Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 38
Finite element formulation for shear deformable beam: Part-I

by
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Hello, welcome to basics of finite element analysis, today is the 7th week second day of this course, and what we will do today in this particular lecture is a continuation of our discussion on shear deformable beams or Timoshenko beams, and in the last lecture what we had done was we had developed weak formulations for the two equilibrium equations, these weak forms are expressed here on your screen and as a follow-up to this what we will start looking at are the boundary terms of these equations and interpret these boundary terms in terms of what do they mean.

So what we look at are the boundary terms.

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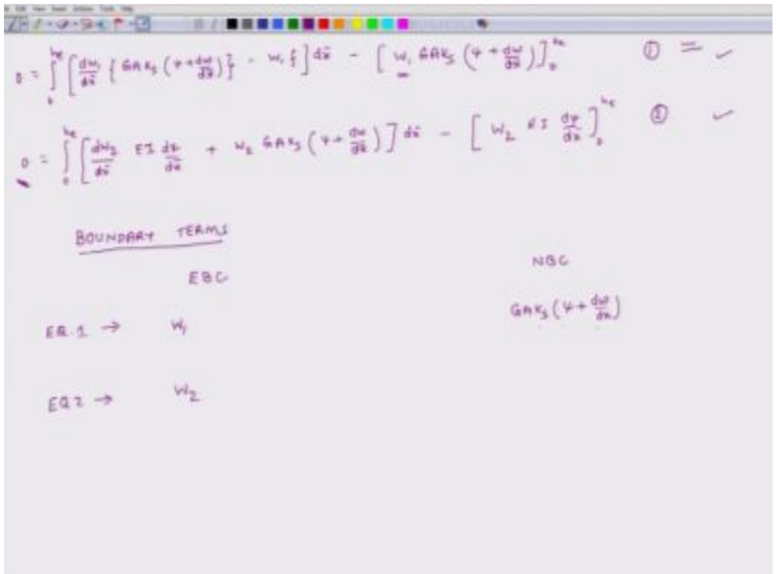

At this point of time okay, so we will look at our aim is to figure out what are the essential boundary conditions and also what are the natural boundary conditions for such a beam, so when we look at equation 1, this is equation 1okay and when we look we see that the weight function is w_1, weight function is w_1 and in equation two the weight function is w_2 okay, the coefficient of weight function in the first equation is GAK_S times ψ+ dw over dx and we had discussed earlier in one of our lectures that the coefficient of the weight function constitutes the natural boundary condition so this is the thing which is the natural boundary condition associated with the first equation.

And physically what does it represent ψ+ dw over dx we had explained earlier is nothing but shear stream right, that is what we had explained in the last class when I multiply the shear strain by G then it becomes shear stress and if I have a beam.

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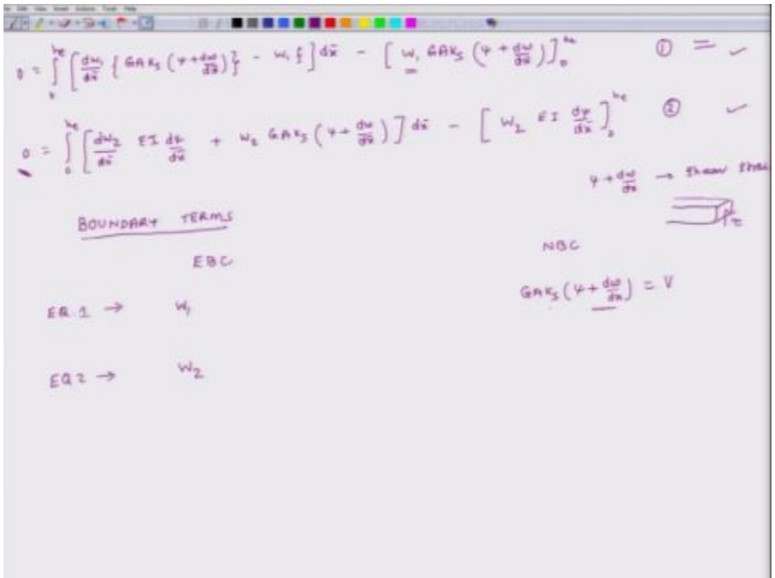

And the shear stress here is T and suppose I multiply this by that area of the cross section of the beam then that shear stress times area is nothing but shear force, so in the first equation this term

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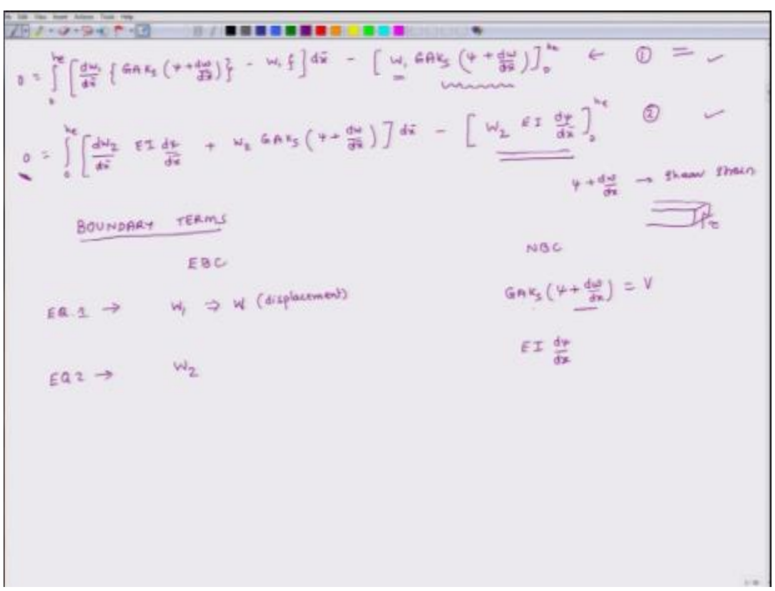

