Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 40
Equal interpolation but reduced integration element

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Hello, welcome to basics of finite element analysis, today is the fourth day of this current week and we will continue our discussion on reduced integration element and specifically we will actually conduct an exercise and see how this reduced integration exercise is actually performed, so.

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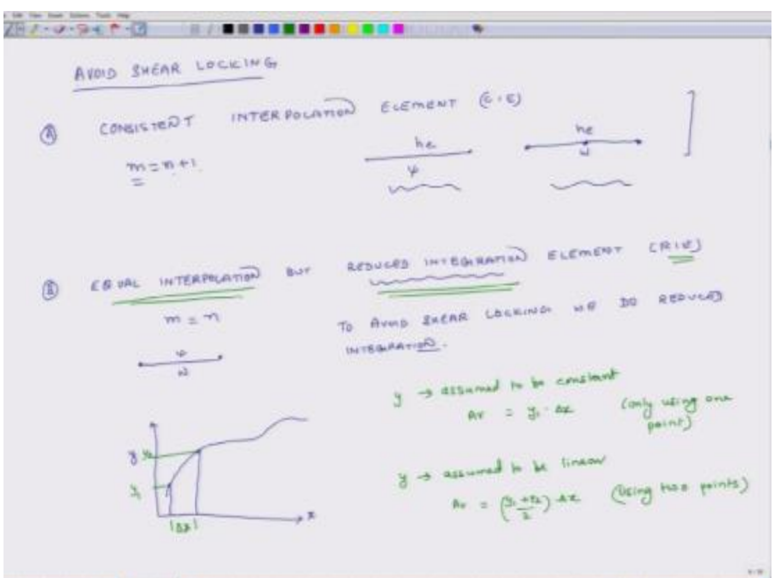

In one of our earlier classes

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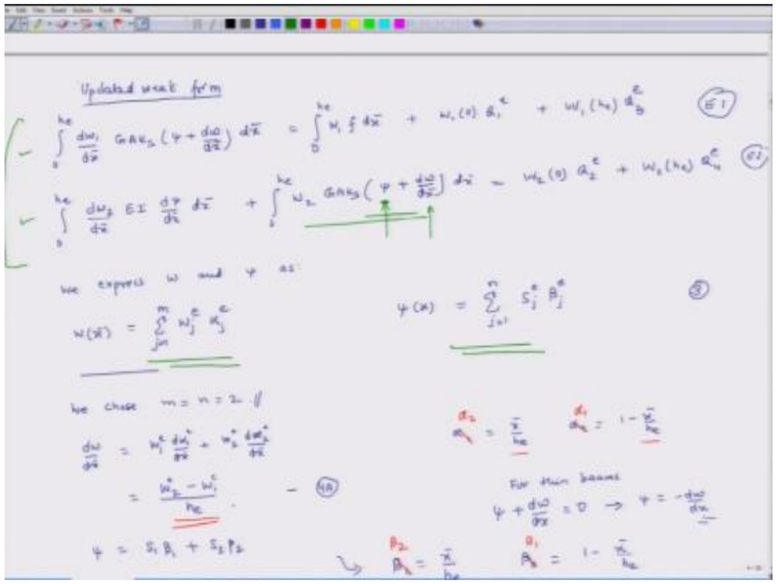

We had seen that the weak form of the two equilibrium equations are given here okay, and using these weak forms and these interpolation functions we will now develop the stiffness matrices for the entire system.

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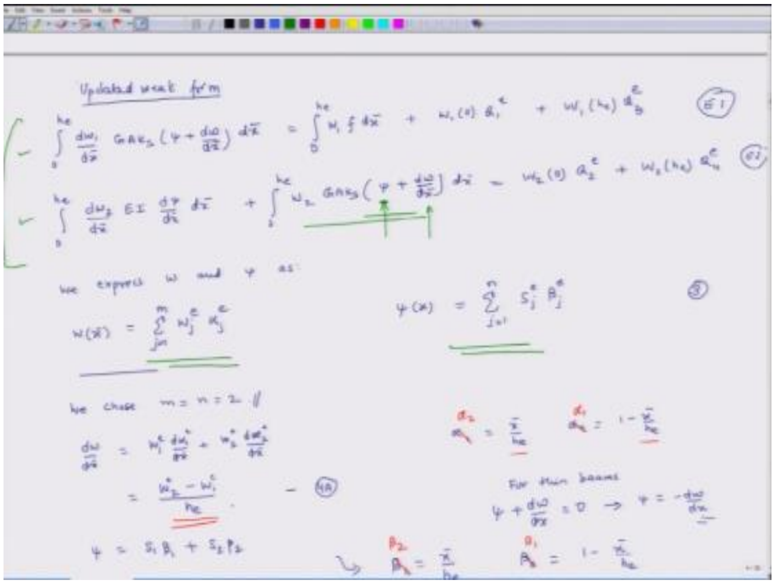

So, so that is what we will do, we will start with the first equation so corresponding to the first equation.

