

**Indian Institute of Technology Kanpur**  
**National Programme on Technology Enhanced Learning (NPTEL)**  
**Course Title**  
**Basics of Finite Element Analysis**

**Lecture – 38**  
**Finite element formulation for shear deformable beam:**  
**Part-I**

**by**  
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Hello, welcome to basics of finite element analysis, today is the 7th week second day of this course, and what we will do today in this particular lecture is a continuation of our discussion on shear deformable beams or Timoshenko beams, and in the last lecture what we had done was we had developed weak formulations for the two equilibrium equations, these weak forms are expressed here on your screen and as a follow-up to this what we will start looking at are the boundary terms of these equations and interpret these boundary terms in terms of what do they mean.

So what we look at are the boundary terms.

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Handwritten mathematical derivations for boundary conditions in a beam. The equations are:

$$0 = \int_0^{L_1} \left[ \frac{dw_1}{dx} \left( GAK_s \left( \psi + \frac{dw}{dx} \right) \right) - w_1 \tau \right] dx - \left[ w_1 GAK_s \left( \psi + \frac{dw}{dx} \right) \right]_0^{L_1} \quad (1) = \checkmark$$

$$0 = \int_0^{L_2} \left[ \frac{dw_2}{dx} \tau + w_2 GAK_s \left( \psi + \frac{dw}{dx} \right) \right] dx - \left[ w_2 \tau \right]_0^{L_2} \quad (2) = \checkmark$$

BOUNDARY TERMS

EBC

EQ 1  $\rightarrow w_1$

EQ 2  $\rightarrow w_2$

NBC

$GAK_s \left( \psi + \frac{dw}{dx} \right)$

At this point of time okay, so we will look at our aim is to figure out what are the essential boundary conditions and also what are the natural boundary conditions for such a beam, so when we look at equation 1, this is equation 1 okay and when we look we see that the weight function is  $w_1$ , weight function is  $w_1$  and in equation two the weight function is  $w_2$  okay, the coefficient of weight function in the first equation is  $GAK_s$  times  $\psi + dw$  over  $dx$  and we had discussed earlier in one of our lectures that the coefficient of the weight function constitutes the natural boundary condition so this is the thing which is the natural boundary condition associated with the first equation.

And physically what does it represent  $\psi + dw$  over  $dx$  we had explained earlier is nothing but shear strain right, that is what we had explained in the last class when I multiply the shear strain by  $G$  then it becomes shear stress and if I have a beam.

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$$0 = \int_0^{h_e} \left[ \frac{dw_1}{dx} \left\{ GAK_s \left( \psi + \frac{dw}{dx} \right) \right\} - w_1 \frac{d}{dx} \right] dx - \left[ w_1 GAK_s \left( \psi + \frac{dw}{dx} \right) \right]_0^{h_e} \quad (1) = \checkmark$$

$$0 = \int_0^{h_e} \left[ \frac{dw_2}{dx} EI \frac{d\psi}{dx} + w_2 GAK_s \left( \psi + \frac{dw}{dx} \right) \right] dx - \left[ w_2 EI \frac{d\psi}{dx} \right]_0^{h_e} \quad (2) = \checkmark$$

BOUNDARY TERMS

EBC

EQ 1  $\rightarrow w_1$

EQ 2  $\rightarrow w_2$

NSC

$GAK_s \left( \psi + \frac{dw}{dx} \right) = V$

$\psi + \frac{dw}{dx} \rightarrow \text{shear strain}$

And the shear stress here is  $T$  and suppose I multiply this by that area of the cross section of the beam then that shear stress times area is nothing but shear force, so in the first equation this term

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$$0 = \int_0^{h_e} \left[ \frac{dw_1}{dx} \left\{ GAK_s \left( \psi + \frac{dw}{dx} \right) \right\} - w_1 \frac{d}{dx} \right] dx - \left[ w_1 GAK_s \left( \psi + \frac{dw}{dx} \right) \right]_0^{h_e} \quad (1) = \checkmark$$

$$0 = \int_0^{h_e} \left[ \frac{dw_2}{dx} EI \frac{d\psi}{dx} + w_2 GAK_s \left( \psi + \frac{dw}{dx} \right) \right] dx - \left[ w_2 EI \frac{d\psi}{dx} \right]_0^{h_e} \quad (2) = \checkmark$$

BOUNDARY TERMS

EBC

EQ 1  $\rightarrow w_1 \rightarrow w$  (displacement)

EQ 2  $\rightarrow w_2$

NSC

$GAK_s \left( \psi + \frac{dw}{dx} \right) = V$

$EI \frac{d\psi}{dx}$

$\psi + \frac{dw}{dx} \rightarrow \text{shear strain}$

Represents physically it represents the shear force okay, now the other thing we had discussed was that this in earlier class also that this is essentially minimum potential energy statement the same thing for the second thing also, so when I multiply  $v$  by a displacement that corresponds to work done by the system, so this  $w_1$  if I replace it by  $w$  which is the displacement then the essential boundary condition associated with the first equation is displacement and the natural boundary condition associated with the first equation is the shear force.

And when I multiply  $w$  by  $v$  I get the work done okay. Next what we will do is we will look at the second weak form which is equation 2, so in equation 2 in the boundary term which is here I have  $w_2$  is the weight function, the coefficient of that  $w_2$  is  $EI \frac{d\psi}{dx}$  over  $dx$  and this so what is  $\psi$ ,  $\psi$  is rotation,  $\psi$  is rotation.