Represents physically it represents the shear force okay, now the other thing we had discussed was that this in earlier class also that this is essentially minimum potential energy statement the same thing for the second thing also, so when I multiply v by a displacement that corresponds to work done by the system, so this w_1 if I replace it by w which is the displacement then the essential boundary condition associated with the first equation is displacement and the national boundary condition associated with the first equation is the shear force.

And when I multiply w by v I get the work done okay. Next what we will do is we will look at the second weak form which is equation 2, so in equation 2 in the boundary term which is here I have w_2 is the weight function, the coefficient of that w_2 is EI dψ over dx and this so what is ψ, ψ is rotation, ψ is rotation.

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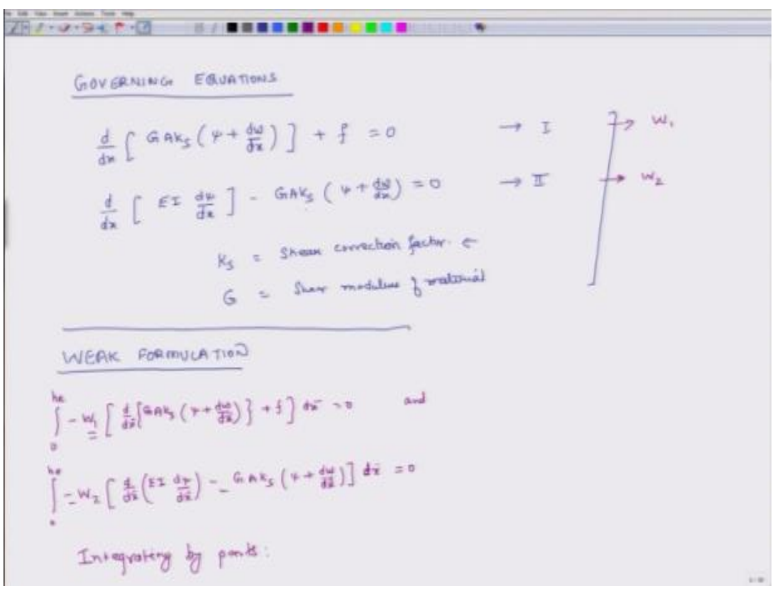

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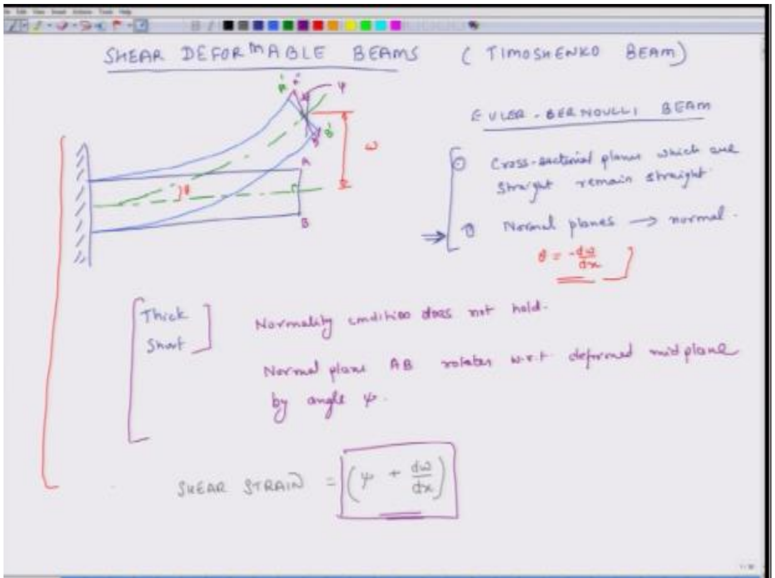

Of the original vertical line a b with respect to it how much it rotates by right, that it is the rotation.

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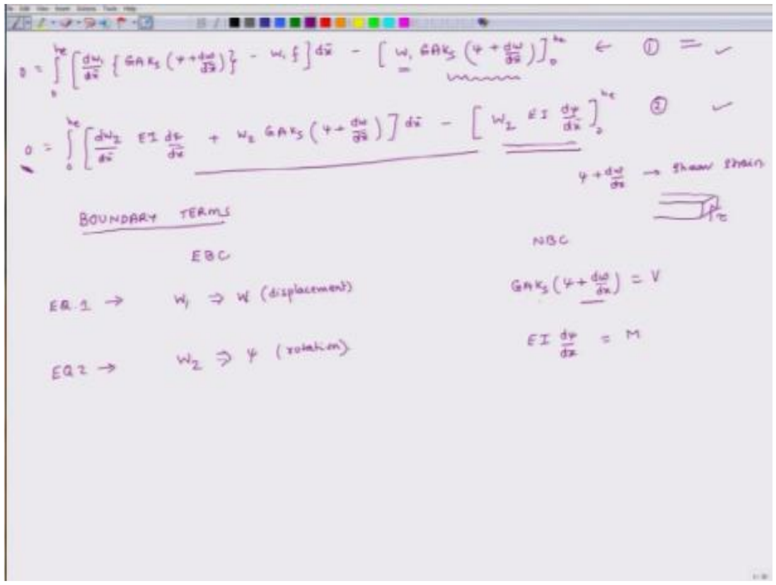