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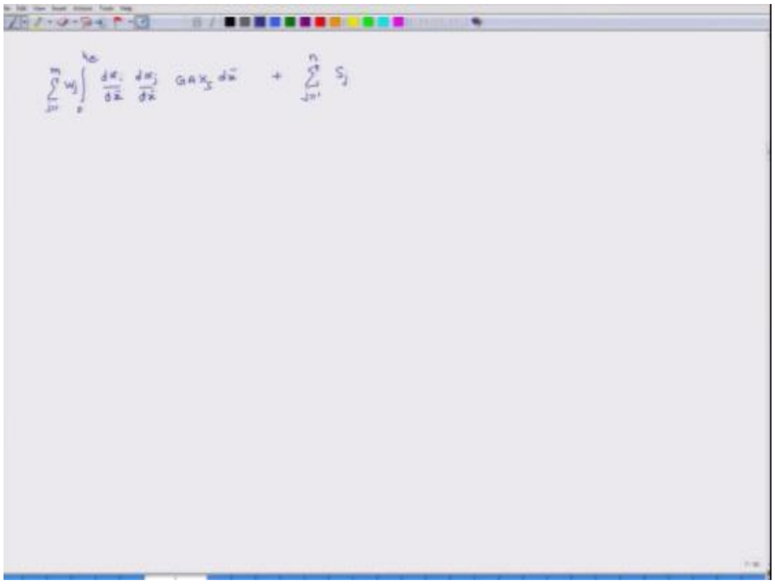

We get integral 0 to h_e $d\alpha_i$ over $d\overline{x}$ $d\alpha_j$ over $d\overline{x}$ GAK_s $d\overline{x}$, and then this entire thing is multiplied by this unknown constant, and I have to sum this up on j is equal to 1 2 m okay. So plus sum on j is equal to one to n S_j so wherever we had ψ's we are replacing them by S times betas and wherever we had W's we are replacing that by alpha times W's right, so that, that is why we are getting.

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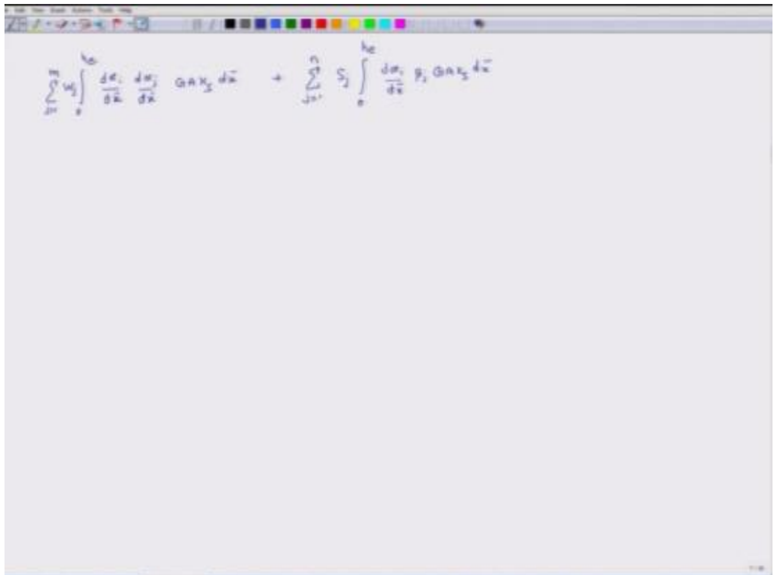

Here the index of u, we are adding from 1 to m, and in the second case we are adding from 1 to n okay, 0to h_e $d\alpha_i$ over dx β_j GAK_s $d\overline{x}$, the other thing is the first equation equilibrium equation was multiplied by weight function w_1 and here we assume that weight function w_1 is same as α_i. So that is why we are getting this.

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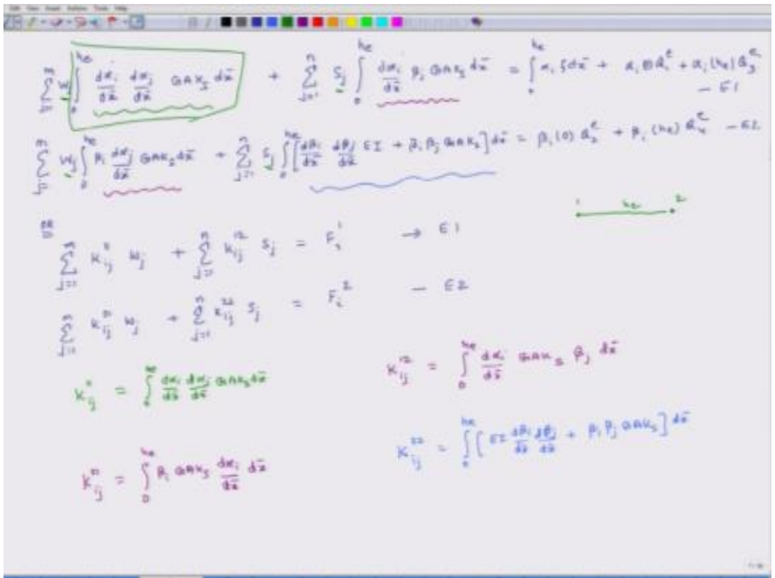