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GOVERNING EQUATIONS

$$\frac{d}{dx} \left[ G A K_S \left( \psi + \frac{dw}{dx} \right) \right] + f = 0 \quad \rightarrow \text{I}$$

$$\frac{d}{dx} \left[ E I \frac{d\psi}{dx} \right] - G A K_S \left( \psi + \frac{dw}{dx} \right) = 0 \quad \rightarrow \text{II}$$

$K_S$  = Shear correction factor  $\leftarrow$   
 $G$  = Shear modulus of material

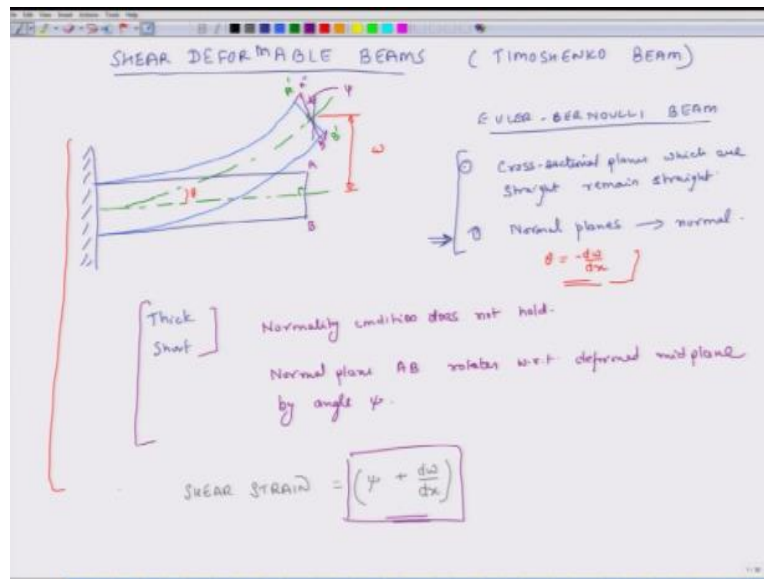
WEAK FORMULATION

$$\int_0^{L_0} -w_1 \left[ \frac{d}{dx} \left( G A K_S \left( \psi + \frac{dw}{dx} \right) \right) + f \right] dx = 0 \quad \text{and}$$

$$\int_0^{L_0} -w_2 \left[ \frac{d}{dx} \left( E I \frac{d\psi}{dx} \right) - G A K_S \left( \psi + \frac{dw}{dx} \right) \right] dx = 0$$

Integrating by parts:

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Of the original vertical line a b with respect to it how much it rotates by right, that it is the rotation.

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Handwritten mathematical derivation of boundary terms for a beam element. The equations are:

$$0 = \int_0^L \left[ \frac{dw_1}{dx} \left\{ GA K_s \left( \psi + \frac{dw_2}{dx} \right) \right\} - w_1 f \right] dx - \left[ w_1 GA K_s \left( \psi + \frac{dw_2}{dx} \right) \right]_0^L \quad \text{--- (1) ---}$$

$$0 = \int_0^L \left[ \frac{dw_2}{dx} \left[ EI \frac{d\psi}{dx} + w_2 GA K_s \left( \psi + \frac{dw_2}{dx} \right) \right] \right] dx - \left[ w_2 EI \frac{d\psi}{dx} \right]_0^L \quad \text{--- (2) ---}$$

Below the equations, a table lists the boundary and natural boundary conditions:

BOUNDARY TERMS		
EBC		NBC
EQ 1 $\rightarrow$	$w_1 \rightarrow w$ (displacement)	$GA K_s \left( \psi + \frac{dw_2}{dx} \right) = V$
EQ 2 $\rightarrow$	$w_2 \rightarrow \psi$ (rotation)	$EI \frac{d\psi}{dx} = M$

A small diagram of a beam element is shown to the right of the table, with a coordinate system  $x$  and a rotation  $\psi$  at the right end.

And if I differentiate it once again I essentially get the curvature right, and when I multiply that curvature by  $EI \, d\psi$  over  $dx$  when I, or  $d^2w / dx^2$  in for Euler Bernoulli beams that times  $EI$  is the moment, so this is our bending moment and because this equation also represents minima of potential energy so I have to multiply moment with the angle or the rotation so  $w_2$  corresponds to  $\psi$  which is the rotation, so in the second boundary second equilibrium equation the essential boundary condition is rotation and the natural boundary condition is moment. So either I specify

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$$0 = \int_0^{L_e} \left[ \frac{dW_1}{dx} \left\{ GAK_s \left( \psi + \frac{dw}{dx} \right) \right\} - W_1 f \right] dx - \left[ W_1 GAK_s \left( \psi + \frac{dw}{dx} \right) \right]_0^{L_e} \quad \text{--- (1) ---}$$

$$0 = \int_0^{L_e} \left[ \frac{dW_2}{dx} \left[ EI \frac{d\psi}{dx} + W_2 GAK_s \left( \psi + \frac{dw}{dx} \right) \right] - W_2 EI \frac{d\psi}{dx} \right] dx - \left[ W_2 EI \frac{d\psi}{dx} \right]_0^{L_e} \quad \text{--- (2) ---}$$

$\psi + \frac{dw}{dx} \rightarrow \text{shear strain}$

BOUNDARY TERMS

EBC

EQ 1  $\rightarrow W_1 \rightarrow W$  (displacement)

EQ 2  $\rightarrow W_2 \rightarrow \psi$  (rotation)

NBC

$GAK_s \left( \psi + \frac{dw}{dx} \right) = V$

$EI \frac{d\psi}{dx} = M$

In the second bound the second differential equation rotation at our given endpoint or the moment and in the same sense if I am dealing with equation 1 either I have to just specify the displacement which is  $w$  or I have to specify the shear force, so this is.

