Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 04
Polynomials as Shape Functions,
Weighted Residuals,
Elements & Assembly level Equations

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Hello again welcome to Basics of Finite Element Analysis. This week we have started with the introductions to the course and in the last lecture what we had discussed was how shape functions and their amplitudes are assigned over an element. And specifically we had developed this relation.

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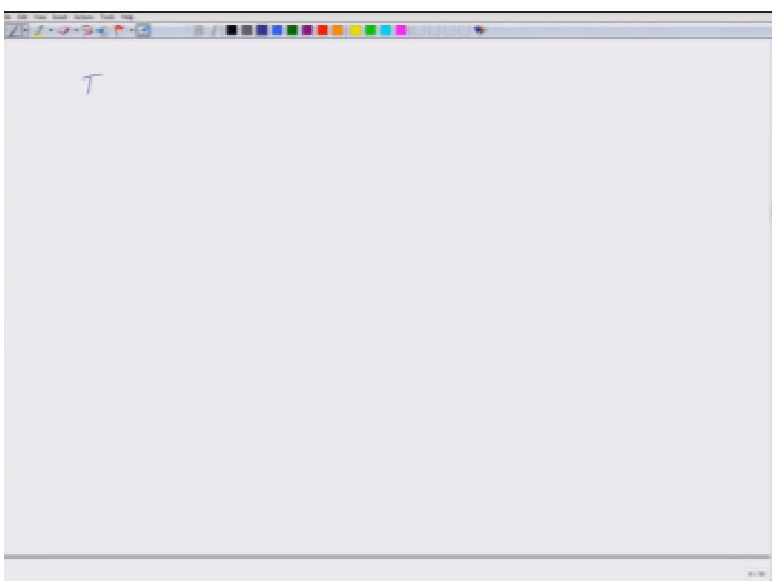

That in context of flow of heat in a pipe which is radially symmetric and all the boundary conditions are symmetric, so how is temperature changing with respect to radius. We had said that.

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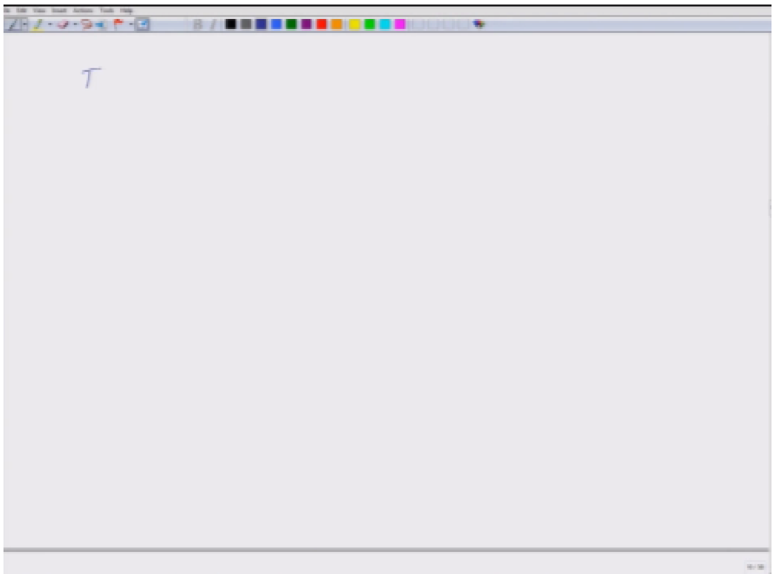

If I break the domain or that is the region of our interest which is from r is equal to R_i to r is equal to R_o inside radius to outside radius. If I break that domain so that is my domain.

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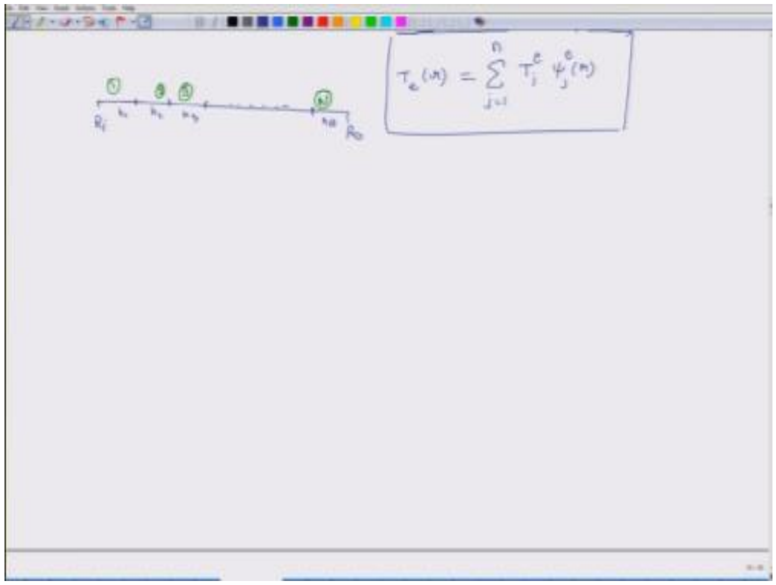

R_o R_i and this is R_o and I break it into several elements such that this is my first element, second, third, and that is the n^{th} element and the size of each element is h_1, h_2, h_3, h_n. Then we had developed an expression that temperature in e^{th} element. Excuse me, this has to be super subscript in e^{th} element and it changes with respect to as so it is a function of R is can be expressed without losing any generality as a sum of some interpolation functions, each of these interpolation functions is a function of R and because its associated with e^{th} element.

I have a superscript e, and these number of elements is from 1 to n and the amplitude of each of these interpolation functions is described by T's, so this is T_j and because again it is associated with e^{th} element it is T_j and subscript, a superscript e. So this is the expression which we have developed and in this development we have not lost any generality. Because I can make the transition within the element as much complicated and as much varying as I want if I increase the order of these shape functions or interpolation functions.

So that is all I have done I have not solved any problem, but I have assumed that temperature changes over the length of an element in a particular way.

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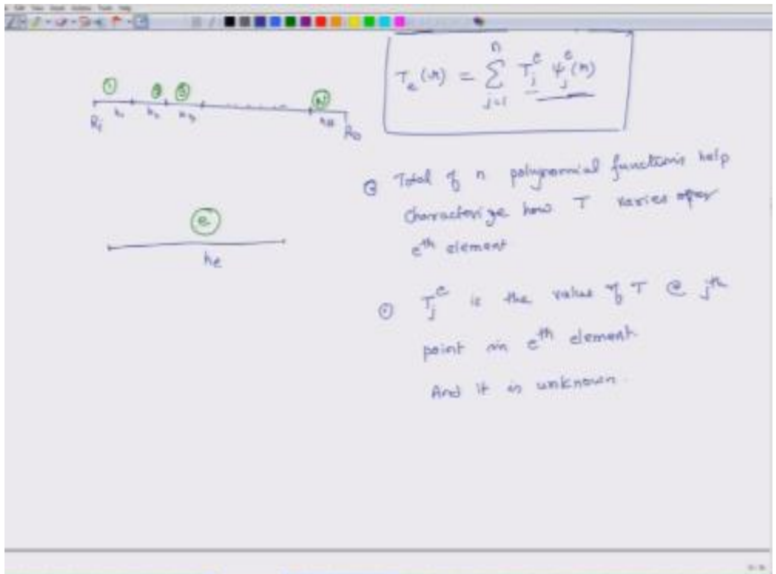

And how it changes is actually defined by these side functions okay. So that is what I have done. Now I can look at this T_j incisor in a little different way also. Suppose I have an element so let us say this is an e^{th} element, this is e^{th} element and its length is h_e right. So, so there are n shaped functions associated with this element which, so the total of n polynomial functions help characterize how T varies over e^{th} element, and we had said that T_j is the amplitude of these polynomial functions.

But another way to look at these is that T_j^e is the value of T at j^{th} point in and why we can say this is will see it later. It is nothing but the, the j^{th} point in e^{th} element and, and it is unknown, and it is unknown we do not know it. We know size because we are assuming if it is constant then it is one, if it is linear then it is ax plus b right. So we know how size are varying, but the values of those constants we do not know okay.

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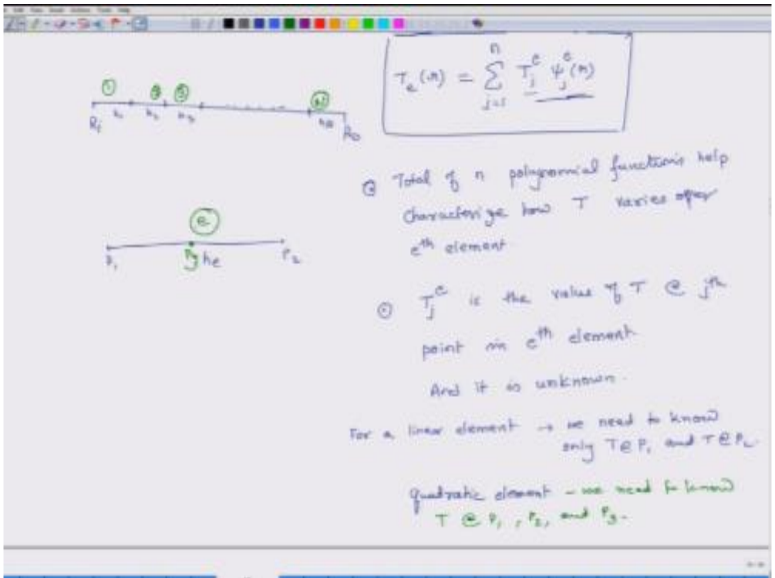

So T_j is the value of t at j at j^{th} point in e^{th} element okay. Now what you are seeing here is that there are only two points, let us say this is P_1 and this is P_2 right. So but we have n such functions so what are the other points, so for a linear element, for a linear element so what is a linear element, where the value of T is changing linearly over its length. Its value of linear it is changing linearly over its length then we have to know the value of T at two points right. For a linear element, we need to know only T at $P_1$and T at P_2 right. And then we can interpolate, then we can develop an interpolation function based on that. If it is a quadratic element then what do we do then we need an extra point. Let us say a P_3, because to perfectly define a quadratic curve we need know T at even P_2 and P_3.