α_i here okay. So this equals integral of α_i f $d\bar{x}$ plus the boundary terms, α_i evaluated at 0 Q_1^e as α_i evaluated at h_e times Q_3^e okay, so that is my first equation. The second equation is again j is equal to 1 to m, W_j 0 to h_e and here the whole equation is being multiplied by weight function W_2 which is beta. So I am multiplying the whole equation by β_i $d\alpha_j$ $d\bar{x}$ GAK_s $d\bar{x}$ plus j is equal to 12 n S_j 0 to h_e $d\beta_i$ over dx $d\beta_j$ over dx and it should be actually $\bar{x}$ times EI and this entire thing is in brackets plus $\beta_i\beta_j$ GAK_s equals β_i evaluated at 0 Q_2^e plus β_i evaluated at h_e times Q_4^e.

Now couple of comments, here W_j is present and S_j is present in the first equation, and also W_j and S_j is also present in the second equation. W_j and S_j are constants of associated with w and ψ and they are unknowns so we have to find them out right, and specifically physically they mean the displacement at node 1, deflection at node 1 will be $w_{1,}$ deflection at node 2 will be w_2 right, similarly rotation at node 1 will be S_1 and rotation at node 2 will be S_2.

So this one, so these are two equations and in W's and S these are two equations which are linear, they are linear equations because the powers of W's and s are 1 and these are coupled equations, so I cannot so these are simultaneous equations in w and s. So from here I can compress all this as K_{ij}^{11} W_j $+K_{ij}^{12}$ S_j $=F_i^1$ okay. So this is the equation I get so this is equation 1, this is equation

2, this is the thing which I get from E1 and what is the definition of K^{11}, I will explain that, so please bear with me and here I am adding these K_{ij}^{21} W_j + K_{ij}^{22} S_j equals F_i^2 where, so this is E2, and now I will define K_{ij}^{11} equals this term, then my this is $K^{12}{}_j$ okay, this is $K^{21}{}_j$, K_{ij}^{21} and finally I have K^{22} this entire thing okay yeah. So if I have a node, an element with two nodes node 1 and node 2, let us say the element is h_e long then I will get essentially two equations from the first weak form E1.

How do I get two equations, when I first in first case I put α_i I give i as the value, I give the value of i as one and I get one equation okay, then I give the second value i is equal to two, I get the second equation okay. Similarly I get two equilibrium equations from the second weak form E2, here also β is the weight function so I have I give two betas, in first case I give β_1 and in the second case I give β_2, so I get two equations, so overall.

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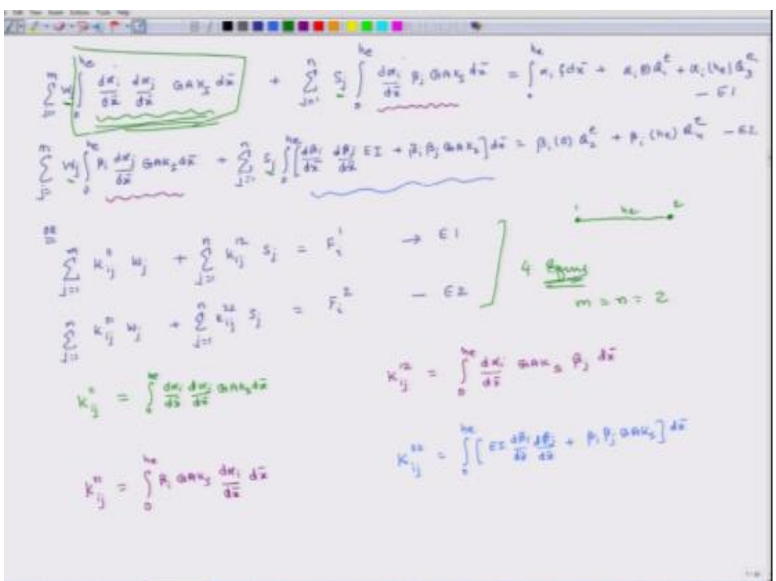

These are four equations right, these are overall four equations associated with two nodes of an element which is h_e long. Now we have talked about reduced integration. So what we note is that when we are integrating this function the function which is underlined in green we do not do reduced integration if M. So here we have deliberately specified m and n as different, now if m is equal to n is equal to 2 then we will have four equations which we talked about right, we will have four equations and we still have two nodes.

