac{d\phi_i}{dz} \text{GAX}_2 \frac{d\bar{w}}{dz}$$

$$k_{ij}^1 = \int_0^{h_0} \frac{d\phi_1}{dz} \text{GAX}_2 \leq p_j \frac{d\bar{w}}{dz}$$

$$K_{ij}^{22} = \int_0^{h_0} \left[E I \frac{d\phi_1}{dz} \frac{d\phi_i}{dz} + p_i p_j \text{GAX}_2 \right] d\bar{w}$$

4 Rungs m = n = 2

Because if we did reduced then if nm was 2, then again we will have the same thing same problem, and here when we are evaluating this β_i times β_i , this is the term which we do reduced

integration. And once we do that we eliminate that shear locking problem and we get our solutions.

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The handwritten notes on the whiteboard show the following steps:

$$\sum_{j=1}^m w_j \int_0^{h_e} \left[\frac{d\sigma_1}{dx} \frac{dw_j}{dx} - \sigma_1 \frac{dw_j}{dx} \right] dx + \sum_{j=1}^n S_j \int_0^{h_e} \frac{d\sigma_1}{dx} \phi_j dx = \int_0^{h_e} \sigma_1 \delta dx + \sigma_1 B \delta + \sigma_1 (h_e) \delta_3 - EI$$

$$\sum_{j=1}^m w_j \int_0^{h_e} \phi_j \frac{d\sigma_1}{dx} dx + \sum_{j=1}^n S_j \int_0^{h_e} \left[\frac{d\phi_j}{dx} EI + \phi_j \frac{d\sigma_1}{dx} \right] dx = \phi_1(0) \delta_1 + \phi_1(h_e) \delta_3 - EI$$

Normal integration is indicated for the first term, and shear integration for the second term.

$$\sum_{j=1}^m K_{1j}^{11} w_j + \sum_{j=1}^n K_{1j}^{12} S_j = F_1 \rightarrow EI$$

$$\sum_{j=1}^m K_{2j}^{11} w_j + \sum_{j=1}^n K_{2j}^{12} S_j = F_2 \rightarrow EI$$

4 terms, $m=n=2$

$$K_{ij}^{11} = \int_0^{h_e} \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx$$

$$K_{ij}^{12} = \int_0^{h_e} \frac{d\phi_i}{dx} \phi_j dx$$

$$K_{ij}^{21} = \int_0^{h_e} \phi_i \frac{d\phi_j}{dx} dx$$

$$K_{ij}^{22} = \int_0^{h_e} \left[EI \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} + \phi_i \phi_j \right] dx$$

Some other points we see that so K^{11} is a okay, so before I do that I will write down the overall.

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The image shows handwritten mathematical derivations for a finite element system of equations. The top part shows the global system of equations for two nodes, \$E1\$ and \$E2\$, where \$m=n=2\$. The equations are:

$$\sum_{j=1}^2 K_{ij}^1 w_j + \sum_{j=1}^2 K_{ij}^2 s_j = F_i^1 \rightarrow E1$$

$$\sum_{j=1}^2 K_{ij}^2 w_j + \sum_{j=1}^2 K_{ij}^1 s_j = F_i^2 \rightarrow E2$$

Below these, the sub-matrices are defined as integrals over the element length \$h\$:

$$K_{ij}^{11} = \int_0^h \frac{dw_i}{dx} \frac{dw_j}{dx} \phi_1 \phi_2 dx$$

$$K_{ij}^{12} = \int_0^h \frac{dw_i}{dx} \phi_1 \phi_2 dx$$

$$K_{ij}^{21} = \int_0^h \left[\phi_1 \frac{ds_j}{dx} \frac{dw_i}{dx} + \phi_1 \phi_2 \phi_1 \phi_2 \right] dx$$

$$K_{ij}^{22} = \int_0^h \left[\phi_2 \frac{ds_j}{dx} \frac{dw_i}{dx} + \phi_1 \phi_2 \phi_1 \phi_2 \right] dx$$

At the bottom, the overall system of equations is written in matrix form:

$$\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{Bmatrix} w \\ s \end{Bmatrix} = \begin{Bmatrix} F^1 \\ F^2 \end{Bmatrix}$$

System of equations, so the overall system of equations looks something like this, so here we have this K^{11} sub-matrix, here I have K^{12} sub-matrix, K^{21} sub-matrix and K^{22} sub-matrix, each of these sub-matrix is two by two because i and j can take.