So we have to find the displacement or in this case the temperature at an intermediate point also at an intermediate node also, for a quadratic if we are using a quadratic element R, so what does a quadratic element mean, that temperature is changing quadratically over the element. If it is if we assume that it is changing quadratically, if we assume that it is changing quadratically but in reality if it is changing linearly then we will do the calculations one of the constants associated with quadratic it will come out as zero okay. It will come out as zero, it will come out as zero.

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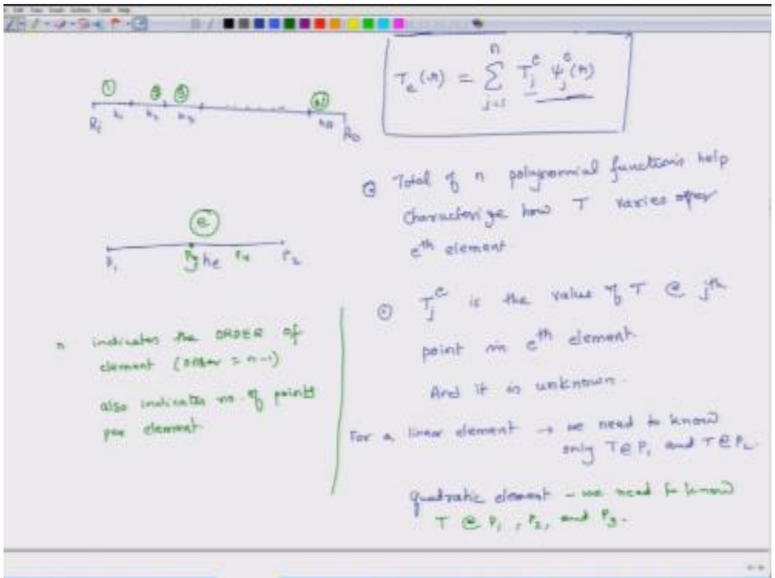

Similarly, if it is a cubic element then we need maybe one more point P_4 okay. So this j is going from 1 to n and n decides, n indicates the order of element agreed. If it is a quadratic element then we n indicates the order of elements and it also indicates number of points per element. So if it is a linear element then n will be 2, and so n is not exactly okay. So and n so order will be n-1 order is equal to n-1 right.

So if it is a linear element n is 2 because we need two points agreed we need two points to make a line, n is 2 and it is a linear curve so order is one. And how many points will be needed on the element 2 points.

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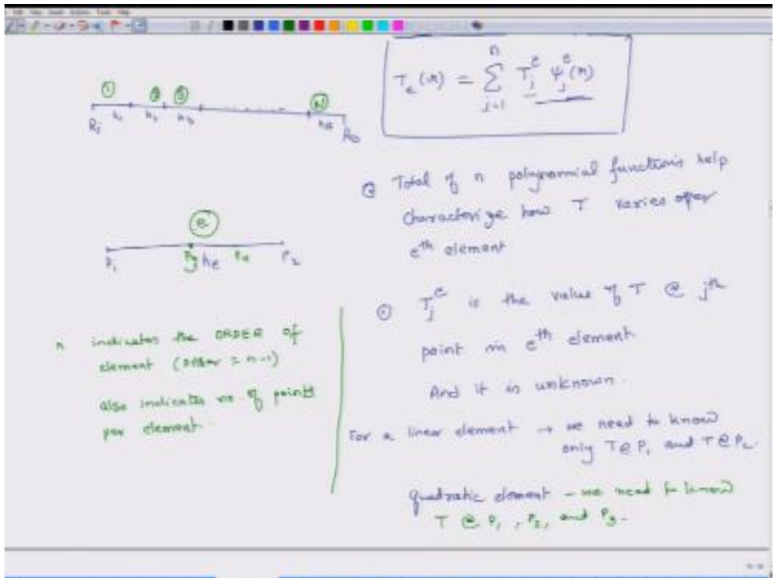

So number of points will be 2. T_j is the value of in this case see if you know the values at three points then in terms of those three points you can construct a polynomial function. So either you can say the other way that their amplitudes of individual functions but here I am saying that if I know the values then I can construct a curve which is a quadratic curve. So T_j is the value of,

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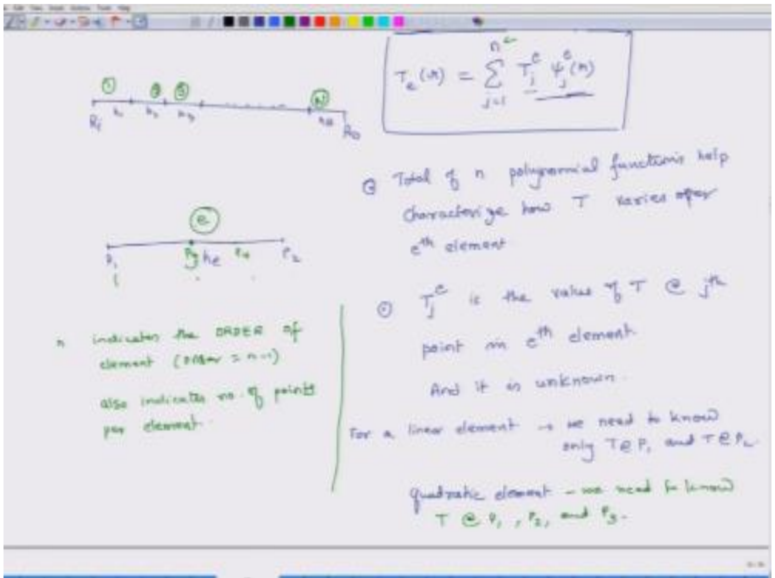

T at j^{th} point, so this is first point, this is second point, this is third point and maybe know my numbering scheme has gone wrong this should, so but this is second point, this is third point, this is fourth point. So T_j will be the value of T at j^{th} point in the e^{th} element okay. So an n indicates, n indicates how many points we will have on the element and n will also help us determine the order of the element.

In finite element codes you will if you explore you will say oh it is a quadratic element, it means that the function or the unknown is varying in a quadratic way over the element that is what it means, it is a linear element, it is a cubic element. So now I am again introducing some extra technology so then you have to become increasingly more comfortable with this terminology.

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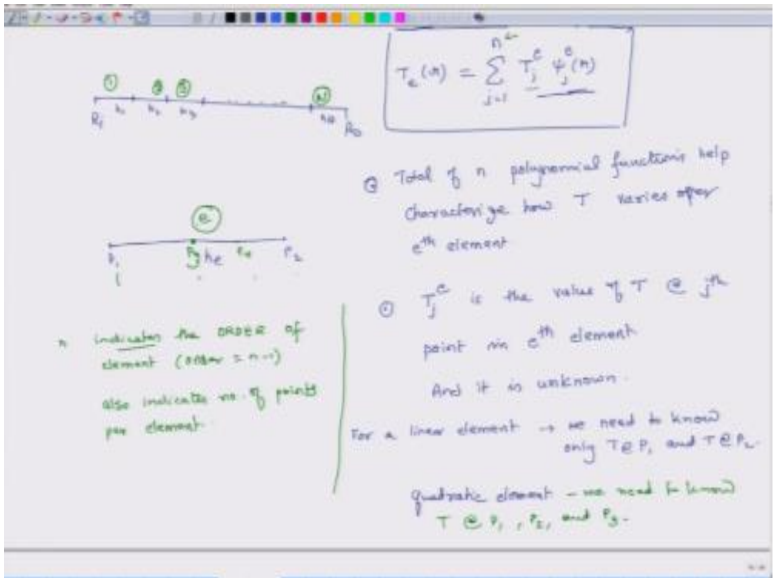

So n indicates the order of the element not it does not mean n is equal to order but n is equal to n - 1 is either order. And it also indicates how many points you have per element. T_j is the value of T at j^{th} point in e^{th} element and based on all this understanding if we have a linear element we need two points and these two points will be at the boundaries, if we have a quadratic element we need three points, two will be at the boundary and one node will be at the somewhere in the middle.

If we have a cubic element two points will be on the boundaries and two points will be in the middle and so on and so forth.