And if I differentiate it once again I essentially get the curvature right, and when I multiply that curvature by EI dψ over dx when I, or d2w / dx^2 in for Euler Bernoulli beams that times EI is te moment, so this is our bending moment and because this equation also represents minima of potential energy so I have to multiply moment with the angle or the rotation so w_2 corresponds to ψ which is the rotation, so in the second boundary second equilibrium equation the essential boundary condition is rotation and the natural boundary condition is moment. So either I specify

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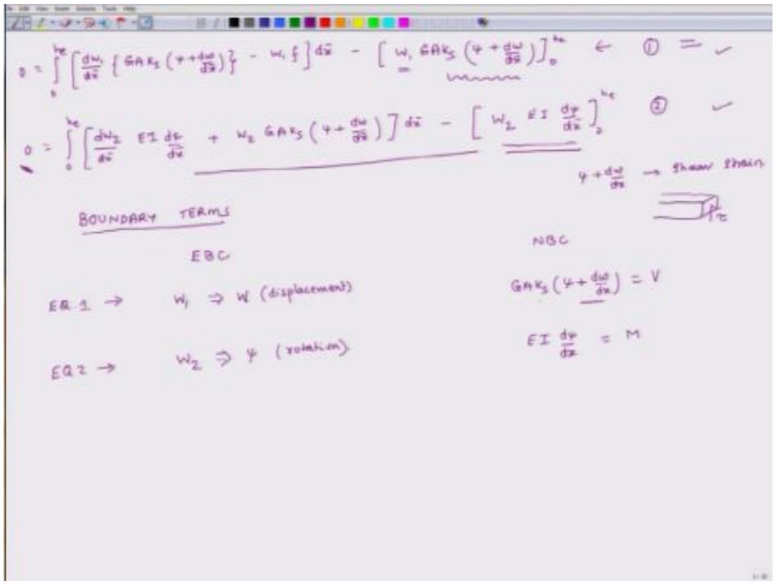

In the second bound the second differential equation rotation at our given endpoint or the moment and in the same sense if I am dealing with equation 1 either I have to just specify the displacement which is w or I have to specify the shear force, so this is.

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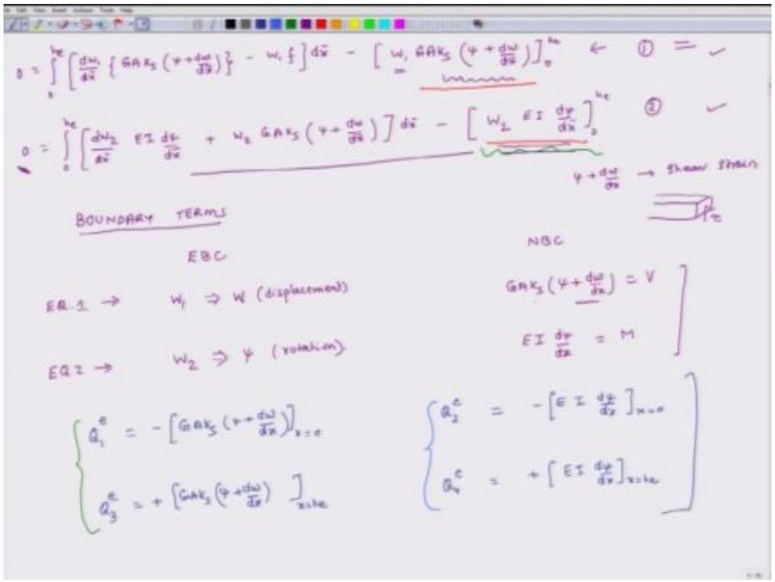

How we interpret these terms in context of the boundary conditions okay and what are the variables, so next what we do is we look at these terms and we define these represent these in terms of Q's because in our FE formulation on the right side of the equality side we always represent something as f + q okay, so that is what we will express these terms as so we will write Q_1^e that is for the eth element is equal to minus GAK_S ψ + dw over dx and this thing I am evaluating at x is equal to 0.

So I am evaluating this term which is underlined in red when you evaluate it at zero and its negative value that is Q_1. Q_2 we do the same thing but put the term in the second differential equation okay, so this is just a question of nomenclature so Q_2 equals – EI dψ over dx and this thing is evaluated at x is equal to 0, then Q_3 for the eth element is not a negative sign but a positive sign and this entire expression GAK_S ψ |+ dw over dx x is evaluated at h_e, and Q_4 is positive of EI dψ over dx evaluated x is equal to h_e, so using this terminology or nomenclature in the first equation I have Q_1 and Q_3 and in the second equation I have, in the second equation I get Q_2 and Q_4

So once again in the first equation I have $Q_1$and Q_3 as the boundary terms in the second equation I have Q_2 and Q_4 as the boundary terms and then we have also seen what are the definitions of our essential as well as natural boundary conditions, so with all this information now what we will do as a next step is once again rewrite our these equations.