So there will not be any problem. Well now to avoid shear locking problem we will go on each of these terms and we will figure out which term has to be done for on using a reduced integration method and which terms when we integrate we do it without reduced integration.

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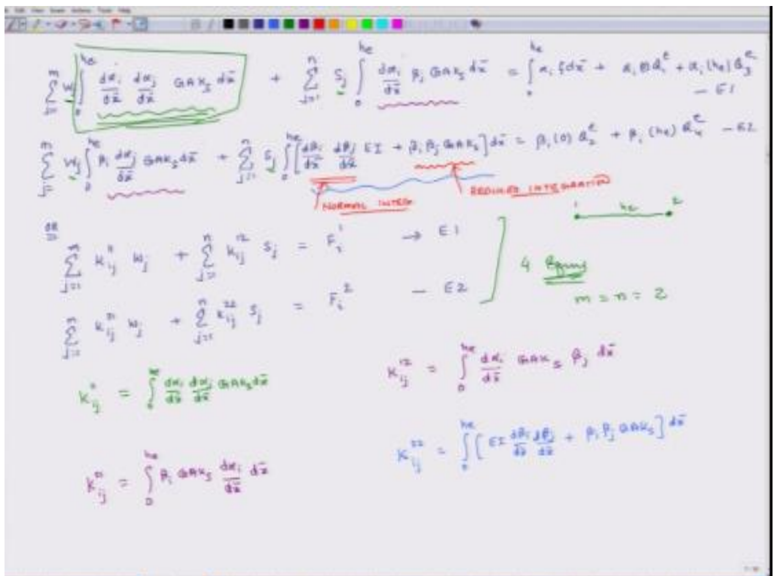

When we look at this term in green which is represented by K^{11} we do not do reduced integration on that, the terms which are underlined in purple which is K^{12} and K^{22} they also do not go through reduced integration. The term which is underlined in light blue we will look at this place, so here we have two terms the first term is dβ_i over dx dβ_j over dx times EI, and the second one is BI or β_i β_j times GAK_s right, this term so this term which is underlined with two lines this we do not do reduced integration normal integration okay.

Because if we did reduced then if nm was 2, then again we will have the same thing same problem, and here when we are evaluating this β_i times β_j, this is the term which we do reduced integration. And once we do that we eliminate that shear locking problem and we get our solutions.

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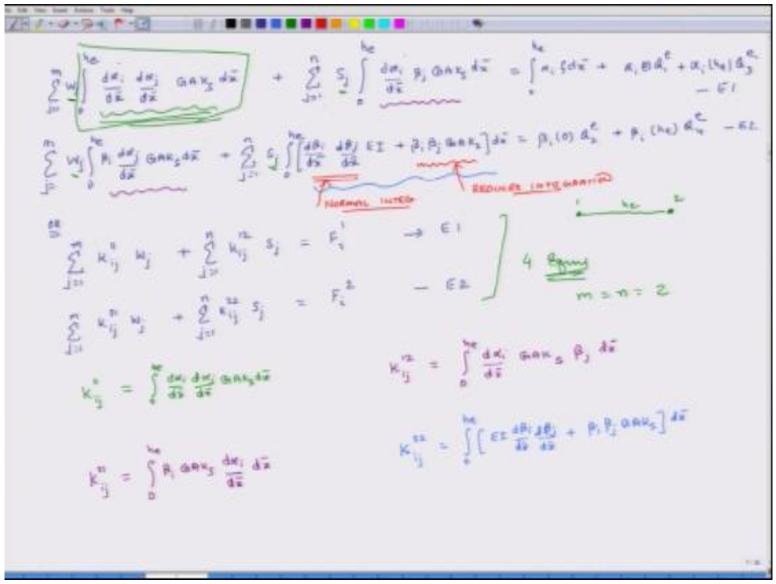

Some other points we see that so K^{11} is a okay, so before I do that I will write down the overall.

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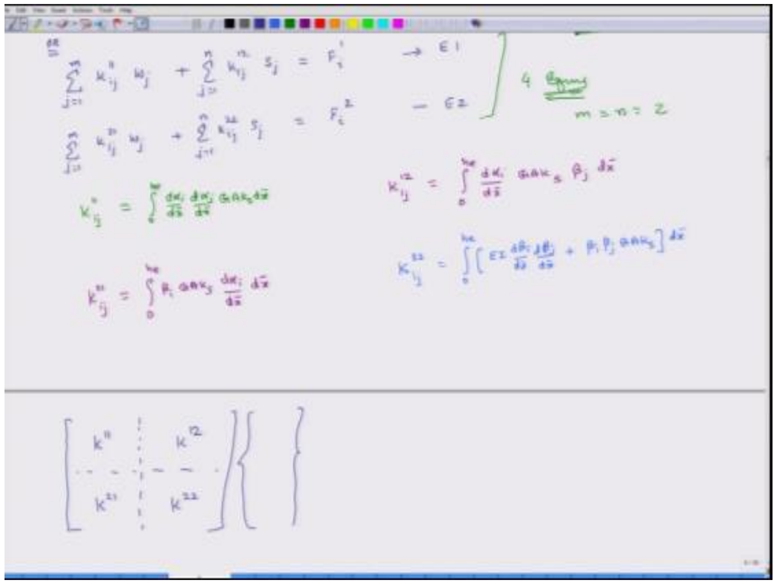

System of equations, so the overall system of equations looks something like this, so here we have this K^{11} sub-matrix, here I have K^{12} sub-matrix, K^{21} sub-matrix and K22 sub-matrix, each of these sub-matrix is two by two because i and j can take.