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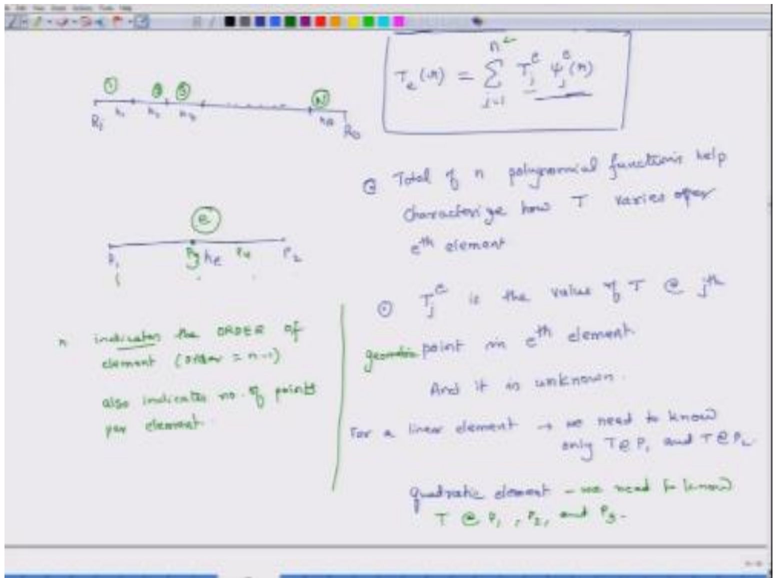

So to make it more explicit it is at j^{th} geometric point, geometric point in the element. But if even if I remove the geometric it does not change much okay. So we will recap.

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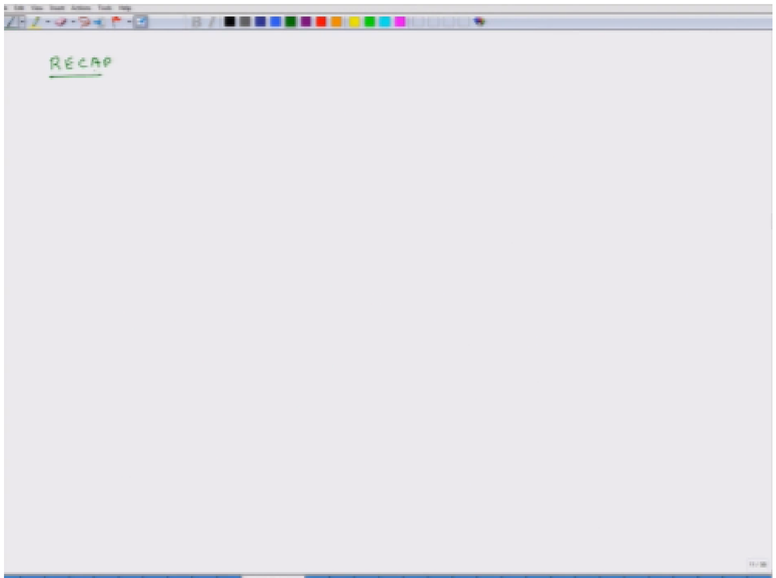

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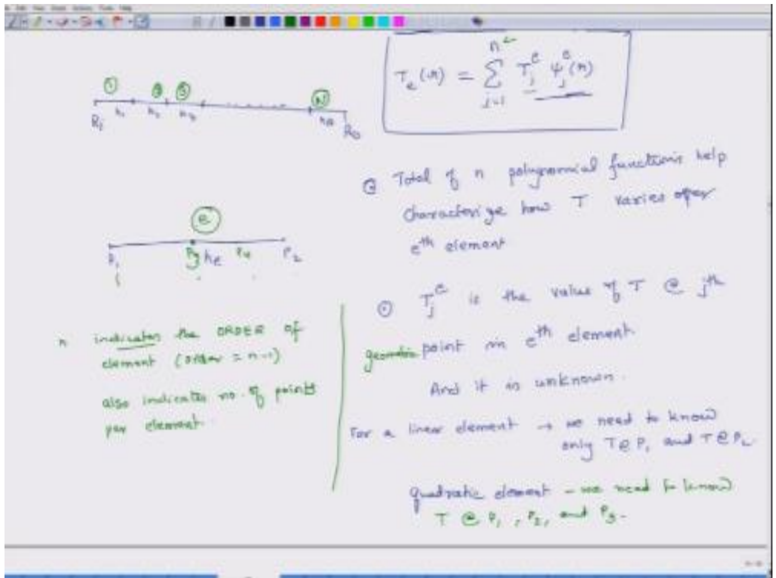

Number of functions yes, it is same for instance ex is squared plus bx plus c it has a c which is one function. x is squared x is another function, x is squared is a third function, it is a sum of those functions, that is why we are summing.

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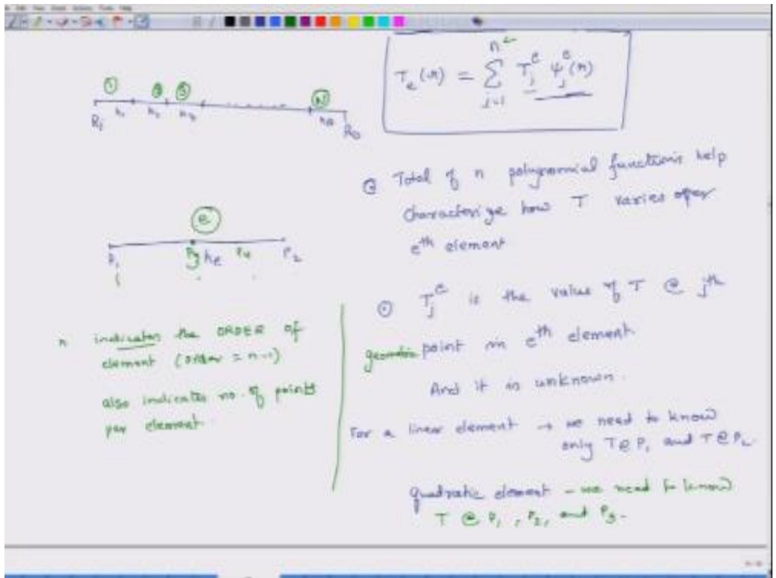

Now these functions will not be as simple as this the way I have explained they will be little more complicated. But ultimately when you do all the analysis it boils down to that if you have n functions and the amplitude of each function is 1 in a normalized way then at each point.

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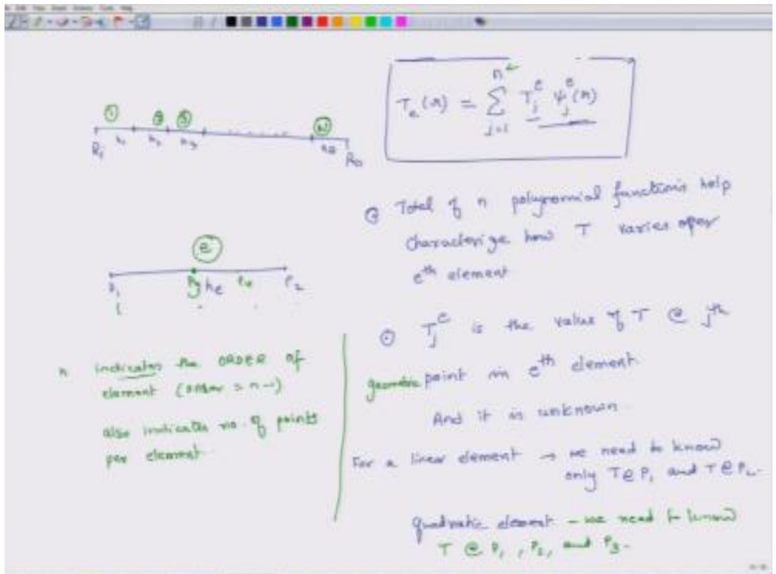

Whatever is the value of that function will be in, it will indicate the value, the displacement, or the temperature or whatever okay. So that is this but right now we are still at a, we are coming from that example area under the curve where we saw that depending on how we interpolated it influences the accuracy. And now we have expanded,

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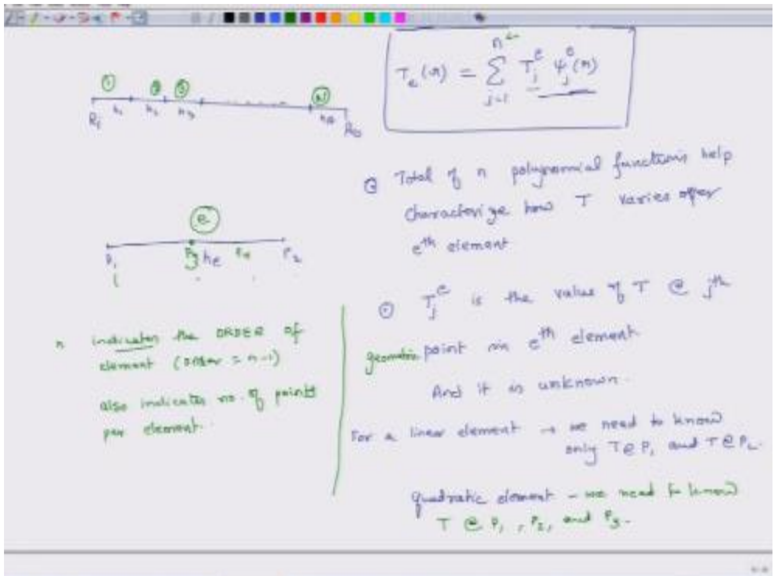

This concept further okay. Now, but so what do we do with these what do we do with these.