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$$\sum_{j=1}^n k_{ij}^e u_j + \sum_{j=1}^n k_{ij}^e s_j = F_i^e \quad \rightarrow E1$$

$$\sum_{j=1}^n k_{ij}^e u_j + \sum_{j=1}^n k_{ij}^e s_j = F_i^e \quad \rightarrow E2$$

$$\sum_{j=1}^n k_{ij} u_j + \sum_{j=1}^n k_{ij} s_j = F_i$$

$$K_{ij} = \int_0^L \frac{dx_i}{dx} \frac{dx_j}{dx} \phi_i \phi_j dx$$

$$F_i = \int_0^L p_i \phi_i dx$$

$$K_{12} = \int_0^L \frac{dx_1}{dx} \frac{dx_2}{dx} \phi_1 \phi_2 dx$$

$$K_{22} = \int_0^L \left[\frac{dx_2}{dx} \frac{dx_2}{dx} \phi_2 \phi_2 + p_2 \phi_2 \phi_2 \right] dx$$

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

At the most two values right for m is equal to n is equal to 2. So K^{11} is two by two, K_{21} is two by two and so on and so forth. So the overall stiffness matrix is four by four, and it boils down to same thing because we have four equations so we should have a four by four matrix. In the unknown vector

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Handwritten mathematical derivations for a 4x4 system matrix K .

Top section (Element level):

$$\sum_{j=1}^2 K_{ij} w_j + \sum_{j=1}^2 K_{ij} s_j = F_i \quad \text{for } i=1, 2$$

$$K_{ij} = \int_0^{h_e} \frac{dw_i}{dx} \frac{dw_j}{dx} dx$$

$$K_{ij}^{12} = \int_0^{h_e} \frac{dw_i}{dx} \frac{dw_j}{dx} dx$$

$$K_{ij}^{22} = \int_0^{h_e} \left[E \frac{dw_i}{dx} \frac{dw_j}{dx} + \rho_i \rho_j \frac{dw_i}{dx} \frac{dw_j}{dx} \right] dx$$

Bottom section (Global system):

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ s_1 \\ s_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

4x4 system

We have w_1 and then w_2 , s_1 and s_2 okay, and on the fourth side we have f_1 and f_2 and 0 and 0, so this is the external force which is being applied and plus if there are some point forces they are q_1 , q_3 , q_2 and q_4 , and all these equations are at element level. All these equations are at element level. Now when you see this equation, this over four by four system you note that K^{11} so now let us look at, so first just see this so we have four sub matrices K_{11} , K_{12} , K_{21} , K_{22} okay.

(Refer Slide Time: 16:33)

The image shows handwritten mathematical derivations for the finite element method, specifically the assembly of the global stiffness matrix K and the global force vector F .

Global Force Vector F :

$$\sum_{i=1}^m w_i \left[\frac{dx_i}{dx} \frac{dx_j}{dx} GA K_2 dx \right] + \sum_{j=1}^n S_j \int_0^{L_e} \frac{d\alpha_i}{dx} \beta_j GA K_3 dx = \int_0^{L_e} \alpha_i f dx + \alpha_i(0) A_1 + \alpha_i(L_e) A_2 - E I$$

$$\sum_{j=1}^n w_j \left[\beta_j \frac{dx_j}{dx} GA K_2 dx \right] + \sum_{j=1}^n S_j \int_0^{L_e} \left[\frac{d\beta_j}{dx} EI + \beta_j \beta_j GA K_3 \right] dx = \beta_j(0) A_2 + \beta_j(L_e) A_1 - E I$$

Global Stiffness Matrix K :

$$K_{ij}^{11} = \int_0^{L_e} \frac{d\alpha_i}{dx} \frac{d\alpha_j}{dx} GA K_2 dx$$

$$K_{ij}^{22} = \int_0^{L_e} \left[EI \frac{d\beta_i}{dx} \frac{d\beta_j}{dx} + \beta_i \beta_j GA K_3 \right] dx$$

Assembly of Global Matrices:

$$\sum_{i=1}^m K_{ij}^{11} w_j + \sum_{j=1}^n K_{ij}^{22} S_j = F_i^1 \rightarrow E I$$

$$\sum_{i=1}^m K_{ij}^{21} w_j + \sum_{j=1}^n K_{ij}^{22} S_j = F_i^2 \rightarrow E I$$

Notes:

- Normal integrals
- Reduces into Gaussian
- 4 Gauss
- $m = n = 2$

Now we will look at K^{11} , so K^{11} is symmetric in i and j which means this sub matrix.