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Handwritten mathematical derivation of boundary terms for a beam element. The derivation shows two equations, (1) and (2), representing the weak form of the governing equations. Equation (1) is for the displacement equation, and Equation (2) is for the rotation equation. Both equations involve integration by parts, leading to boundary terms. The boundary terms are then defined as  $Q_1^e$ ,  $Q_2^e$ ,  $Q_3^e$ , and  $Q_4^e$ . A diagram of a beam element is shown with nodes at  $x=0$  and  $x=h_e$ , and a coordinate system  $x$ .

Equation (1): 
$$0 = \int_0^{h_e} \left[ \frac{dw_1}{dx} \left\{ GAK_S \left( \psi + \frac{dw}{dx} \right) \right\} - w_1 \frac{f}{S} \right] dx - \left[ w_1 GAK_S \left( \psi + \frac{dw}{dx} \right) \right]_0^{h_e} \quad \text{--- (1) = } \checkmark$$

Equation (2): 
$$0 = \int_0^{h_e} \left[ \frac{dw_2}{dx} \left[ EI \frac{d\psi}{dx} + w_2 GAK_S \left( \psi + \frac{dw}{dx} \right) \right] - w_2 \left[ EI \frac{d\psi}{dx} \right]_0^{h_e} \right] dx - \left[ w_2 \left[ EI \frac{d\psi}{dx} \right]_0^{h_e} \right] \quad \text{--- (2) = } \checkmark$$

BOUNDARY TERMS

EBC

EQ 1  $\rightarrow w_1 \rightarrow w$  (displacement)

EQ 2  $\rightarrow w_2 \rightarrow \psi$  (rotation)

NBC

$GAK_S \left( \psi + \frac{dw}{dx} \right) = V$

$EI \frac{d\psi}{dx} = M$

$$\begin{cases} Q_1^e = - \left[ GAK_S \left( \psi + \frac{dw}{dx} \right) \right]_{x=0} \\ Q_3^e = + \left[ GAK_S \left( \psi + \frac{dw}{dx} \right) \right]_{x=h_e} \end{cases}$$

$$\begin{cases} Q_2^e = - \left[ EI \frac{d\psi}{dx} \right]_{x=0} \\ Q_4^e = + \left[ EI \frac{d\psi}{dx} \right]_{x=h_e} \end{cases}$$

How we interpret these terms in context of the boundary conditions okay and what are the variables, so next what we do is we look at these terms and we define these represent these in terms of  $Q$ 's because in our FE formulation on the right side of the equality side we always represent something as  $f + q$  okay, so that is what we will express these terms as so we will write  $Q_1^e$  that is for the  $e^{\text{th}}$  element is equal to minus  $GAK_S \psi + dw$  over  $dx$  and this thing I am evaluating at  $x$  is equal to 0.

So I am evaluating this term which is underlined in red when you evaluate it at zero and its negative value that is  $Q_1$ .  $Q_2$  we do the same thing but put the term in the second differential equation okay, so this is just a question of nomenclature so  $Q_2$  equals  $-EI d\psi$  over  $dx$  and this thing is evaluated at  $x$  is equal to 0, then  $Q_3$  for the  $e^{\text{th}}$  element is not a negative sign but a positive sign and this entire expression  $GAK_S \psi + dw$  over  $dx$   $x$  is evaluated at  $h_e$ , and  $Q_4$  is positive of  $EI d\psi$  over  $dx$  evaluated  $x$  is equal to  $h_e$ , so using this terminology or nomenclature in the first equation I have  $Q_1$  and  $Q_3$  and in the second equation I have, in the second equation I get  $Q_2$  and  $Q_4$



So once again in the first equation I have  $Q_1$  and  $Q_3$  as the boundary terms in the second equation I have  $Q_2$  and  $Q_4$  as the boundary terms and then we have also seen what are the definitions of our essential as well as natural boundary conditions, so with all this information now what we will do as a next step is once again rewrite our these equations.

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The image shows handwritten mathematical derivations for beam equations. At the top, two equations are written:

$$0 = \int_0^{L_0} \left[ \frac{dw_1}{dz^2} \left[ GA K_S \left( \psi + \frac{dw_2}{dz} \right) \right] - w_1 f \right] dz - \left[ w_1 GA K_S \left( \psi + \frac{dw_2}{dz} \right) \right]_0^{L_0} \quad \text{--- (1) } \checkmark$$

$$0 = \int_0^{L_0} \left[ \frac{dw_2}{dz^2} \left[ EI \frac{d\psi}{dz} + w_2 GA K_S \left( \psi + \frac{dw_2}{dz} \right) \right] - \left[ w_2 EI \frac{d\psi}{dz} \right]_0^{L_0} \right] dz - \left[ w_2 EI \frac{d\psi}{dz} \right]_0^{L_0} \quad \text{--- (2) } \checkmark$$

Below these, the boundary terms are categorized into Essential Boundary Conditions (EBC) and Natural Boundary Conditions (NBC).

**BOUNDARY TERMS**

**EBC:**

- EQ 1  $\rightarrow w_1 \Rightarrow w$  (displacement)
- EQ 2  $\rightarrow w_2 \Rightarrow \psi$  (rotation)

**NBC:**

- $GA K_S \left( \psi + \frac{dw_2}{dz} \right) = V$
- $EI \frac{d\psi}{dz} = M$

The boundary terms are then written as:

$$\begin{cases} Q_1^e = - \left[ GA K_S \left( \psi + \frac{dw_2}{dz} \right) \right]_{z=0} \\ Q_3^e = + \left[ GA K_S \left( \psi + \frac{dw_2}{dz} \right) \right]_{z=L_0} \end{cases} \quad \begin{cases} Q_2^e = - \left[ EI \frac{d\psi}{dz} \right]_{z=0} \\ Q_4^e = + \left[ EI \frac{d\psi}{dz} \right]_{z=L_0} \end{cases}$$

A small diagram of a beam element is shown on the right, with a coordinate system  $z$  and a shear strain  $\psi + \frac{dw_2}{dz}$  indicated.