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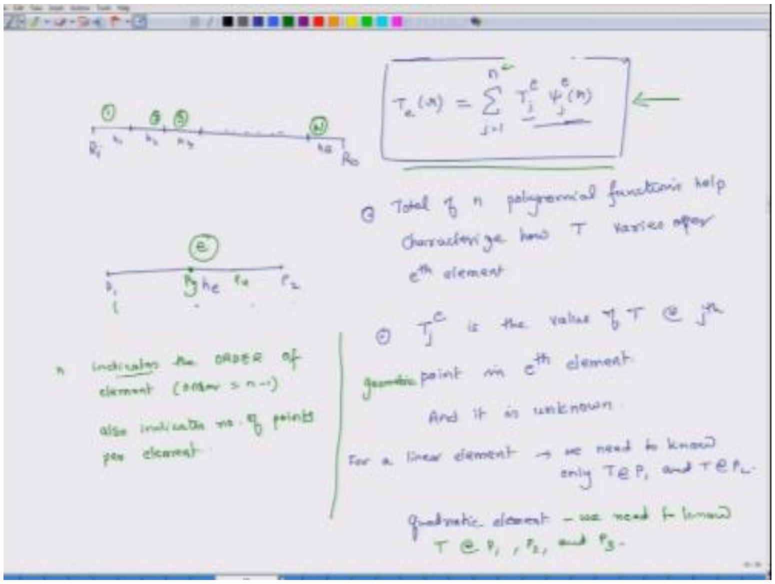

What do we do with these things, this equation.

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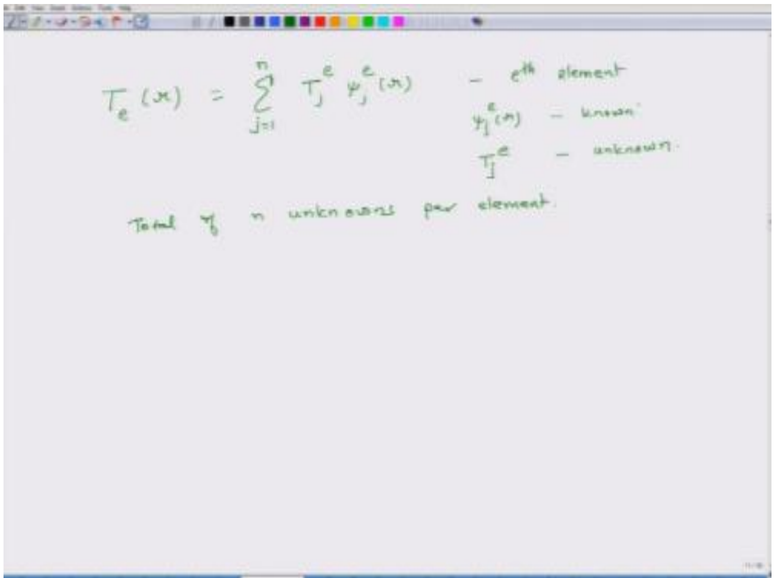

$T_e^{(r)}$ $=\sum T_j\, \psi_j(r)$ e and e j is equal to 1 to n this is for e^{th} element. Here $\psi_j^e(r)$ is known, right it is known and $T_j^{\ e}$ is unknown, T_j^e is unknown right. How many unknowns we have now for each element, we have small n so total of n unknows per element. In this case in mechanics you will not only have one variable temperature, you will have u,v,w it will be three n right. Because they will be n use n w's n uv's and also and so on and so forth.

But in this we have only one variable temperature, so we have to find n unknown's per element.

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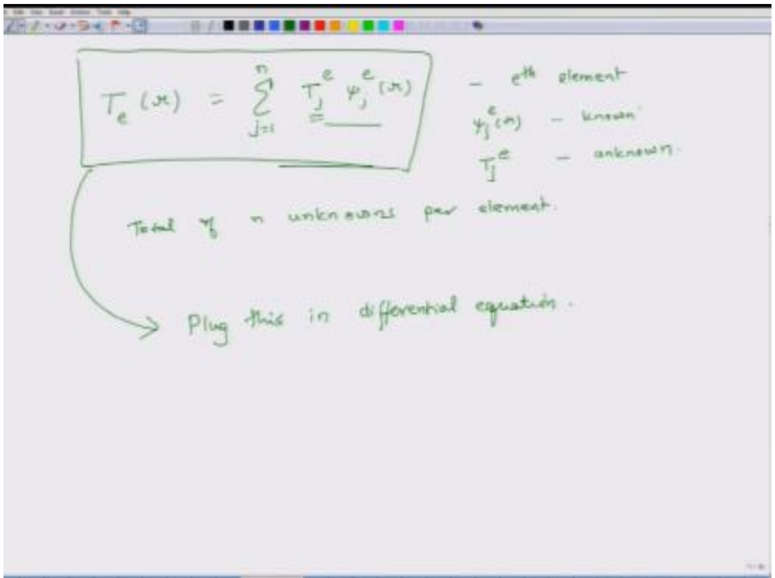

We have to find n unknowns per element, if we choose a linear element then we have to calculate two unknowns. If we choose a quadratic element we have to find the values of n points and also add one middle point. If we have to, if we choose cubic element we have to find four unknowns and so on and so forth right. But the point is that we have assumed that this is how T varies over radius.

So what do we do, so I am giving again now up to this I have given you in detail now I will give you an overview because all the details we will go step-by-step, we will develop step-by-step, so what do we do? We know these functions, these constants we do not know, we plug this. We plug this in the differential equation like this in the differential equation. So what do we get, will the equation be hundred percent satisfied?

This is a very general thing there is no reason to think that the equation is going to be a hundred percent satisfied. Unless our choice of these functions is exact and correct these functions this for T^e this is a function T_j I_j it is a function, ψj could be x, ψ1 could be x, ψ2 would be x squared, ψ3 could be cube and so on and so forth right. So this is some general function, there is no guarantee that it is going to satisfy this differential equation.

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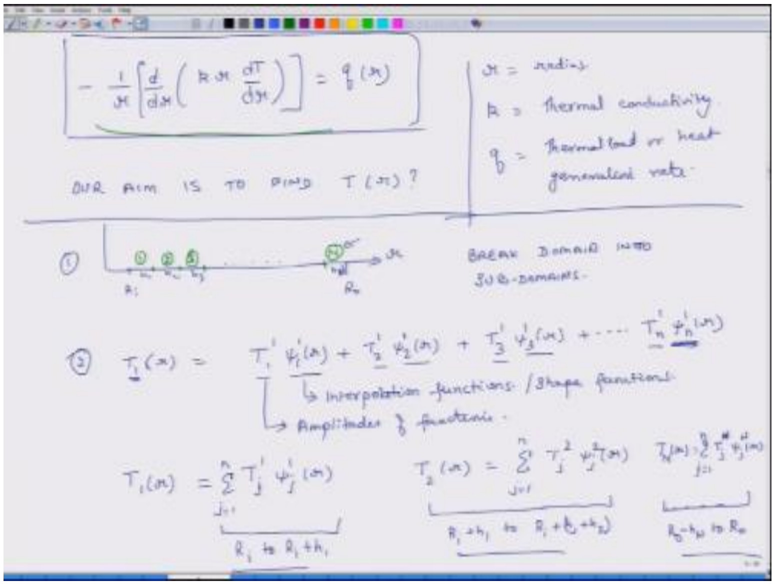

There is no guarantee, so if it does not satisfy then what do we get?

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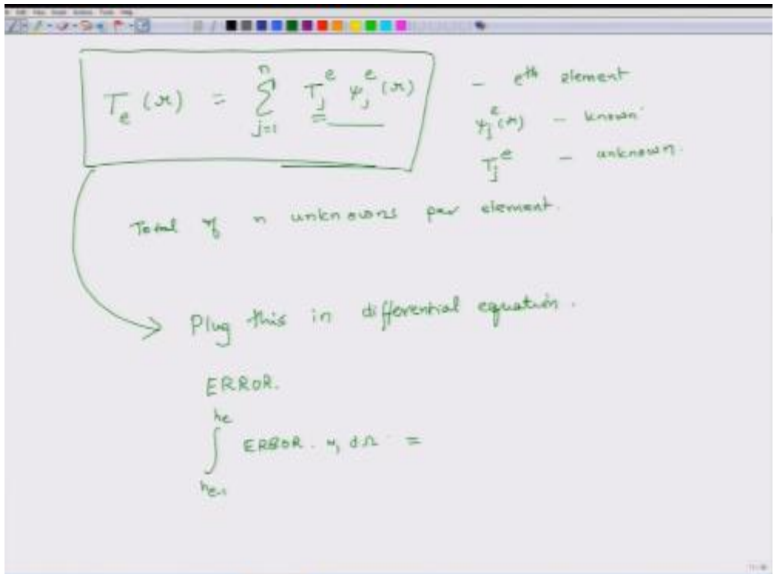

Plug this and then we get error okay. And what we do is we integrate that error iteratively, we integrate that error with some other function. Now and do not get confused with some other because maybe in 4,5, lecture then things will become very clear. So some other, so we first time, so we integrate this error iteratively, so first time we multiply it by 1 weight function. And we integrate it over the length of the element.