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The image shows handwritten mathematical derivations for a 4x4 system matrix K . At the top, a nodal equation is given: $\sum_{j=1}^3 K_{ij} u_j + \sum_{j=1}^2 K_{ij}^s s_j = F_i$, with a note $m = 3, n = 2$. Below this, the stiffness coefficient K_{ij}^s is derived as $K_{ij}^s = \int_0^{L_e} \frac{dx_1}{dx} \frac{dN_i}{dx} \frac{dN_j}{dx} dx$. The global stiffness coefficient K_{ij}^{12} is then defined as $K_{ij}^{12} = \int_0^{L_e} \frac{dN_i}{dx} \frac{dN_j}{dx} \frac{dN_2}{dx} dx$. The global stiffness coefficient K_{ij}^{22} is defined as $K_{ij}^{22} = \int_0^{L_e} \left[E \frac{dN_i}{dx} \frac{dN_j}{dx} + P_i P_j \frac{dN_i}{dx} \frac{dN_j}{dx} \right] dx$. At the bottom, the final 4x4 system is presented as a matrix equation: $\begin{bmatrix} K^{11} & \vdots & K^{12} \\ \vdots & \ddots & \vdots \\ K^{21} & \vdots & K^{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ s_1 \\ s_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ 0 \\ 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}$, with a note "4x4 system" to the right.

Is symmetric in i and j, okay.

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$$\sum_{j=1}^n w_j \left[\int_0^{L_e} \frac{d^2 u_j}{dx^2} \frac{d^2 u}{dx^2} dx + \int_0^{L_e} \frac{d u_j}{dx} \frac{d u}{dx} dx \right] + \sum_{j=1}^n S_j \int_0^{L_e} \frac{d u_j}{dx} \frac{d u}{dx} dx = \int_0^{L_e} u_j f dx + u_j B \frac{d u}{dx} + u_j (u_e) \frac{d u}{dx} - E I$$

$$\sum_{j=1}^n w_j \left[\int_0^{L_e} \frac{d^2 u_j}{dx^2} \frac{d^2 u}{dx^2} dx + \int_0^{L_e} \frac{d u_j}{dx} \frac{d u}{dx} dx \right] + \sum_{j=1}^n S_j \int_0^{L_e} \left[\frac{d^2 u_j}{dx^2} \frac{d^2 u}{dx^2} E I + \frac{d u_j}{dx} \frac{d u}{dx} \right] dx = \beta_1 (u) \frac{d u}{dx} + \beta_1 (u_e) \frac{d u}{dx} - E I$$

Normal Integ. REDUCES INTEGRATION

$$\sum_{j=1}^n w_j \left[\int_0^{L_e} \frac{d^2 u_j}{dx^2} \frac{d^2 u}{dx^2} dx + \int_0^{L_e} \frac{d u_j}{dx} \frac{d u}{dx} dx \right] + \sum_{j=1}^n S_j \int_0^{L_e} \left[\frac{d^2 u_j}{dx^2} \frac{d^2 u}{dx^2} E I + \frac{d u_j}{dx} \frac{d u}{dx} \right] dx = \beta_1 (u) \frac{d u}{dx} + \beta_1 (u_e) \frac{d u}{dx} - E I$$

4 Springs $m = n = 2$

$$\sum_{j=1}^n w_j \left[\int_0^{L_e} \frac{d^2 u_j}{dx^2} \frac{d^2 u}{dx^2} dx + \int_0^{L_e} \frac{d u_j}{dx} \frac{d u}{dx} dx \right] + \sum_{j=1}^n S_j \int_0^{L_e} \left[\frac{d^2 u_j}{dx^2} \frac{d^2 u}{dx^2} E I + \frac{d u_j}{dx} \frac{d u}{dx} \right] dx = \beta_1 (u) \frac{d u}{dx} + \beta_1 (u_e) \frac{d u}{dx} - E I$$

$$K_{ij} = \int_0^{L_e} \frac{d u_i}{dx} \frac{d u_j}{dx} dx$$

$$K_{ij} = \int_0^{L_e} \left[E I \frac{d^2 u_i}{dx^2} \frac{d^2 u_j}{dx^2} + \frac{d u_i}{dx} \frac{d u_j}{dx} \right] dx$$

$$K_{ij} = \int_0^{L_e} \frac{d u_i}{dx} \frac{d u_j}{dx} dx$$

Now we look at K^{22} this is also symmetric in i and j okay, does not matter whether it is i equals j , j equals i we will get the same answer.