Equations 1 and equations, equation 1 and 2 in terms of  $Q$ 's also, so we will replace these boundary terms by  $Q$ 's, so that is what we get.

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Updated weak form

$$\int_0^{h_e} \frac{dw_1}{dx} GAK_S (\psi + \frac{dw_1}{dx}) dx = \int_D w_1 f dx + w_1(0) Q_1^e + w_1(h_e) Q_3^e \quad (E1)$$

$$\int_0^{h_e} \frac{dw_2}{dx} EI \frac{d\psi}{dx} dx + \int_0^{h_e} w_2 GAK_S (\psi + \frac{dw_2}{dx}) dx = w_2(0) Q_2^e + w_2(h_e) Q_4^e \quad (E2)$$

So what we are doing is updated weak form, so 0 to  $h_e$   $\frac{dw_1}{dx}$  over  $dx$   $GAK_S \psi + \frac{dw_1}{dx}$  over  $dx$ ,  $dx$  and now I am moving the force term to the right side so it is 0 to  $h_e$  weight function 1 times  $f$  times  $dx$  + weight function evaluated at location 0 times  $Q_1^e$  + weight function the first weight function evaluated at location  $h_e$  times  $Q_3^e$  okay, so this is my equation 1 and the second equation when I put in  $Q$ 's is partial, differential of  $w_2$  with respect to  $x$ .

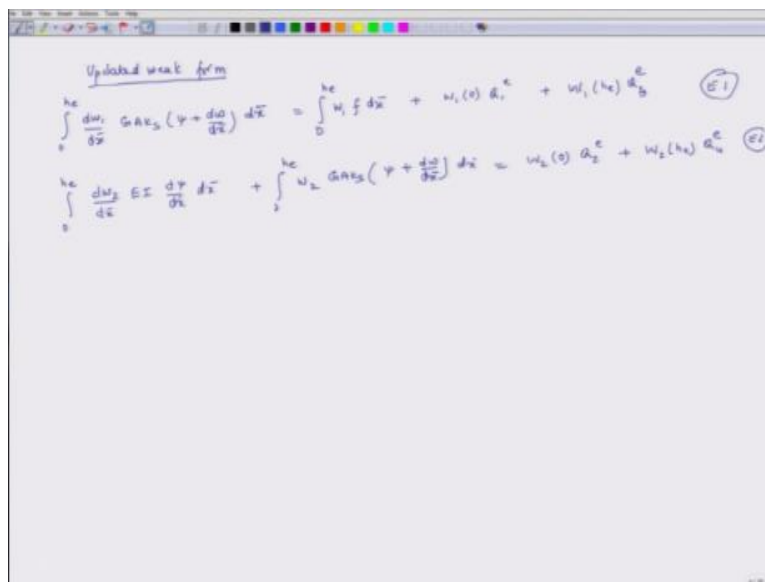
Oh so I forgot everywhere I should have  $\bar{x}$  so  $\frac{dw_2}{dx}$  over  $dx$  times  $EI \frac{d\psi}{dx}$  over  $dx$   $dx$  + 0 to  $h_e$   $w_2$   $GAK_S \psi + \frac{dw_2}{dx}$  over  $dx$   $\bar{x}$  equals weight function second weight function evaluated at 0 times  $Q_2^e$  + second weight function evaluated at  $h_e$  times  $Q_4^e$  this my E2 okay, so these are the two equations and at this stage we are almost ready to introduce interpolation functions because we have we can we have created a weak form for both the differential equations, we have extracted the boundary terms which are  $Q$ 's and at this stage we can replace  $\psi$  and  $w$  by respective interpolation functions like we did in the earlier case.

Now couple of points which I would like to make at this point of time, first  $w$  and  $\psi$  they are independent variables they do not depend on each other, so when I select interpolation functions

then these interpolation functions has to have to be mutually independent okay so this is one thing, so in traditional so I have to say I have to have one set of interpolation functions multiplied by some constants for  $w$ , another set of interpolation functions for  $\psi$ , they cannot be identical that is one thing.

Second thing is what we see here is that.

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The image shows a handwritten derivation of the updated weak form for a 1D element. The title is "Updated weak form". The derivation consists of two equations, (E1) and (E2).

Equation (E1) is:

$$\int_0^{h_e} \frac{dw_1}{dx} \text{GAx}_2 \left( \psi + \frac{dw_2}{dx} \right) dx = \int_D w_1 f dx + w_1(0) a_1^e + w_1(h_e) a_2^e \quad (E1)$$

Equation (E2) is:

$$\int_0^{h_e} \frac{dw_2}{dx} \text{EA} \frac{d\psi}{dx} dx + \int_0^{h_e} w_2 \text{GAx}_2 \left( \psi + \frac{dw_2}{dx} \right) dx = w_2(0) a_2^e + w_2(h_e) a_3^e \quad (E2)$$

To have a we have the highest derivative of  $\psi$  is  $d\psi$  over  $dx$  so  $\psi$  has to be at least linear in nature so that its derivative exists, we do not have a second order derivative, so any element which has at least a linear variation of  $\psi$  that is the type of function we have to choose, second thing is  $w$ , so in case of  $w$  also the highest order derivative in these equations is  $dw$  over  $dx$ , so  $w$  should also be at least linear in nature so that is there.