So that is my domain, domain is what h_e- 1 to h_e right, and we say that this integral over of the error not it is not zero at point by point basis. But over the whole domain it is 0

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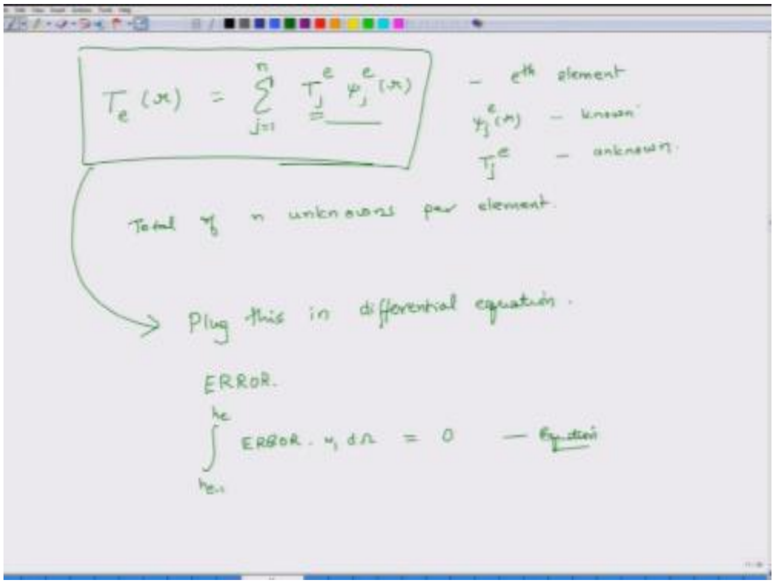

Integral of that error, then we second time we multiply it by another function and we get another equation so this gives us an equation. What kind of equation this is going to be? These are known functions and most likely we will choose polynomial functions, we will choose polynomial functions, we will not use complicated, polynomial we can integrate very easily. So when we integrate what do we get, we will get some algebraic equations.

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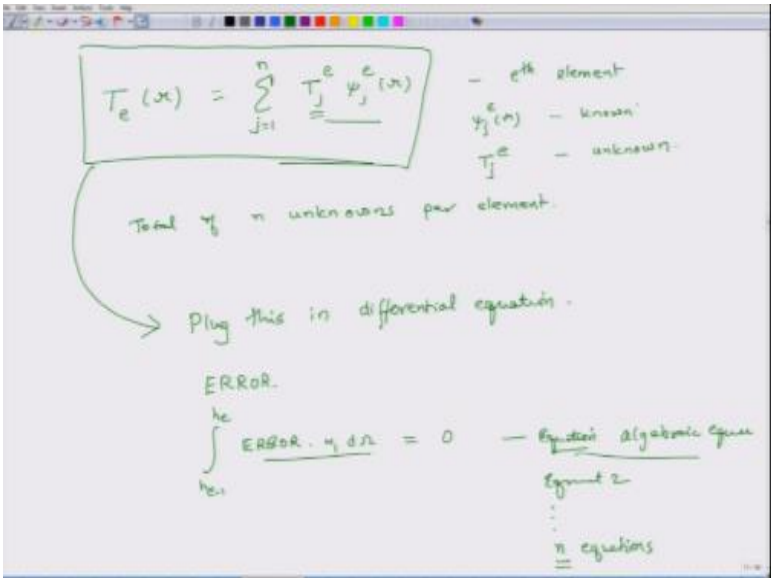

Now how this process happens and what is the mathematics we will explain it later. So we multiply it by 1 function integrated over the domain we get one equation, then we multiply it by another basis another function and again integrate it over the thing we get equation number 2, and we keep on doing it till we get n equations. How many unknowns were there, small n, so we do this process n times, why we do it n times what is the mathematics we will learn later.

But I am just giving you an overview, so and how many unknowns were there, small n.

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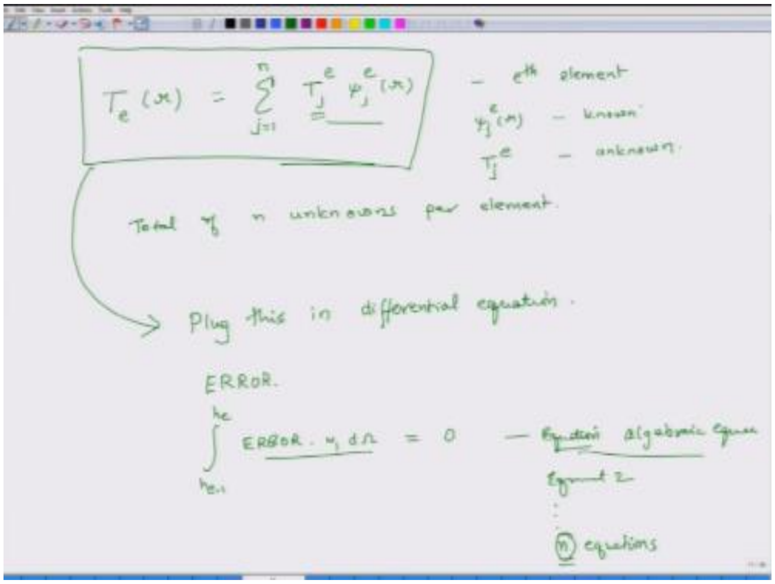

So we are small n variables and small n equations. So at this stage we have atleast the same number of unknowns and same number of equations okay. And then maybe we will be able to solve that.

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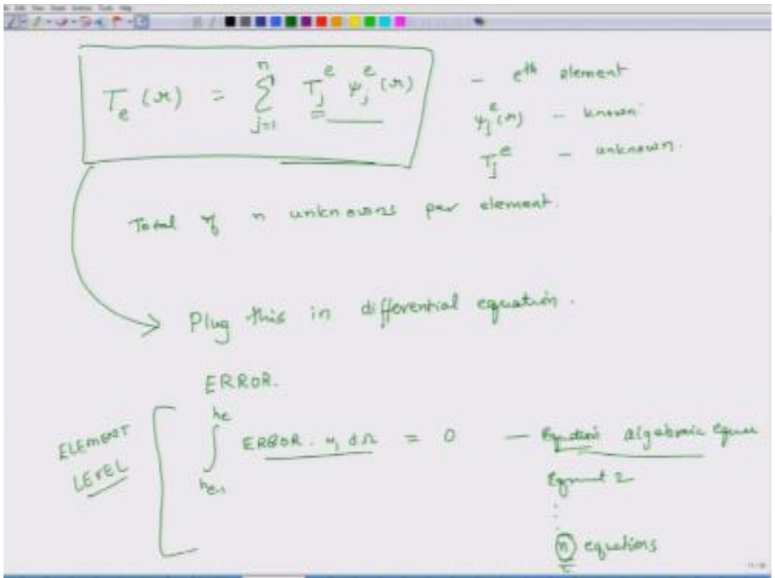

But right now we are at what level, element level. So we developed n equations at element level okay. And at this stage we do not solve it, why? Because we do not, we have to connect our element.

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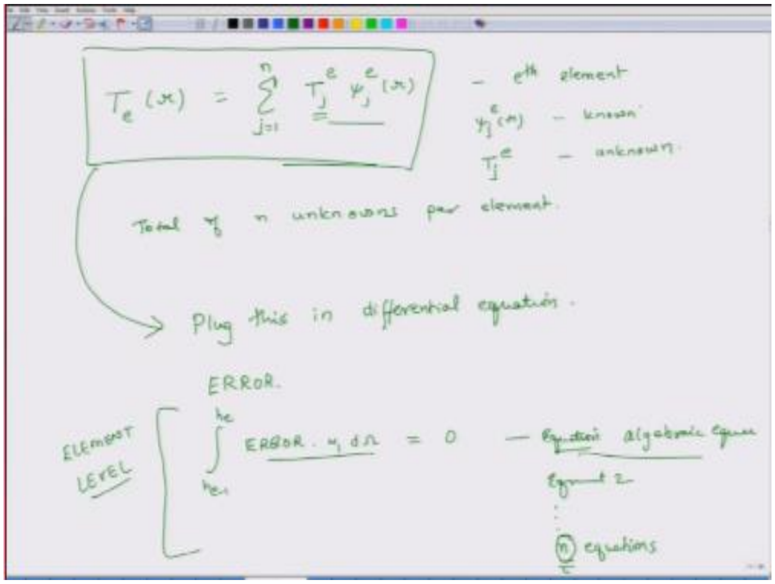

To the neighboring element, only then we will know how what are the inter-relationships you know only then we will know what are the inter-relationships. So we will connect first element to the second element, second element to the third element, third element to the fourth element right.

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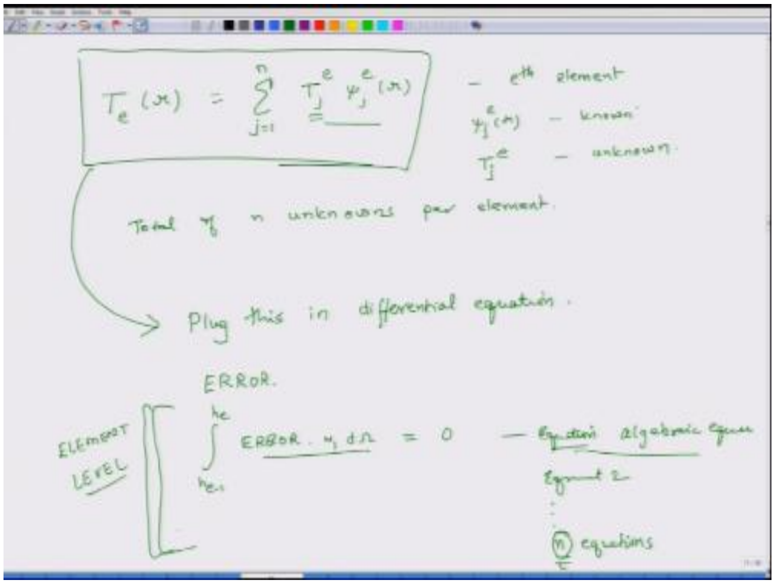

So first we will develop equation set element level.