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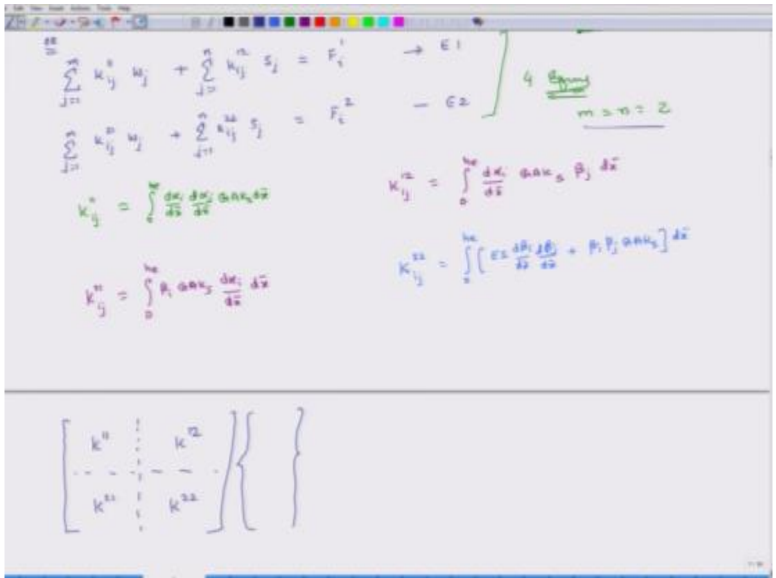

At the most two values right for m is equal to n is equal to 2. So K^{11} is two by two, K_{21} is two by two and so on and so forth. So the overall stiffness matrix is four by four, and it boils down to same thing because we have four equations so we should have a four by four matrix. In the unknown vector

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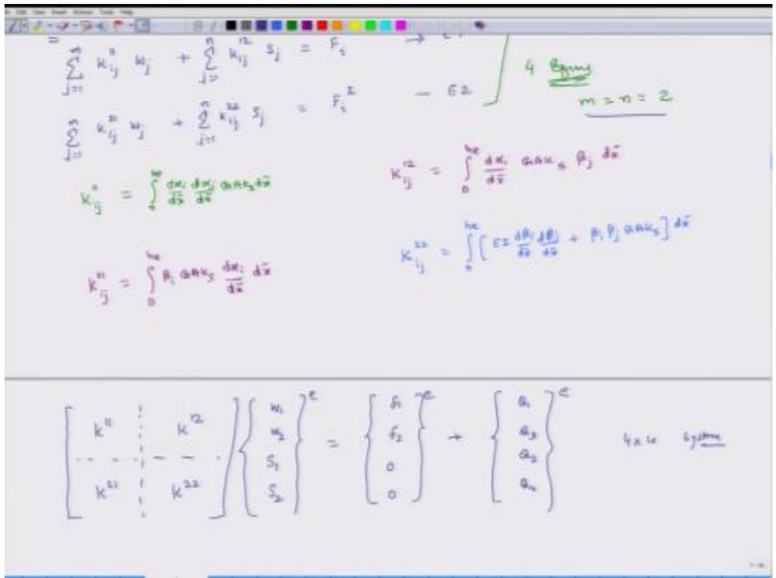

We have W_1 and then W_2, S_1 and S_2 okay, and on the fourth side we have f_1 and f_2 and 0 and 0, so this is the external force which is being applied and plus if there are some point forces they are Q_1 Q_3 Q_2 and Q_4, and all these equations are at element level. All these equations are at element level. Now when you see this equation, this over four by four system you note that K^{11} so now let us look at, so first just see this so we have four sub matrices K_{11}, K_{12}, K_{21}, K_{22} okay.