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Updated weak form

$$\int_0^{h_e} \frac{dw_1}{dx} G A E_s \left( \psi + \frac{dw}{dx} \right) dx = \int_D w_1 f dx + w_1(0) a_1^e + w_1(h_e) a_b^e \quad (E1)$$

$$\int_0^{h_e} \frac{dw_2}{dx} E I \frac{d\psi}{dx} dx + \int_0^{h_e} w_2 G A E_s \left( \psi + \frac{dw}{dx} \right) dx = w_2(0) a_2^e + w_2(h_e) a_u^e \quad (E2)$$

The third thing is that we there is no rule at this point of time which says that the order of  $w$  and order of  $\psi$  should be same,  $\psi$  could be a linear function and  $w$  could be quadratic. The mathematics at this stage does not restrict us to impose conditions where  $\psi$  and  $w$  they have to be of the same order so based on all this understanding and we have seen that  $\psi$  has to be continuous it has to be a unique thing and the nature of  $\psi$  is same as the nature of  $u$  which we had seen in case of heat conduction of bar equation.

So what essentially what I am trying to say is that this stage the Lagrangian interpolation functions which we had developed earlier will work in this case okay.

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The image shows handwritten mathematical derivations for the updated weak form of the Euler-Bernoulli beam problem. The derivations are as follows:

Updated weak form

$$\int_0^{h_e} \frac{dw_1}{dx} GAx_3 \left( \psi + \frac{dw}{dx} \right) dx = \int_D w_1 f dx + w_1(0) a_1^e + w_1(h_e) a_b^e \quad (E1)$$

$$\int_0^{h_e} \frac{dw_2}{dx} EI \frac{d\psi}{dx} dx + \int_0^{h_e} w_2 GAx_3 \left( \psi + \frac{dw}{dx} \right) dx = w_2(0) a_2^e + w_2(h_e) a_u^e \quad (E2)$$

So for an Euler Bernoulli beam Lagrangian interpolation functions do not work because at every node we have two degrees of freedom right, here also we have two degrees of freedom at every node but they are expressed through two different functions, so in case of Euler Bernoulli beam we have you developed hermetic functions because  $w$  was same but we had to extract two independent degrees of freedom from here but here we have two independent functions  $\psi$  and  $w$  so Lagrangian functions work.

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Updated weak form

$$\int_0^{h_e} \frac{dw_1}{dz} G A E_s \left( \psi + \frac{dw_2}{dz} \right) dz = \int_0^{h_e} w_1 f dz + w_1(0) A_1 \epsilon + w_1(h_e) A_2 \epsilon \quad (E1)$$

$$\int_0^{h_e} \frac{dw_2}{dz} E I \frac{d\psi}{dz} dz + \int_0^{h_e} w_2 G A E_s \left( \psi + \frac{dw_2}{dz} \right) dz = w_2(0) A_2 \epsilon + w_2(h_e) A_4 \epsilon \quad (E2)$$

we express  $w$  and  $\psi$  as:

$$w(\bar{x}) = \sum_{j=1}^m w_j^e \alpha_j^e \quad \psi(x) = \sum_{j=1}^n$$

So we express, so we express  $w$  and  $\psi$  as, so  $w$  is a function of  $\bar{x}$  and that is equals  $j$  equals 1 to  $m$  some constants  $w_j^e$  times interpolation functions and here we call this interpolation functions  $\alpha$  okay, and similarly  $\psi$  is a function of  $x$  and that is equal to  $j$  is equal to one to  $n$  because there is no rule which says that the order of  $\psi$  order of  $w$  they should be same okay.

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Updated weak form

$$\int_0^{h_e} \frac{dw_1}{dz} G A E_2 \left( \psi + \frac{dw_2}{dz} \right) dz = \int_D w_1 f dz + w_1(0) a_1^e + w_1(h_e) a_2^e \quad (E1)$$

$$\int_0^{h_e} \frac{dw_2}{dz} E I \frac{d\psi}{dz} dz + \int_0^{h_e} w_2 G A E_2 \left( \psi + \frac{dw_2}{dz} \right) dz = w_2(0) a_2^e + w_2(h_e) a_3^e \quad (E2)$$

we express  $w$  and  $\psi$  as:

$$w(\bar{x}) = \sum_{j=1}^m w_j^e \beta_j^e \quad \psi(\bar{x}) = \sum_{j=1}^n S_j^e \beta_j^e$$

And this equals some constant  $S_j$  times  $\beta_j$  and  $\beta$  is the interpolation function and all this is for the  $e^{\text{th}}$  element for the  $e^{\text{th}}$  element. Now if  $m$  is equal to one if  $m$  if  $m$  equals 1 then  $w$  is constant over the element if  $m$  equals 2 then  $w$  is a linear interpolation function right, if  $m$  equals three then  $w$  is quadratic, same thing is true for  $\psi$  also if  $n$  is equal to one.

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Updated weak form

$$\int_0^{h_e} \frac{dw_1}{dz} G A E_2 \left( \psi + \frac{dw_1}{dz} \right) dz = \int_D w_1 f dz + w_1(0) a_1^e + w_1(h_e) a_2^e \quad (E1)$$

$$\int_0^{h_e} \frac{dw_2}{dz} E I \frac{d\psi}{dz} dz + \int_0^{h_e} w_2 G A E_2 \left( \psi + \frac{dw_2}{dz} \right) dz = w_2(0) a_2^e + w_2(h_e) a_3^e \quad (E2)$$

we express  $w$  and  $\psi$  as:

$$w(\bar{x}) = \sum_{j=1}^m w_j^e \phi_j^e \quad \psi(\bar{x}) = \sum_{j=1}^n \xi_j^e \beta_j^e$$

Then it is constant, n equals 2 then it is linear and so on and so forth, so now at this stage.