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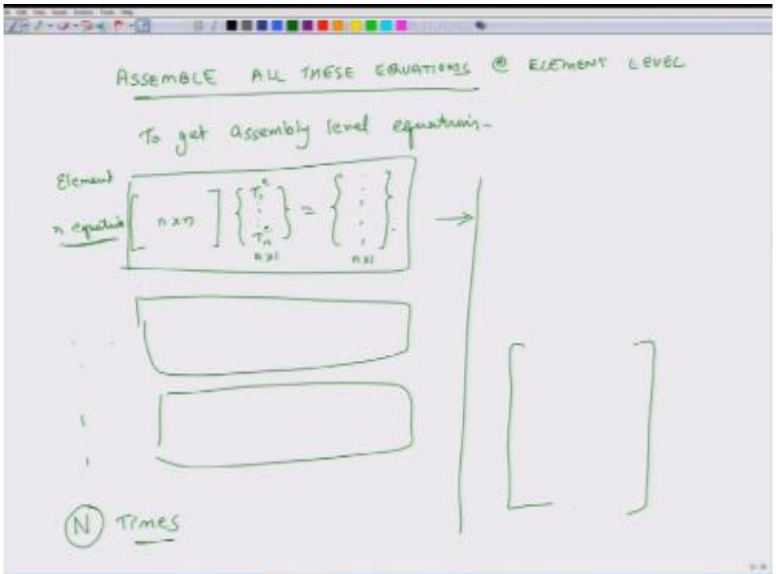

And then we will assemble and how this assembly process happens we will learn it. But I am now just giving you the algorithm, assemble all these equations at element level. So we will assemble all these elements at to get assembly level equations okay. We will assemble all these equations to get an assembly level equation, so at element level we have a, when you have n questions now this is element level.

So if there are n equations then there will be n variables T_1 T_n , and this is for the e^{th} element right. And this will be when we get there will be some constants here you know some, some vector will be there so this is the, this will be the format of those equations in matrix form agreed. This will be an n-by-n matrix, this will be a vector or a column matrix which will be having n rows and one column, and this will also have n rows and one column.

But at this is stage we cannot solve it if you solve it, if you solve it actually you will get, everything will be trivial solution 0000. Because right now you have not assembled it, you have not connected this matrix with another set of matrix equations for the next element with another set of, so you will write you will create all these equations. How many times, n times, understood, what is n begin, number of elements? So you will write all these element level equations for n different being different, different times and then using some principles of mathematics we will figure out how to assemble these equations okay. We will figure out how to assemble those equations and finally when you assemble these, these equations you will get one big matrix okay.

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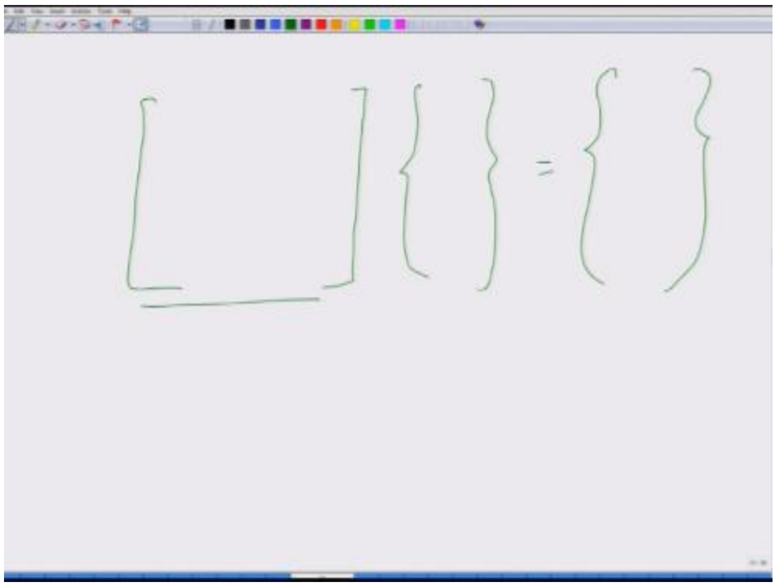

So we will get one big matrix of this form. What will be the size of this matrix, no not capital n, n is the number of elements.

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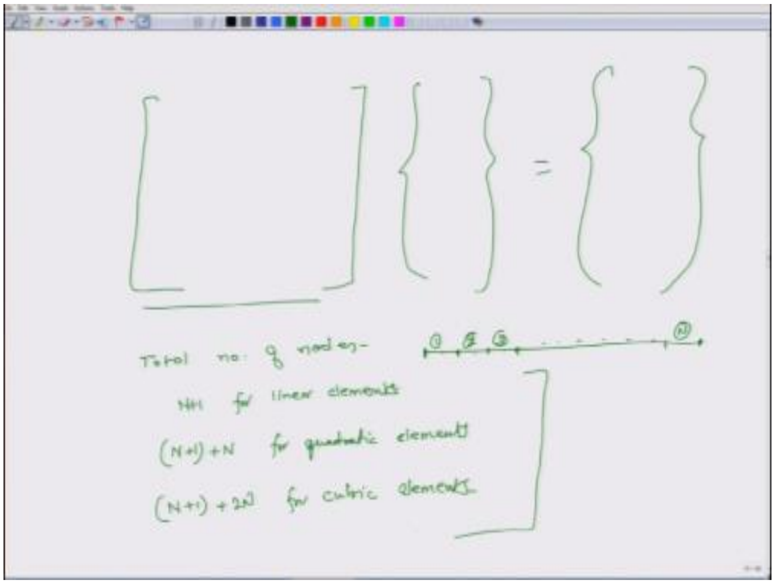

This will be the total number of nodes. If all the elements were linear how many nodes would be there? No not two, not two you have suppose no, so this is this is the n^{th} element, first element, second element, third element like this so how many n nodes, it will be n+ 1 right. So total number of nodes will be n+ 1 for linear elements, so this size of the metrics will be N+ 1 by N+ 1 agreed.

If you have quadratic elements then how much will be there N + 1 and then there will be one node in the middle for each element, so it will be (N+1)+N agreed for quadratic elements. If it is a cubic element then it will be (N + 1) +2N in for cubic elements agreed. So the size of the matrix will be based on how many elements we have n how many, what is the order of each element it will depend on number of elements.

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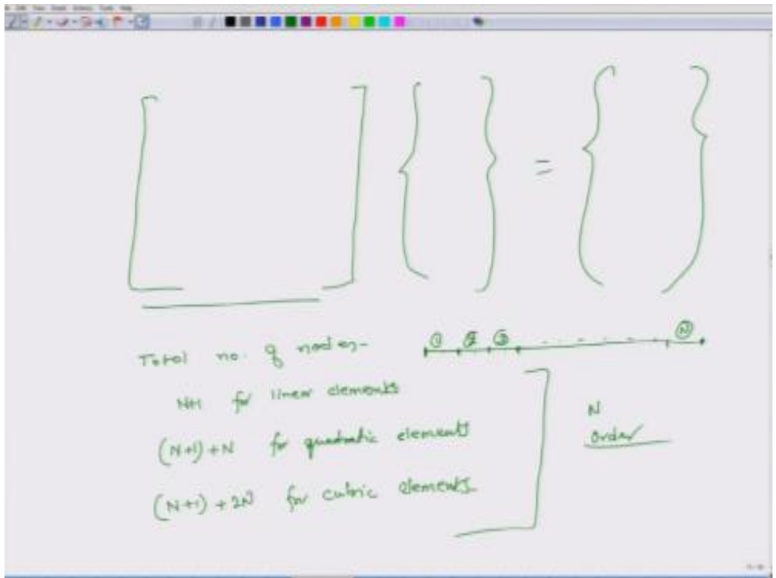

And order okay. So at least for line 1d ements it will be yeah, so this is there and this is only if there is one single variable T. If there is one single variable T, this is a single variable problem okay. You can have several variables u, v, w, then they have three equations so everything get multiplied okay. So this is the next step so before,

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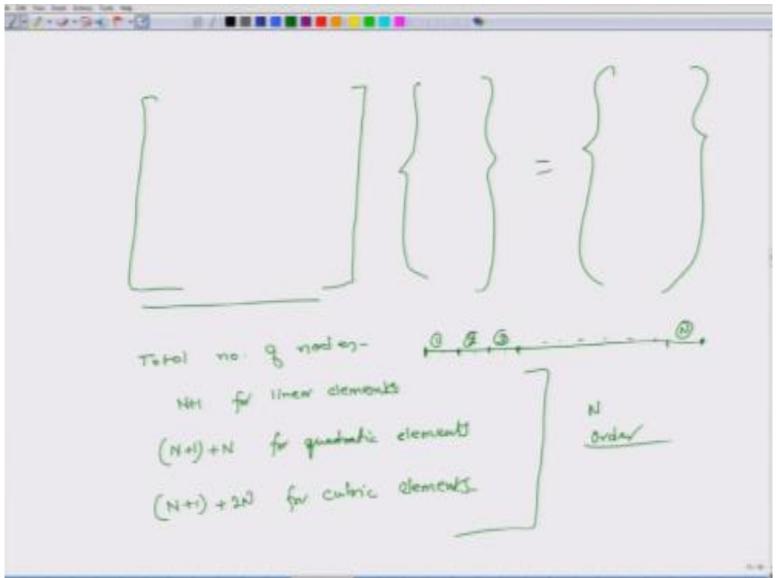