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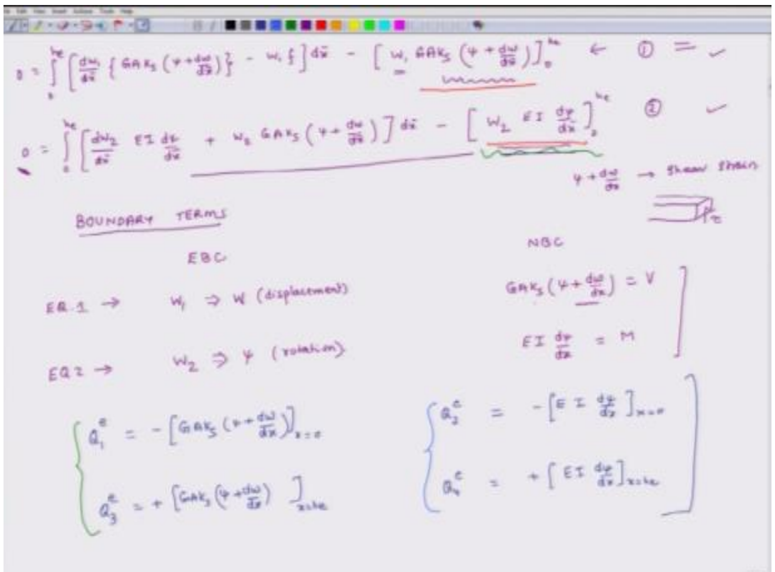

Equations 1 and equations, equation 1 and 2 in terms of Q's also, so we will replace these boundary terms by Q’s, so that is what we get.

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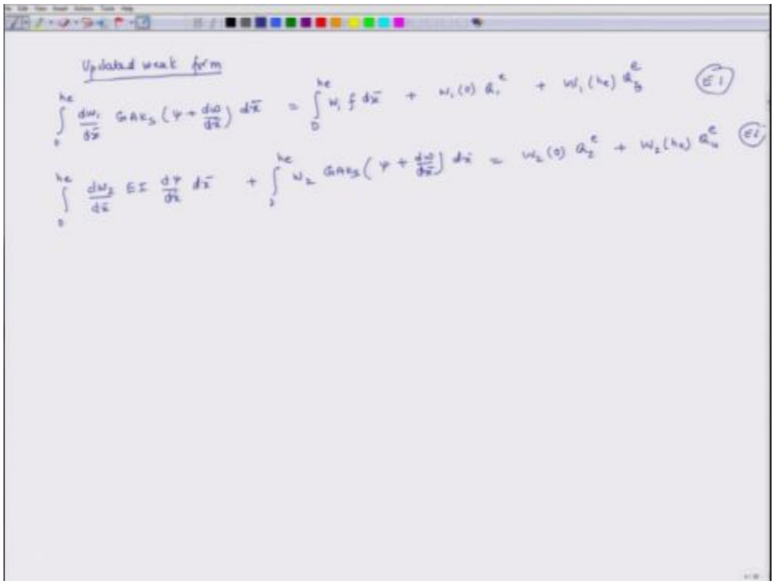

So what we are doing is updated weak form, so 0 to h_e dw1 over dx GAK_S ψ+ dw over dx, dx and now I am moving the force time to the right side so it is0 to h_e weight function 1times f times dx + weight function evaluated at location 0 times Q_1^e + weight function the first weight function evaluated at location h_e times Q_3^e okay, so this is my equation 1 and the second equation when I put in Q's is partial, differential of w_2 with respect to x.

Oh so I forgot everywhere I should have $\bar{x}$ so dw2 over dx times EI dψ over dx dx + 0 to h_e w_2 GAK_S ψ+ dw over dx $\bar{x}$ equals weight function second weight function evaluated at 0 times Q_2^e + second weight function evaluated at h_e times Q_4^e this my E2 okay, so these are the two equations and at this stage we are almost ready to introduce interpolation functions because we have we can we have created a weak form for both the differential equations, we have extracted the boundary terms which are Q's and at this stage we can replace ψ and w by respective interpolation functions like we did in the earlier case.

Now couple of points which I would like to make at this point of time, first w and ψ they are independent variables they do not depend on each other, so when I select interpolation functions then these interpolation functions has to have to be mutually independent okay so this is one thing, so in traditional so I have to say I have to have one set of interpolation functions multiplied by some constants for w, another set of interpolation functions for ψ, they cannot be identical that is one thing.

Second thing is what we see here is that.