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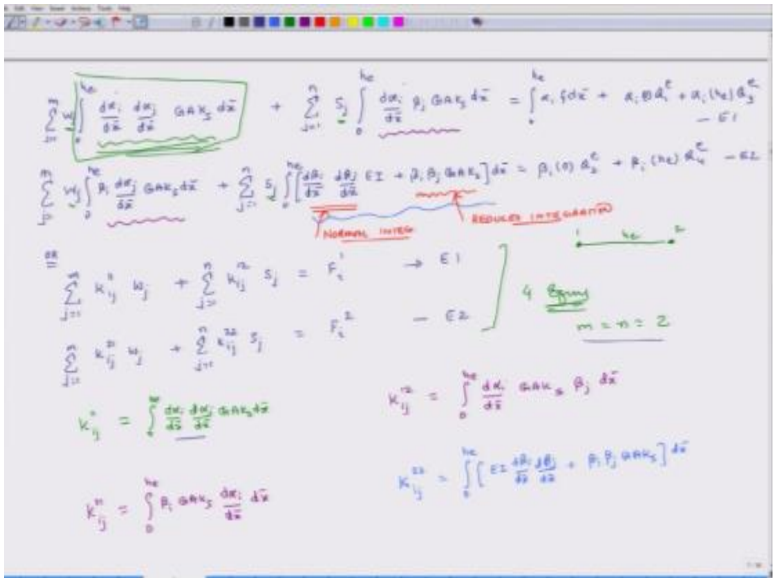

Now we will look at K^{11}, so K^{11} is symmetric in i and j which means this sub matrix.

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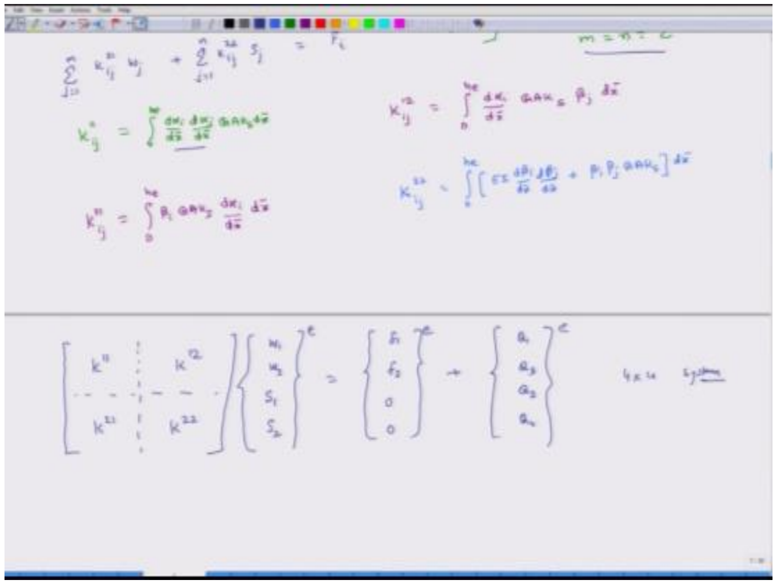

Is symmetric in i and j, okay.

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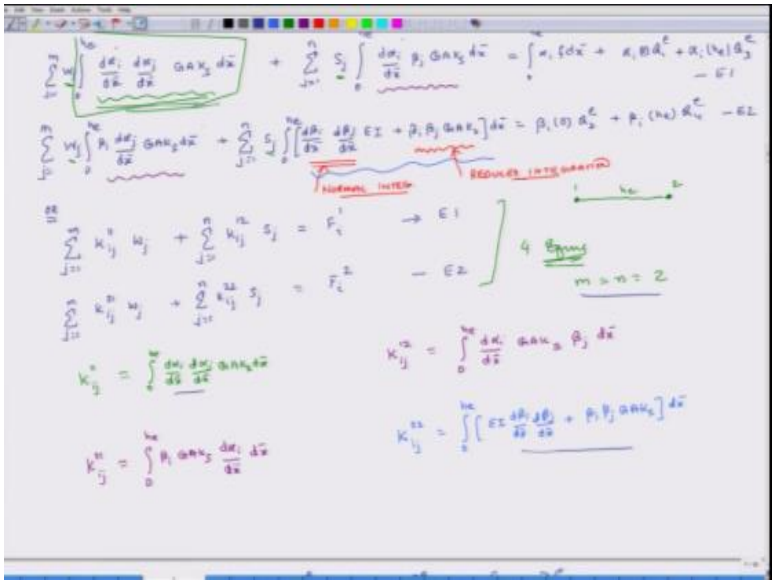

Now we look at K^{22} this is also symmetric in i and j okay, does not matter whether it is i equals j, j equals i we will get the same answer.