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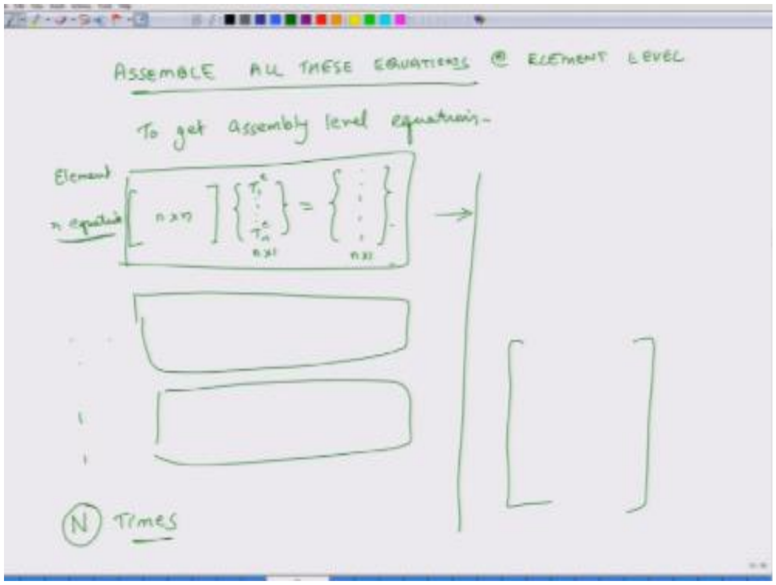

First you develop element level equations, we learn how the mathematics of it later, but now I am giving you the procedure you develop matter element level equation by somehow computing.

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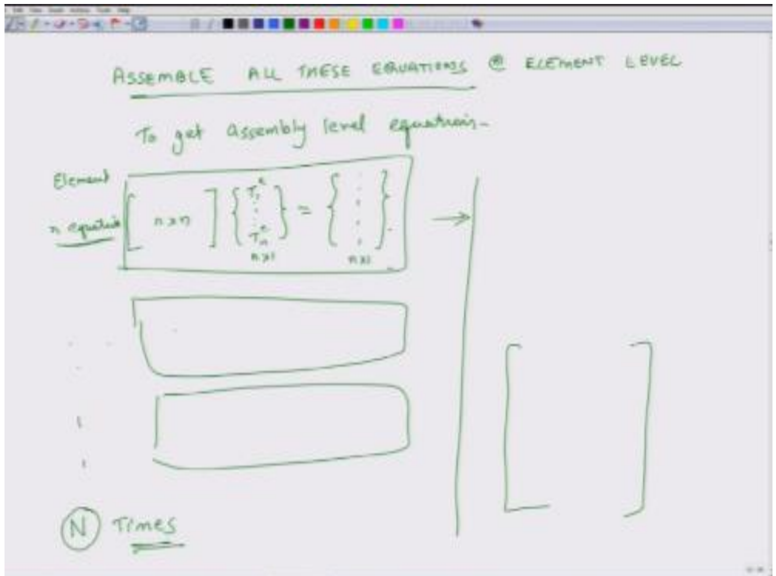

The error multiplying it by some function w 1 integrating it over the domain you get an algebraic set of equations. And because everything error is basically nothing but a function of polynomial functions you can integrate it very easily. And so you get one algebraic, one set of algebraic equation then you find it for the next first element, second element, third element and you find element level equations for all the n's.

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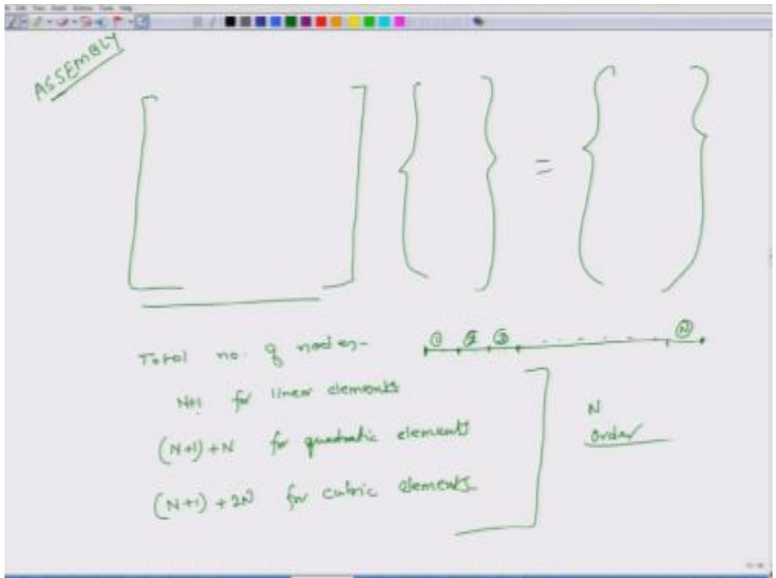

Then you do the assembly then the total size of your metrics will be N+1 for linear (N+1)+N4 quadratic and so on and so forth. So this is by assembly process, so once we have done the assembly I have captured the whole geometry of the system in my equations. We have started from R is equal to R_i to R is equal to R_0, pura पूरा geometry capture kiya किया, also I have captured.

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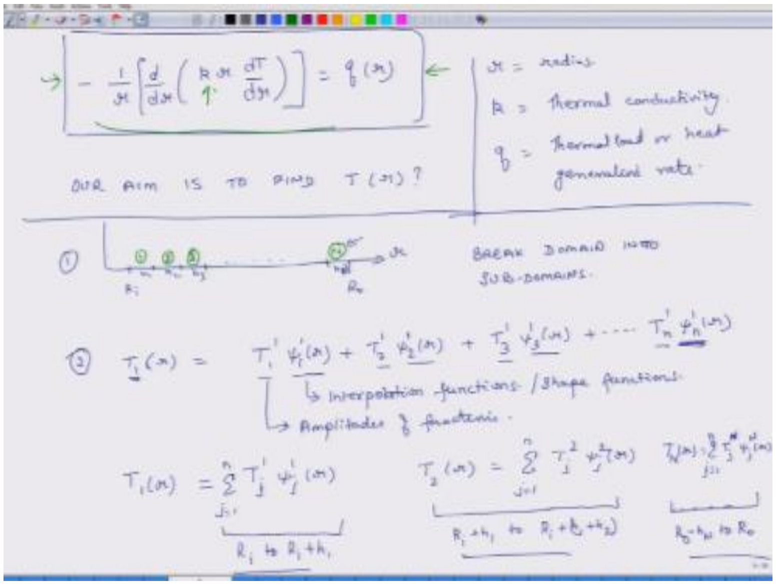

Material properties, because when I am finding the error material properties coming here okay. So I have captured the role of material properties also, I have captured the role of material property also in my equation. Also I have captured the heat this is actually heat transfer equation. So, so I have captured the overall physical phenomena, material properties also embedded in the in the matrix equation, geometry is also captured in the matrix equation is this clear.

What is it that we have not yet captured there is one thing which we have not captured, boundary condition you know we are not captured boundary condition. Because when we are using this equation.

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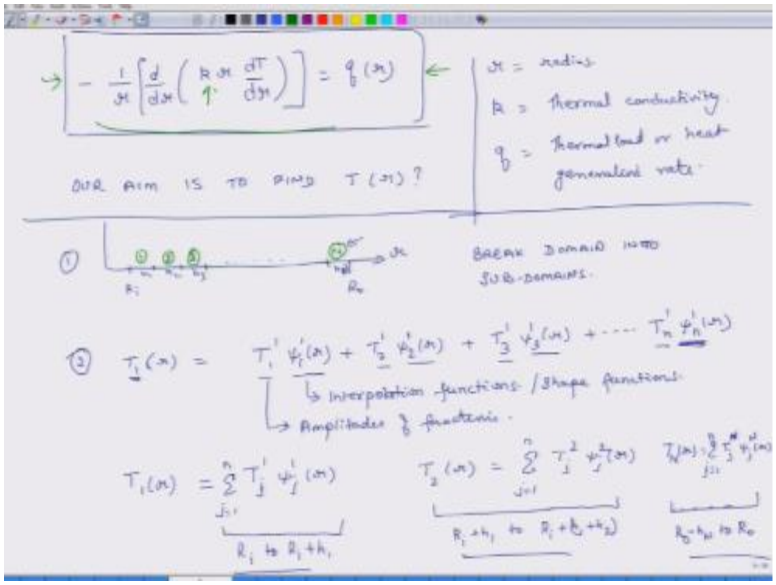

We have not used any boundary condition.