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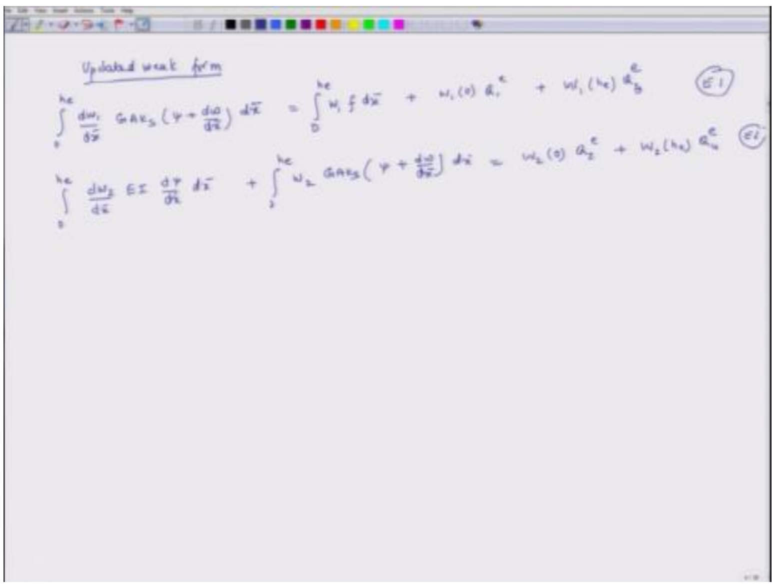

To have a we have the highest derivative of ψ is dψ over dx so ψ has to be at least linear in nature so that its derivative exists, we do not have a second order derivative, so any element which has at least a linear variation of ψ that is the type of function we have to choose, second thing is w, so in case of w also the highest order derivative in these equations is dw over dx, so w should also be at least linear in nature so that is there.

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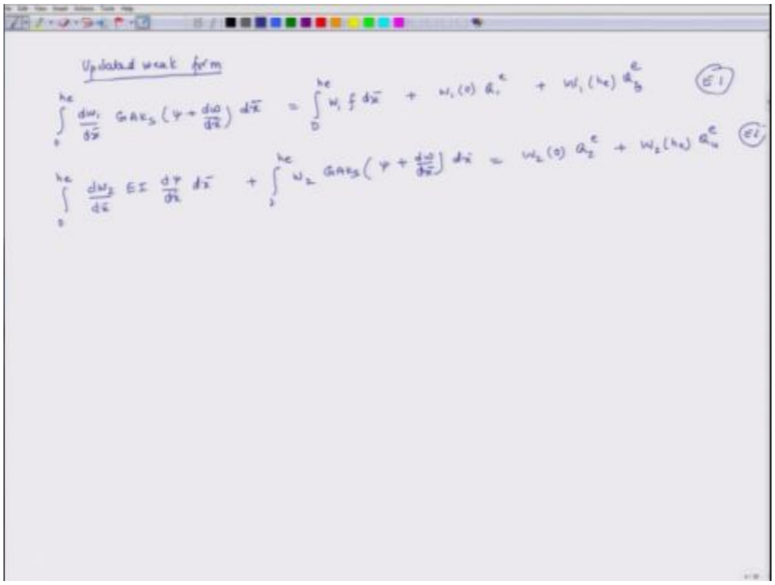

The third thing is that we there is no rule at this point of time which says that the order of w and order of ψ should be same, ψ could be a linear function and w could be quadratic. The mathematics at this stage does not restrict us to impose conditions where ψ and w they have to be of the same order so based on all this understanding and we have seen that ψ has to be continuous it has to be a unique thing and the nature of ψ is same as the nature of u which we had seen in case of heat conduction of bar equation.

So what essentially what I am trying to say is that this stage the Lagrangian interpolation functions which we had developed earlier will work in this case okay.

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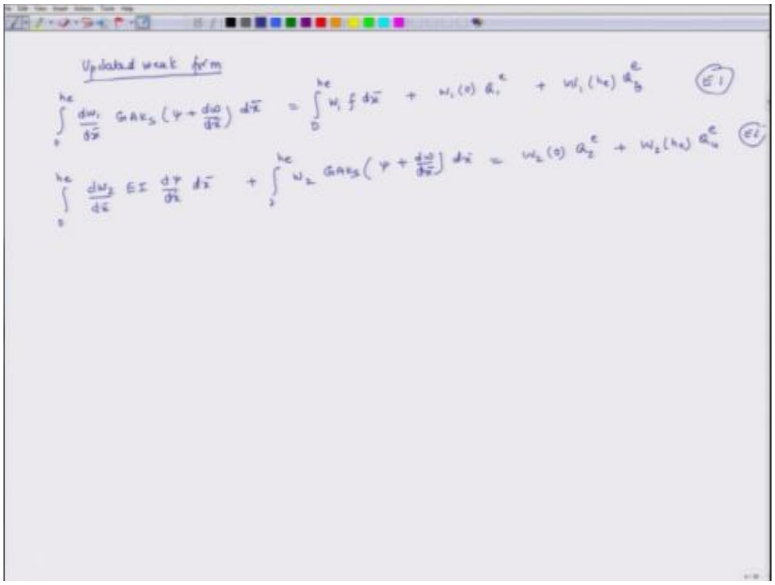

So for an Euler Bernoulli beam Lagrangian interpolation functions do not work because at every node we have two degrees of freedom right, here also we have two degrees of freedom at every node but they are expressed through two different functions, so in case of Euler Bernoulli beam we have you developed hermetic functions because w was same but we had to extract two independent degrees of freedom from here but here we have two independent functions ψ and w so Lagrangian functions work.