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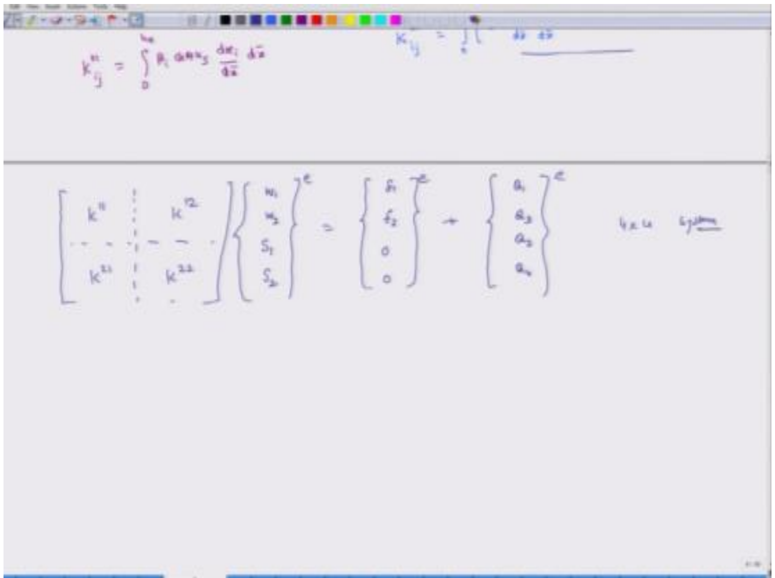

So the other diagonal sub-metrics which is K^{22} is also symmetric for this overall matrix to be symmetric now what we want is that K^{12} should be transpose of K^{21}, then the overall matrix will be symmetric.

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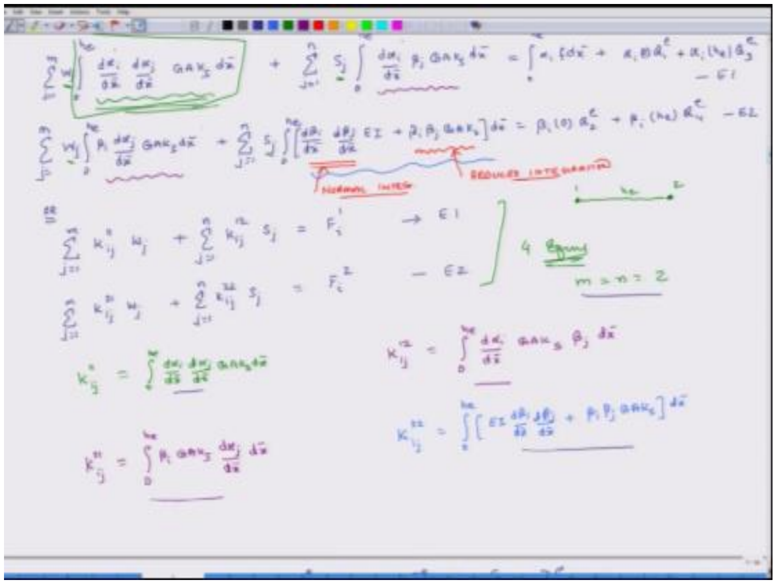

So let us see whether that is the case or not, so here we see that K^{12} is $d\alpha_i$ over $d\overline{x}$ GAK_s β_j integral, and K^{21} is β_i GAK_s and this should be j' dx, so the definition of K^{12} and K^{21} is such that these sub matrices are also.

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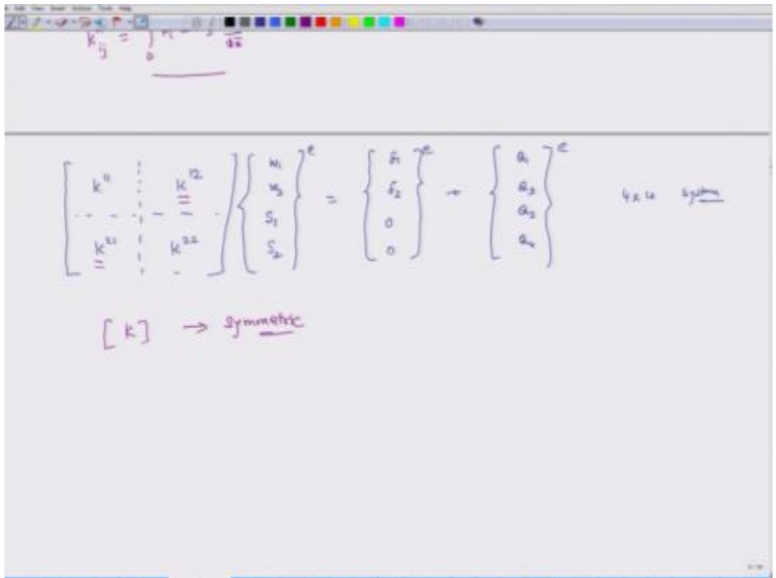

They are not symmetric in themselves, but this sub matrix is transpose of this sub matrix. So the overall K matrix is symmetric okay, so once again.