(Refer Slide Time: 17:11)

Updated weak form

$$\int_0^{h_e} \frac{dw_1}{dx} G A E_2 \left( \psi + \frac{dw}{dx} \right) dx = \int_0^{h_e} w_1 f dx + w_1(x) a_1^e + w_1(h_e) a_b^e \quad (E1)$$

$$\int_0^{h_e} \frac{dw_2}{dx} E I \frac{d\psi}{dx} dx + \int_0^{h_e} w_2 G A E_2 \left( \psi + \frac{dw}{dx} \right) dx = w_2(x) a_2^e + w_2(h_e) a_u^e \quad (E2)$$

we express  $w$  and  $\psi$  as:

$$w(x) = \sum_{j=1}^m w_j^e a_j^e \quad \psi(x) = \sum_{j=1}^n S_j^e \beta_j^e \quad (3)$$

we choose  $m = n = 2$ .

$$\frac{dw}{dx} =$$

We have to make a choice so at this stage we choose  $m$  equals  $n$  equals 2 just to make our life simple and we have to go from here somewhere so we have to choose some values of  $m$  and  $n$  we are just choosing that  $m$  and  $n$  are same and they are equal two, so let us call these equations as equations 3 so if that is the case then  $dw$  over  $dx$   $m$  is equal to two so  $w$  is  $w_1$  times  $\alpha_1 + w_2$  times  $\alpha_2$  right so in that is the case

(Refer Slide Time: 17:57)

Updated weak form

$$\int_0^{h_e} \frac{dw}{dx} G A E_2 \left( \psi + \frac{dw}{dx} \right) dx = \int_0^{h_e} w_1 f dx + w_1(0) A_1 + w_1(h_e) A_2 \quad (E1)$$

$$\int_0^{h_e} \frac{dw}{dx} E I \frac{d\psi}{dx} dx + \int_0^{h_e} w_2 G A E_2 \left( \psi + \frac{dw}{dx} \right) dx = w_2(0) A_2 + w_2(h_e) A_2 \quad (E2)$$

we express  $w$  and  $\psi$  as:

$$w(x) = \sum_{j=1}^n w_j^e A_j^e$$

$$\psi(x) = \sum_{j=1}^n s_j^e \beta_j^e$$

we chose  $n = 2$

$$\frac{dw}{dx} = w_1^e \frac{dA_1^e}{dx} + w_2^e \frac{dA_2^e}{dx}$$

$$= \frac{w_2 - w_1}{h_e} \quad (4A)$$

$$\psi = s_1 \beta_1 + s_2 \beta_2$$

Then  $dw$  over  $dx$  is  $w_1$  the  $d\alpha_1$  over  $d\bar{x}$  +  $w_2$   $d\alpha_2$  over  $d\bar{x}$  okay and earlier, in our classes we had seen that a linear function over an element can be expressed as  $\alpha_1$  is equal to  $\bar{x}$  over  $h_e$  and  $\alpha_2$  is equal to  $-\bar{x}$  over  $h_e$ . We had, we have developed these functions so we will just, we do not have to reinvent the wheel so if that is the case then  $dw$  over  $dx$  equals if I differentiate I get  $w_2 - w_1$  over  $h_e$  okay so  $dw$  over  $dx$  is difference in  $w$ 's  $w_2 - w_1$  over  $h_e$  so we call it 4A. Also let us look at rotation so rotation equals  $s_1$  and I have called them as  $\beta$  and okay actually I will fine

So this is equals  $\beta_1 + s_2 \beta_2$  okay and  $\beta_1$  and they  $\beta_2$  depend on  $x$  they change with  $x$  so they depend on  $x$ ,  $s_1$  and  $s_2$  are constant  $\beta_1 \beta_2$  are functions of  $x$  okay. Now we know that for thin beams we had seen that in the case of thin beam there is a shear deformation but in thin beams the shear deformation is not there so the planes which are normal initially they still remain normal and if that is the case then what is our shear strain, we had said the shear strain is  $\psi + dw$  over  $dx$  and if the planes normal planes still remain normal which means shear strains are zero which means  $\psi + dw$  over  $dx$

(Refer Slide Time: 20:40)

Updated weak form

$$\int_0^{h_e} \frac{dw_1}{dz} G A E_2 \left( \psi + \frac{dw_1}{dz} \right) dz = \int_0^{h_e} w_1 f dz + w_1(0) A_1 \epsilon + w_1(h_e) A_2 \epsilon \quad (E1)$$

$$\int_0^{h_e} \frac{dw_2}{dz} E I \frac{d\psi}{dz} dz + \int_0^{h_e} w_2 G A E_2 \left( \psi + \frac{dw_1}{dz} \right) dz = w_2(0) A_2 \epsilon + w_2(h_e) A_2 \epsilon \quad (E2)$$

we express  $w$  and  $\psi$  as:

$$w(x) = \sum_{j=1}^m w_j^e A_j^e \quad \psi(x) = \sum_{j=1}^n S_j^e B_j^e \quad (3)$$

we chose  $m = n = 2$ .

$$\frac{dw}{dz} = w_1^e \frac{dA_1^e}{dz} + w_2^e \frac{dA_2^e}{dz}$$

$$= \frac{w_2^e - w_1^e}{h_e} \quad (4)$$

$$\psi = S_1 B_1 + S_2 B_2$$

Is equal to 0 and this is for thin beams which means  $\psi$  equals-  $dw$  over  $dx$  okay.