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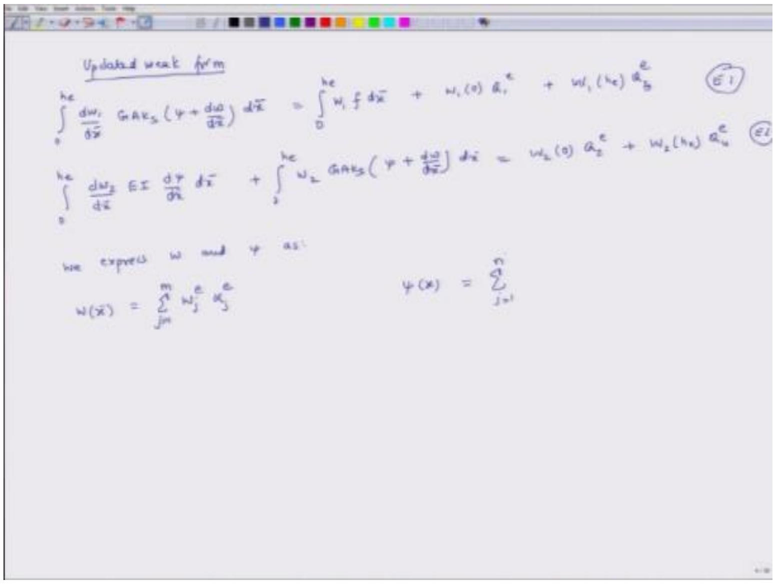

So we express, so we express w and ψ as, so w is a function of $\bar{x}$ and that is equals j equals 1 to m some constants w_j^e times interpolation functions and here we call this interpolation functions α okay, and similarly ψ is a function of x and that is equal to j is equal to one to n because there is no rule which says that the order of ψ order of w they should be same okay.

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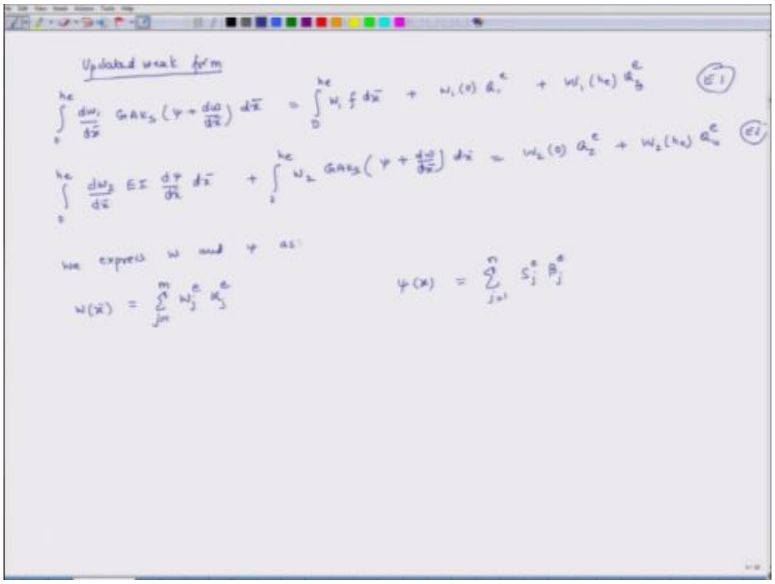

And this equals some constant S_j times β_j and β is the interpolation function and all this is for the e^{th} element for the e^{th} element. Now if m is equal to one if m if m equals 1 then w is constant over the element if m equals to then w is a linear interpolation function right, if m equals three then w is quadratic, same thing is true for ψ also if n is equal to one.

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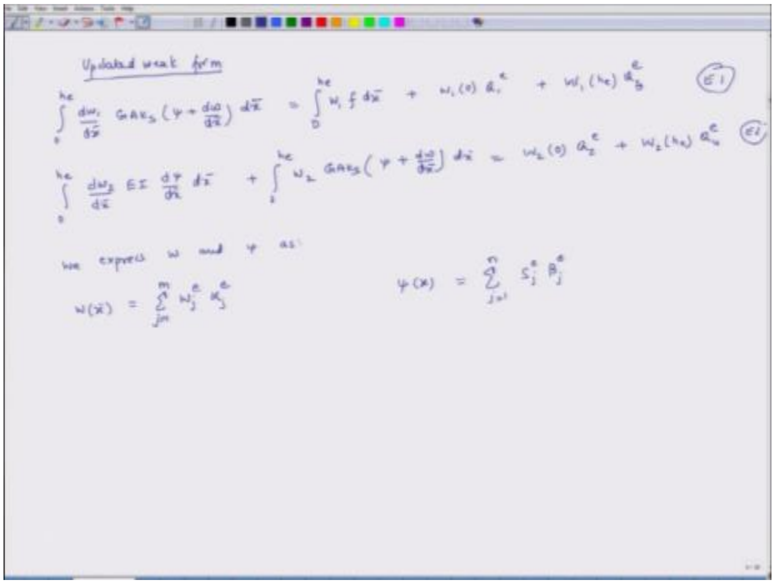

Then it is constant, n equals 2 then it is linear and so on and so forth, so now at this stage.