(Refer Slide Time: 16:58)

$$K_{ij} = \int_{\Omega} B_i^T B_j d\Omega$$

$$K_{ij} = \int_{\Omega} \frac{dN_i}{dx} \frac{dN_j}{dx} d\Omega$$

$$\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ 0 \\ 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \quad 4 \times 4 \text{ system}$$

So the other diagonal sub-metrics which is K^{22} is also symmetric for this overall matrix to be symmetric now what we want is that K^{12} should be transpose of K^{21} , then the overall matrix will be symmetric.

(Refer Slide Time: 17:14)

The image shows handwritten mathematical derivations on a whiteboard, likely for a finite element analysis of a beam element. The derivations are as follows:

$$\sum_{j=1}^n w_j \int_0^{L_e} \left[\frac{d\alpha_i}{dx} \frac{dw_j}{dx} GAK_s \right] dx + \sum_{j=1}^n S_j \int_0^{L_e} \frac{dw_j}{dx} \beta_j GAK_s dx = \int_0^{L_e} \alpha_i f dx + \alpha_i B \dot{\alpha}_i + \alpha_i (h_e) \dot{\alpha}_i - EI$$

$$\sum_{j=1}^n w_j \int_0^{L_e} \left[\beta_i \frac{dw_j}{dx} GAK_s \right] dx + \sum_{j=1}^n S_j \int_0^{L_e} \left[\frac{d\beta_i}{dx} \frac{dw_j}{dx} EI + \beta_i \beta_j GAK_s \right] dx = \beta_i (f) \dot{\alpha}_i + \beta_i (h_e) \dot{\alpha}_i - EI$$

Below these, the nodal forces are given as:

$$\sum_{j=1}^n K_{ij}^1 w_j + \sum_{j=1}^n K_{ij}^2 S_j = F_i^1 \rightarrow EI$$

$$\sum_{j=1}^n K_{ij}^3 w_j + \sum_{j=1}^n K_{ij}^4 S_j = F_i^2 \rightarrow EI$$

Then, the stiffness matrices are defined as:

$$K_{ij}^1 = \int_0^{L_e} \frac{d\alpha_i}{dx} \frac{dw_j}{dx} GAK_s dx$$

$$K_{ij}^2 = \int_0^{L_e} \beta_i \frac{dw_j}{dx} GAK_s dx$$

$$K_{ij}^3 = \int_0^{L_e} \left[EI \frac{d\beta_i}{dx} \frac{d\beta_j}{dx} + \beta_i \beta_j GAK_s \right] dx$$

$$K_{ij}^4 = \int_0^{L_e} \beta_i \beta_j GAK_s dx$$

There are also some notes and diagrams: "REDUCES INTEGRATION" with an arrow pointing to the second equation, and a diagram of a beam element of length \$L_e\$ with nodes 1 and 2.

So let us see whether that is the case or not, so here we see that K^{12} is $d\alpha_i$ over $d\bar{x}$ GAK_s β_j integral, and K^{21} is β_i GAK_s and this should be $j' dx$, so the definition of K^{12} and K^{21} is such that these sub matrices are also.

(Refer Slide Time: 17:43)

The image shows a handwritten derivation on a digital whiteboard. At the top, a small equation is written: $k_{ij} = \frac{\partial^2 U}{\partial x_i \partial x_j}$. Below this, the main derivation is shown. It starts with a 4x4 matrix $[K]$ partitioned into four 2x2 blocks: k^{11} , k^{12} , k^{21} , and k^{22} . This matrix is multiplied by a column vector of nodal displacements $\begin{Bmatrix} u_1 \\ u_2 \\ s_1 \\ s_2 \end{Bmatrix}^e$. The result is equated to a column vector of nodal forces $\begin{Bmatrix} F_1 \\ F_2 \\ 0 \\ 0 \end{Bmatrix}^e$. This is then rearranged to show the matrix multiplication: $\begin{Bmatrix} k^{11} & k^{12} \\ k^{21} & k^{22} \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ s_1 \\ s_2 \end{Bmatrix}^e = \begin{Bmatrix} F_1 \\ F_2 \\ 0 \\ 0 \end{Bmatrix}^e \rightarrow \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}^e$. To the right of this, it is noted that this is a "4x4 system". Below the main equation, it is concluded that $[K] \rightarrow \text{Symmetric}$.

They are not symmetric in themselves, but this sub matrix is transpose of this sub matrix. So the overall K matrix is symmetric okay, so once again.