(Refer Slide Time: 20:50)

Handwritten notes on a whiteboard:

$$\frac{dw}{dx} = w_1^e \frac{dw_1}{dx} + w_2^e \frac{dw_2}{dx}$$

$$= \frac{w_1^e - w_2^e}{h_e} \quad \text{--- (1A)}$$

$$\psi = S_1 \beta_1 + S_2 \beta_2$$

For thin beams:

$$\psi + \frac{dw}{dx} = 0 \rightarrow \psi = -\frac{dw}{dx}$$

$$\beta_1 = \frac{\bar{x}}{h_e} \quad \beta_2 = 1 - \frac{\bar{x}}{h_e}$$


---

For thin beams:

$$\psi = S_1 \frac{\bar{x}}{h_e} + S_2 \left(1 - \frac{\bar{x}}{h_e}\right) = \frac{w_1^e - w_2^e}{h_e}$$

So if that is the case then for thin beams, for thin beams and again once again  $\beta_1$  is equal to  $\bar{x}$  over  $h_e$  and  $\beta_2$  is equal to  $1 - \bar{x}$  over  $h_e$  right, so for thin beams  $\psi$  equals  $S_1 \bar{x}$  over  $h_e + S_2 1 - \bar{x}$  over  $h_e + S_2 1 - \bar{x}$  over  $h_e$  and this equals  $-(dw \text{ over } dx \text{ and } dw \text{ over } dx)$  is the thing which is underlined in red okay, so this one is equal to  $w_1 - w_2$  over  $h_e$ . Now these two sides so what you see is that the right side is a constant and the left side depends on  $x$ , the left side depends on  $x$  and the right side is independent of  $x$ .

(Refer Slide Time: 22:17)

Integrating by parts:

$$0 = \int_0^{L_e} \left[ \frac{d}{dx} \left\{ G A K_s \left( \psi + \frac{dw}{dx} \right) \right\} - w_1 f \right] dx - \left[ w_1 G A K_s \left( \psi + \frac{dw}{dx} \right) \right]_0^{L_e} \quad \text{--- (1) ---}$$

$$0 = \int_0^{L_e} \left[ \frac{dw_2}{dx} E I \frac{d\psi}{dx} + w_2 G A K_s \left( \psi + \frac{dw}{dx} \right) \right] dx - \left[ w_2 E I \frac{d\psi}{dx} \right]_0^{L_e} \quad \text{--- (2) ---}$$

BOUNDARY TERMS

EBC

EA 1  $\rightarrow w_1 \rightarrow w$  (displacement)

EA 2  $\rightarrow w_2 \rightarrow \psi$  (rotation)

NBC

$$\left. \begin{aligned} G A K_s \left( \psi + \frac{dw}{dx} \right) &= V \\ E I \frac{d\psi}{dx} &= M \end{aligned} \right\}$$

$$\left\{ \begin{aligned} Q_1^e &= - \left[ G A K_s \left( \psi + \frac{dw}{dx} \right) \right]_{x=0} \\ Q_2^e &= - \left[ E I \frac{d\psi}{dx} \right]_{x=0} \end{aligned} \right\}$$

This can be true only if the coefficient of  $x$  on the left side is 0. So what is the coefficient of  $x$ , so this can be true only if coefficient of  $x$  on LHS is 0 only then it will be independent of  $x$  so that will be there when  $s_1 - s_2$  is equal to 0, this is the coefficient of  $x$  okay which means  $s_1$  equals  $s_2$ , so what this means is that for thin beams  $s_1$  which is a constant should be same as  $s_2$ ,  $s_1$  is the constant and it is multiplied by  $\psi$  this  $\beta_1 s_2$  is a constant which is multiplied by  $\beta_2$  and they have to be same okay.

Now if  $s_1$  is equal to  $s_1$  is equal to  $s_1$  is same as  $s_2$  what does that mean, suppose you have an element and if I am going to plot in the  $y$ -direction the rotation  $\psi$  then  $\psi$  will be constant over the element right because  $s_1$  is the coefficient that first node  $s_2$  is the coefficient at second node the values at both these nodes are same and it is a linear element this means that  $\psi$  is equal to constant, and this is  $s_1$  this is  $s_2$ , this is node one this is node two understood, if  $\psi$  is constant then  $d\psi$  over  $dx$  will be 0.

So for thin beams if we are doing all this process and we are choosing that  $m$  is equal  $n$  is equal two and if we you all this process ultimately what all that means is that on in a thin beam  $d\psi$  over  $dx$  will be using this formulation it will come out as 0. If it comes out as 0 then we go back

and look at our equation we look at this term this is the equation in we look at this equation second equation, and we see that here this term  $E \frac{dw}{dx} \frac{d\psi}{dx}$  this will be 0 and this term is essentially related to the bending energy of the beam. It is related to the bending energy of the beam so which means that the beam has very little bending energy which is not the reality because when you are bending being a lot of energy is spent right.

(Refer Slide Time: 25:36)

Handwritten mathematical derivation of the weak form for a beam element. The derivation starts with the integration by parts of the bending energy term:

$$0 = \int_0^{l_e} \left[ \frac{dw}{dx} \left( GAK_s \left( \psi + \frac{dw}{dx} \right) \right) - w_1 \frac{dw}{dx} \right] dx - \left[ w_1 GAK_s \left( \psi + \frac{dw}{dx} \right) \right]_0^{l_e} \quad \text{--- (1)}$$

$$0 = \int_0^{l_e} \left[ \frac{dw_2}{dx} \frac{EI}{dx} \frac{d\psi}{dx} + w_2 GAK_s \left( \psi + \frac{dw}{dx} \right) \right] dx - \left[ w_2 \frac{EI}{dx} \frac{d\psi}{dx} \right]_0^{l_e} \quad \text{--- (2)}$$

The term  $\psi + \frac{dw}{dx}$  is identified as the shear strain, illustrated with a diagram of a beam element under shear.