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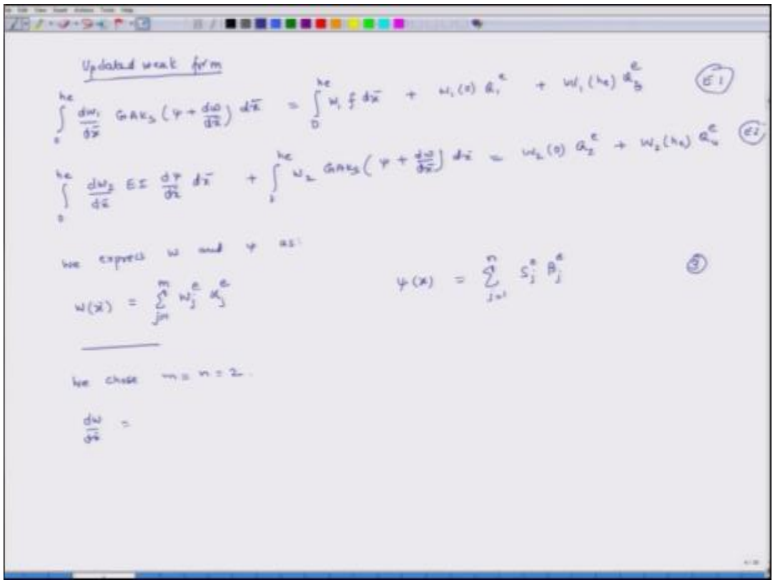

We have to make a choice so at this stage we choose m equals n equals 2 just to make our life simple and we have to go from here somewhere so we have to choose some values of m and n we are just choosing that m and n are same and they are equal two, so let us call these equations as equations 3 so if that is the case then dw over d$\bar{x}$ m is equal to two so w is w_1 times α_1 + w_2 times α_2 right so in that is the case

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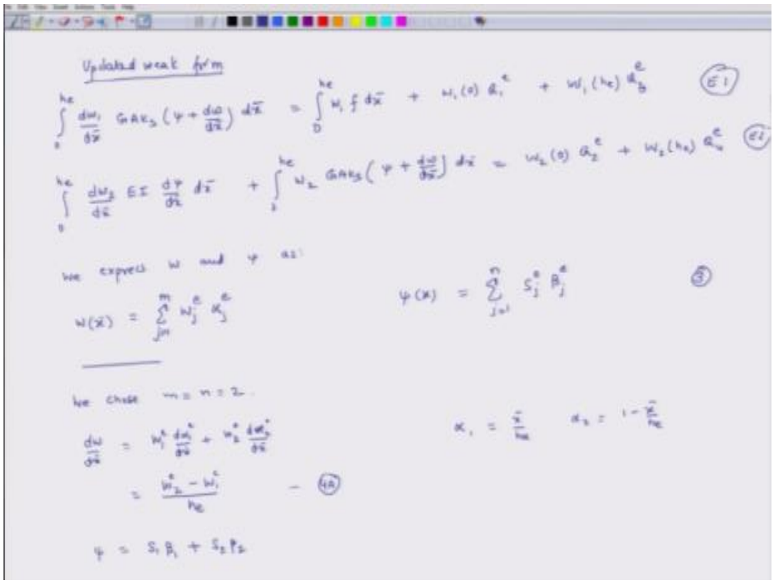

Then dw over dx is w_1 the $d\alpha_1$ over $d\bar{x}$ + w_2 $d\alpha2$ over $d\bar{x}$ okay and earlier, in our classes we had seen that a linear function over an element can be expressed as α_1 is equal to $\bar{x}$ over h_e and α_2 is equal to - $\bar{x}$ over $h_{e.}$ We had, we have developed these functions so we will just, we do not have to reinvent the wheel so if that is the case then dw over dx equals if I differentiate I get $w_2 - w_1$ over h_e okayso dw over dx is difference in w's $w_2 - w_1$ over h_e so we call it 4A. Also let us look at rotation so rotation equals s_1 and I have called them as β and okay actually I will fine

So this is equals β_1 + s_2 β_2 okay and β_1 and they β_2 depend on x they change with x so they depend on x, $s_,$ s_1 and s_2 are constant β_1 β_2 are functions of x okay. Now we know that for thin beams we had seen that in the case of thin beam there is a shear deformation but in thin beams the shear deformation is not there so the planes which are normal initially they still remain normal and if that is the case then what is our shear strain, we had said the shear strain is ψ + dw over dx and if the planes normal planes still remain normal which means shear strains are zero which means ψ + dw over dx

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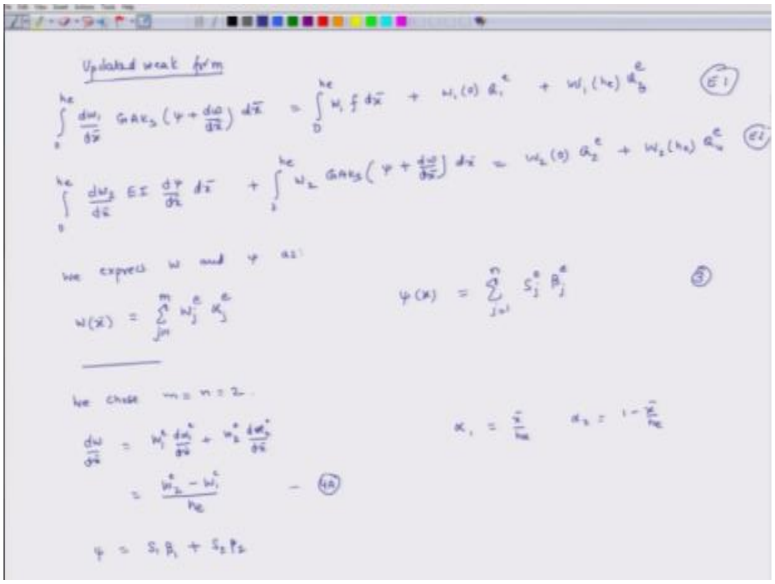

Is equal to 0 and this is for thin beams which means ψ equals- dw over dx okay.