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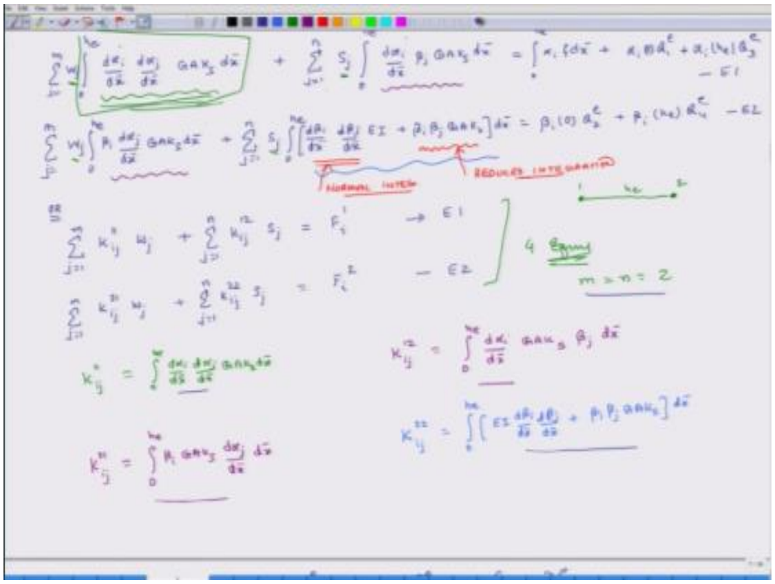

Our weak formulation and its sum matrix, so this is something we saw also in our earlier formulations that if the system was conservative and if the system could be represented as a functional you know as a bilinear functional then whatever we will get out will be something of this nature.

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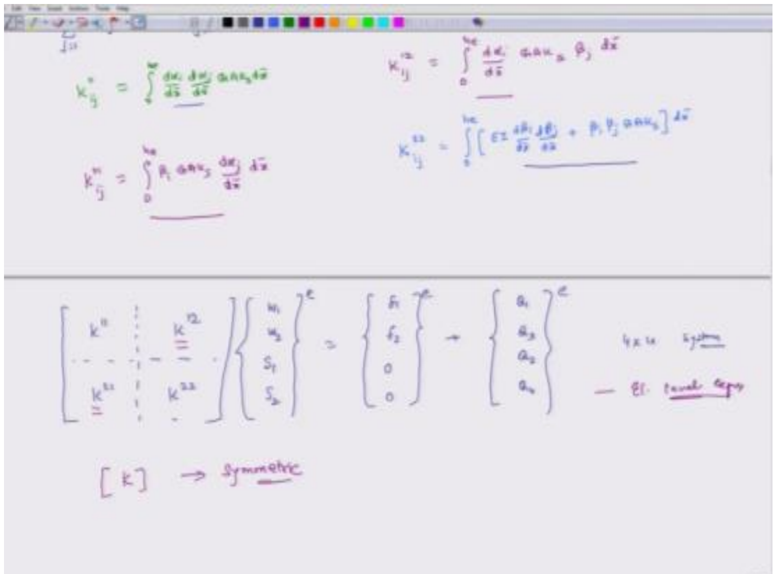

So we have a symmetric matrix and the way we, so we can compute each element of these of this symmetric matrix, and then we find so this is element level equation, and then the next step is that if we have broken our beam into n number of elements so we write such, n such set of equations, assemble them using conditions of continuity and force equilibrium and then impose the boundary conditions to get a reduced set of equations and solve them to get our solution. So this completes our treatment of the shear deformable beam which is also known as the Timoshenko beam.

And in this treatment we have learned a couple of several, we have learned several things, one is how do we handle equations handle more than one differential equations if they are coupled, they maybe linear but they are coupled equations, so that is what we saw in this case that ψ and W were appearing in both the differential equations, and we could construct a weak form and from that we were able to develop a set of equations which could be used to solve these unknowns. So that is first thing, the second thing which we learned through this exercise is that if one particular variable requires that its interpolation function should be one level higher than the other one then we can address that, then we can address that requirement either using the consistent interpolation element method where m was defined as n+1 or the other one was where m equals n but we still take care of the differentiability requirement by using reduced integration method.

We will learn more about this reduced integration method in the next week when we deal with a numerical integration procedures, because in computer implementation of finite element analysis we do not actually go around analytically integrating all the functions but rather we use some fast numerical procedures to integrate these functions. So we will learn more about that and then we will see how we can use reduced integration approaches to address these kinds of problems. So this is pretty much what I wanted to cover in context of emotion go beams, and in the next class or the next coming classes we will discuss Eigen value problems. So that completes our discussion for today and we will meet tomorrow. Thank you.

Acknowledgement
Ministry of Human Resource & Development

Prof. Satyaki Roy
Co-ordinator, NPTEL IIT Kanpur

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Sanjay Pal
Ashish Singh
Badal Pradhan
Tapobrata Das
Ram Chandra
Dilip Tripathi
Manoj Shrivastava
Padam Shukla
Sanjay Mishra
Shubham Rawat
Shikha Gupta
K. K. Mishra
Aradhana Singh
Sweta
Ashutosh Gairola
Dilip Katiyar
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Bhadra Rao
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Lalty Dutta

Ajay Kanaujia
Shivendra Kumar Tiwari

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