(Refer Slide Time: 18:02)

The image shows a handwritten derivation of the weak formulation and the resulting sum matrix for a beam element. The derivation starts with the strong form of the beam equation and integrates it by parts to obtain the weak form. The weak form is then discretized using shape functions to derive the element stiffness matrix and the element load vector.

$$\sum_{j=1}^m w_j \int_0^{L_e} \left[\frac{d w_j}{d x} \frac{d u}{d x} \right] d x + \sum_{j=1}^n S_j \int_0^{L_e} \frac{d w_j}{d x} p_j u d x = \int_0^{L_e} w_1 \left[d x + u_1 \theta \frac{d}{d x} + u_2 \left(\frac{d}{d x} \right) \right] d x - E I$$

$$\sum_{j=1}^m w_j \int_0^{L_e} p_j \frac{d w_j}{d x} u d x + \sum_{j=1}^n S_j \int_0^{L_e} \left[\frac{d p_j}{d x} E I + p_j \theta \frac{d}{d x} \right] d x = p_1(0) u_1^e + p_2(L_e) u_2^e - E I$$

Normal Integ. REDUCES INTEGRATION

$$\sum_{j=1}^m K_{ij}^e w_j + \sum_{j=1}^n K_{ij}^e S_j = F_i^e \quad \rightarrow E I$$

$$\sum_{j=1}^m K_{ij}^e w_j + \sum_{j=1}^n K_{ij}^e S_j = F_i^e \quad \rightarrow E I$$

$$K_{ij}^e = \int_0^{L_e} \frac{d w_i}{d x} \frac{d w_j}{d x} d x$$

$$K_{ij}^e = \int_0^{L_e} \left[E I \frac{d p_i}{d x} \frac{d p_j}{d x} + p_i p_j \theta \frac{d}{d x} \right] d x$$

4 Equations m = n = 2

Our weak formulation and its sum matrix, so this is something we saw also in our earlier formulations that if the system was conservative and if the system could be represented as a functional you know as a bilinear functional then whatever we will get out will be something of this nature.

(Refer Slide Time: 18:21)

$$K_{ij}^e = \int_0^h \frac{dx}{ds} \frac{du_i}{ds} \frac{du_j}{ds} ds$$

$$K_{ij}^s = \int_0^h \frac{d\psi_i}{ds} \frac{d\psi_j}{ds} G A k_s ds$$

$$K_{ij}^b = \int_0^h \left[EI \frac{d^2 \theta_i}{ds^2} \frac{d^2 \theta_j}{ds^2} + P_i P_j \frac{d\psi_i}{ds} \frac{d\psi_j}{ds} \right] ds$$

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \psi_1 \\ \psi_2 \end{Bmatrix}^e = \begin{Bmatrix} F_1 \\ F_2 \\ 0 \\ 0 \end{Bmatrix}^e \rightarrow \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}^e$$

$[K] \rightarrow \text{symmetric}$

4x4 system
— El. level eqns

So we have a symmetric matrix and the way we, so we can compute each element of these of this symmetric matrix, and then we find so this is element level equation, and then the next step is that if we have broken our beam into n number of elements so we write such, n such set of equations, assemble them using conditions of continuity and force equilibrium and then impose the boundary conditions to get a reduced set of equations and solve them to get our solution. So this completes our treatment of the shear deformable beam which is also known as the Timoshenko beam.

And in this treatment we have learned a couple of several, we have learned several things, one is how do we handle equations handle more than one differential equations if they are coupled, they maybe linear but they are coupled equations, so that is what we saw in this case that ψ and W were appearing in both the differential equations, and we could construct a weak form and from that we were able to develop a set of equations which could be used to solve these unknowns. So that is first thing, the second thing which we learned through this exercise is that if one particular variable requires that its interpolation function should be one level higher than the other one then we can address that, then we can address that requirement either using the consistent

interpolation element method where m was defined as $n+1$ or the other one was where m equals n but we still take care of the differentiability requirement by using reduced integration method.

We will learn more about this reduced integration method in the next week when we deal with a numerical integration procedures, because in computer implementation of finite element analysis we do not actually go around analytically integrating all the functions but rather we use some fast numerical procedures to integrate these functions. So we will learn more about that and then we will see how we can use reduced integration approaches to address these kinds of problems. So this is pretty much what I wanted to cover in context of emotion go beams, and in the next class or the next coming classes we will discuss Eigen value problems. So that completes our discussion for today and we will meet tomorrow. Thank you.

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