**BOUNDARY TERMS**

**EBG:**

EQ 1  $\rightarrow w_1 \rightarrow w$  (displacement)

EQ 2  $\rightarrow w_2 \rightarrow \psi$  (rotation)

**NBC:**

$GAK_s \left( \psi + \frac{dw}{dx} \right) = V$

$EI \frac{d\psi}{dx} = M$

The boundary terms are summarized as:

$$\left\{ \begin{aligned} Q_1^e &= - \left[ GAK_s \left( \psi + \frac{dw}{dx} \right) \right]_{x=0} \\ Q_2^e &= - \left[ EI \frac{d\psi}{dx} \right]_{x=0} \\ Q_3^e &= + \left[ GAK_s \left( \psi + \frac{dw}{dx} \right) \right]_{x=l_e} \\ Q_4^e &= + \left[ EI \frac{d\psi}{dx} \right]_{x=l_e} \end{aligned} \right.$$

So this formulation creates a problem because of this numerical problem.

(Refer Slide Time: 25:41)

The image shows a whiteboard with handwritten mathematical notes. At the top, the equation  $\psi = S_1 \frac{x}{h_e} + S_2 (1 - \frac{x}{h_e}) = \frac{W_1 - W_2}{h_e}$  is written. Below this, a note states "can be true only if coefficient of  $\bar{x}$  on LHS is zero." This leads to the equation  $(S_1 - S_2) = 0 \Rightarrow \boxed{S_1 = S_2}$ . To the right of this, a diagram shows two horizontal lines labeled 1 and 2, with a bracket above them labeled  $S_1$  and a bracket below them labeled  $S_2 = W_1$ . Below the diagram, the expression  $\frac{dW}{dx} = 0$  is written and underlined.

$$\psi = S_1 \frac{x}{h_e} + S_2 (1 - \frac{x}{h_e}) = \frac{W_1 - W_2}{h_e}$$

can be true only if coefficient of  $\bar{x}$  on LHS is zero.

$$(S_1 - S_2) = 0 \Rightarrow \boxed{S_1 = S_2}$$

$\frac{dW}{dx} = 0$

It is not a real problem but when we bend the beam it is bending and takes some energy.

(Refer Slide Time: 25:46)

$$\psi = S_1 \frac{x}{h_e} + S_2 \left(1 - \frac{x}{h_e}\right) = \frac{w_1 - w_2}{h_e}$$

Can be true only if coefficient of  $x$  on RHS is zero.

$(S_1 - S_2) = 0 \Rightarrow \boxed{S_1 = S_2}$

$\frac{dw}{dx} = 0$

SHEAR LOCKING

Diagram: A beam element of length  $h_e$  with nodes 1 and 2. The displacement field is  $\psi = \text{const}$ . The shear strain is  $S_1 = S_2$ .

But this problem the way we have formulated it creates this problem because it forces  $\psi$  to become constant over the element which causes the energy related to bending vanish in the equilibrium formula in the weak formulation and this problem is known as shear locking, it is known as shear locking okay, and it is known as shear locking and the reason for this is that  $\psi$  because of all our assumptions it comes as constant at the end of the day over the domain of the element so to solve this to avoid this numerical problem, so this is a numerical problem.



(Refer Slide Time: 26:37)

The image shows a handwritten derivation on a whiteboard. At the top, the equation  $\psi = S_1 \frac{\bar{x}}{h_e} + S_2 (1 - \frac{\bar{x}}{h_e}) = \frac{w_1 - w_2}{h_e}$  is written. Below it, a note states: "Can be true only if coefficient of  $\bar{x}$  on LHS is zero." This leads to the equation  $(S_1 - S_2) = 0 \Rightarrow S_1 = S_2$ . To the right of this, a diagram shows two horizontal arrows representing shear forces. The top arrow is labeled  $S_1$  and the bottom arrow is labeled  $S_2$ . A bracket on the right side of the arrows is labeled  $S_2 = S_1$ . Below the diagram, the text  $\frac{dw}{dx} = 0$  is written. At the bottom, the phrase "SHEAR LOCKING" is underlined, followed by an arrow pointing to the text "Numerical problem."

$$\psi = S_1 \frac{\bar{x}}{h_e} + S_2 (1 - \frac{\bar{x}}{h_e}) = \frac{w_1 - w_2}{h_e}$$

Can be true only if coefficient of  $\bar{x}$  on LHS is zero.

$$(S_1 - S_2) = 0 \Rightarrow S_1 = S_2$$

$\frac{dw}{dx} = 0$

SHEAR LOCKING  $\rightarrow$  Numerical problem.

To avoid this numerical problem we take we take two methods and we will discuss about these two approaches in our next lecture so all what we have discussed in today's lecture is the cause of shear locking which causes this bending energy term to vanish because the way we have formulated it forces.

(Refer Slide Time: 27:09)

Handwritten notes on a whiteboard:

$$\psi = S_1 \frac{\bar{x}}{h_e} + S_2 \left(1 - \frac{\bar{x}}{h_e}\right) = \frac{w_1 - w_2}{h_e}$$

Can be true only if coefficient of  $\bar{x}$  on LHS is zero.

$$(S_1 - S_2) = 0 \Rightarrow \boxed{S_1 = S_2}$$

Diagram showing a linear element with nodes 1 and 2. The displacement at node 1 is  $w_1$  and at node 2 is  $w_2$ . The element length is  $h_e$ . The displacement  $\psi$  is constant over the element.

$\frac{dw}{dx} = 0$

SHEAR LOCKING  $\rightarrow$  Numerical problem.

$\psi$  to become constant over the element and to avoid this we have to adopt some methods and what we do about those methods is something we will discuss in our next class, thank you.

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