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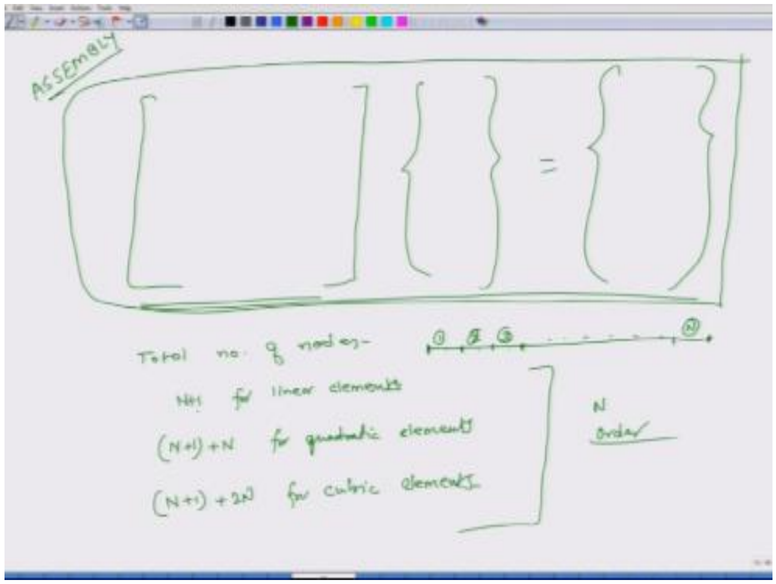

We are not used any boundary condition and that is what we do once the assembly has been done. Once the whole thing all the equations have been assembled then I say okay. What are my boundary conditions? So I say okay, my first boundary condition is where is a boundary condition?

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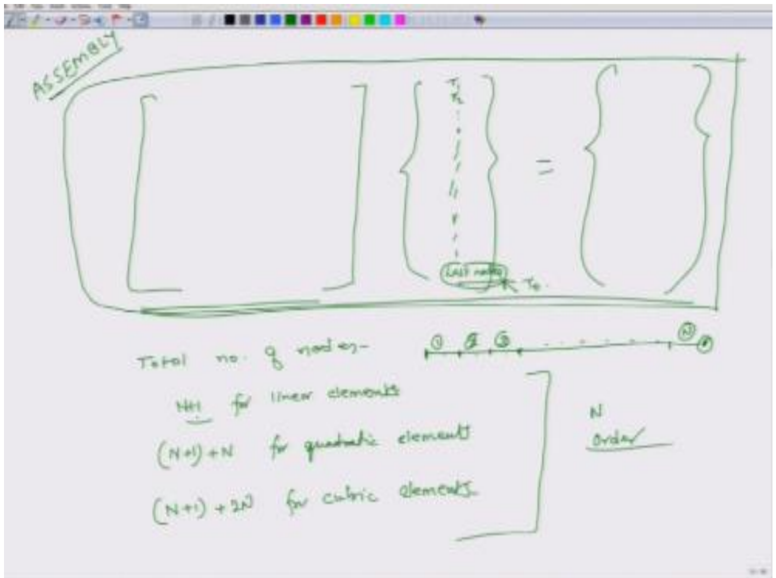

That temperature, that outside radius is T_o okay. So I go to my assembly and T_o will correspond to the last node right, and this is a column of all the temperatures right. This is T_1 first node T_2 and whatever is n + 1 nodes for linear elements or 2N+ 1 for quardratically so this is the last node. So I say okay I will put in this vector it will not be unknown now it is known and its value is T_0 its value is $T_{0.}$

So I put that boundary condition here and similarly the other boundary condition was that there is no, at the inner radius there is no flow of heat happening.

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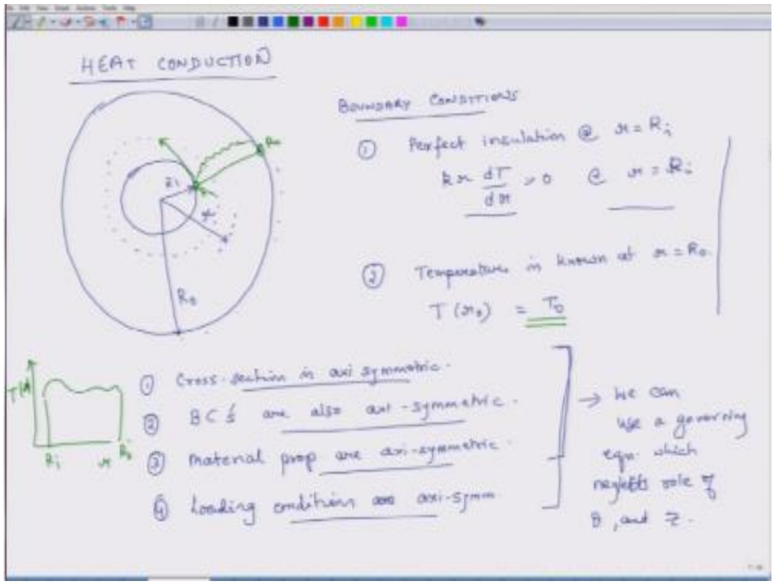

Which means that this mathematical condition has to be prescribed and that is also what we do somehow in this set of equations.

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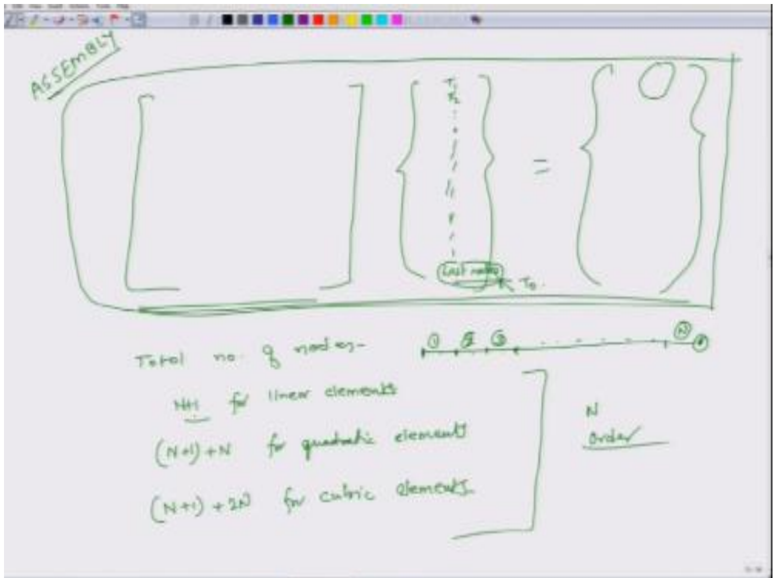

What do we do, how do we do we will learn later, we do something on the side okay. We do something on this side we will understand that mathematics later. So now we have assembled, when we assemble we capture the geometry and the phenomena and then.

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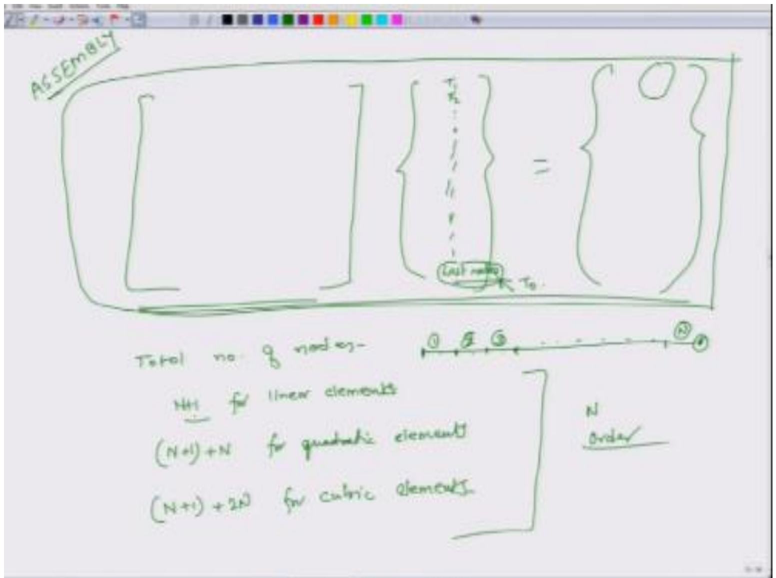

At the assembly level we put in equations, we put in the boundary conditions and then we have captured the whole physics of the problem. And when then we solve these equations and we calculate.

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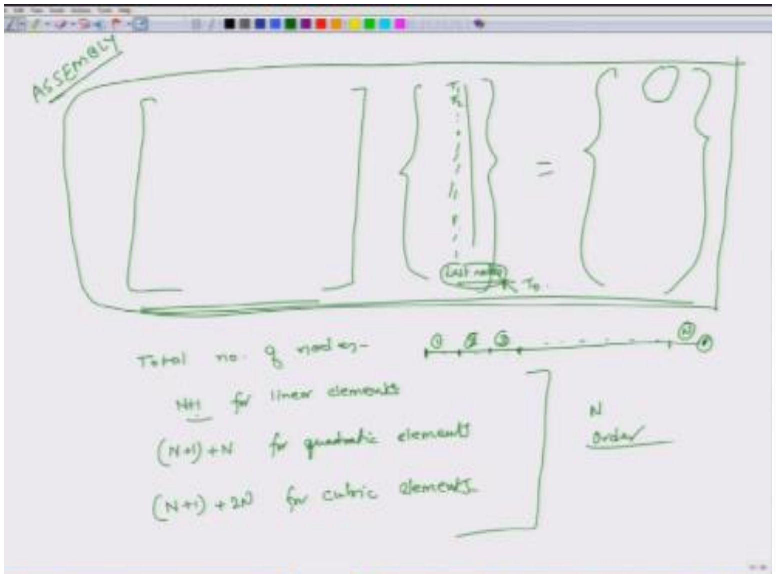

T_1, T_2 because there are number of variables unknowns and number of equations is same. This is what we accomplished through assembly process. How it happens we will learn later, but we so at, this is a stage we compute the all the unknowns and we get the numerical solution of, for the problem. We do not get a formula we get the numerical solution because when you are using matrices you will get some numbers okay. So that is what I wanted to discuss. Thanks.

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