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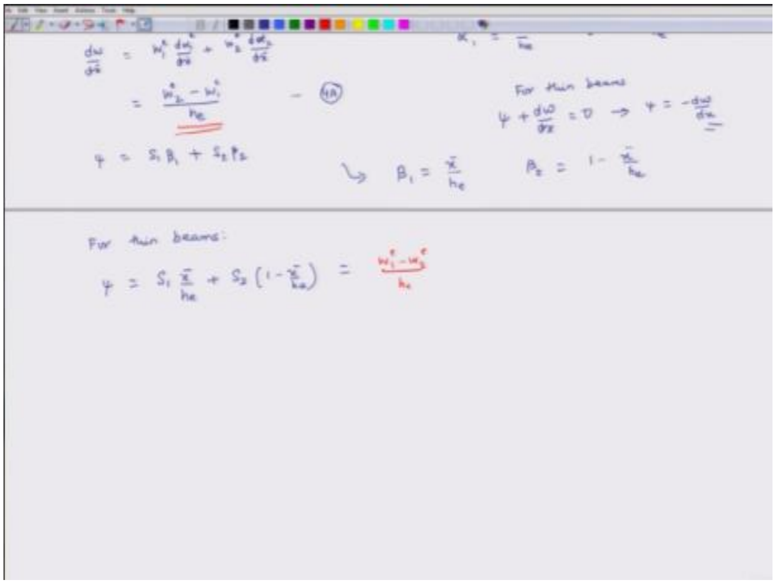

So if that is the case then for thin beams, for thin beams and again once again β_1 is equal to $\bar{x}$ over h_e and β_2 is equal to 1- $\bar{x}$ over h_e right, so for thin beams ψ equals s_1 $\bar{x}$ over h_e + s_2 1 - $\bar{x}$ over h_e + s_2 1- $\bar{x}$ over h_e and this equals – (dw over dx and dw over dx) is the thing which is underlined in red okay, so this one is equal to w_1 - w_2 over h_e. Now these two sides so what you see is that the right side is a constant and the left side depends on x, the left side depends on x and the right side is independent of x.

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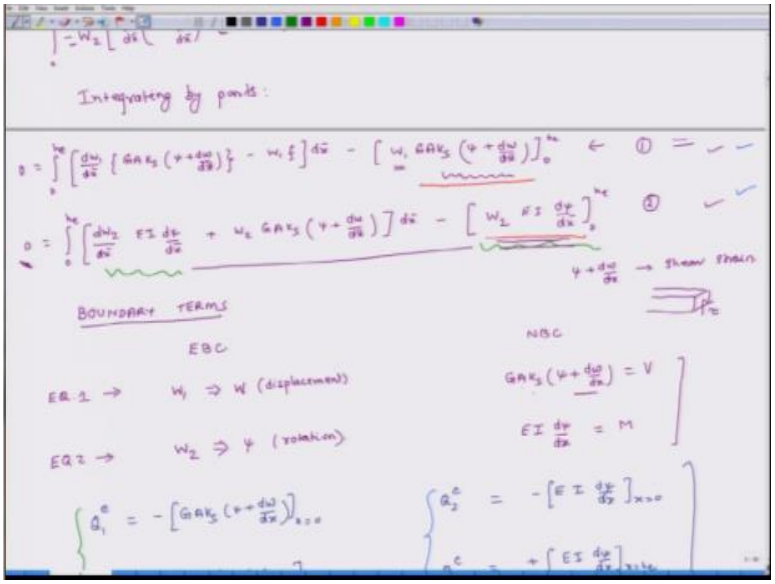

This can be true only if the coefficient of x on the left side is 0. So what is the coefficient of x, so this can be true only if coefficient of x on LHS is 0 only then it will be independent of x so that will be there when s_1- s_2 is equal to 0, this is the coefficient of x okay which means s_1 equals s_2, so what this means is that for thin beams s_1 which is a constant should be same as s_2, s_1 is the constant and it is multiplied by ψ this β_1 s_2 is a constant which is multiplied by β_2 and they have to be same okay.

Now if s_1 is equal to s_1 is equal to s_1 is same as s_2 what does that mean, suppose you have an element and if I am going to plot in the y-direction the rotation ψ then ψ will be constant over the element right because s_1 is the coefficient that first node s_2 is the coefficient at second node the values at both these nodes are same and it is a linear element this means that ψ is equal to constant, and this is s_1 this is s_2, this is node one this is node two understood, if ψ is constant then d ψ over dx will be 0.

So for thin beams if we are doing all this process and we are choosing that m is equal n is equal two and if we you all this process ultimately what all that means is that on in a thin beam d ψ over dx will be using this formulation it will come out as 0. If it comes out as 0 then we go back and look at our equation we look at this term this is the equation in we look at this equation second equation, and we see that here this term E di over dx e times dψ over dx this will be 0 and this term is essentially related to the bending energy of the beam. It is related to the bending energy of the beam so which means that the beam has very little bending energy which is not the reality because when you are bending being a lot of energy is spent right.

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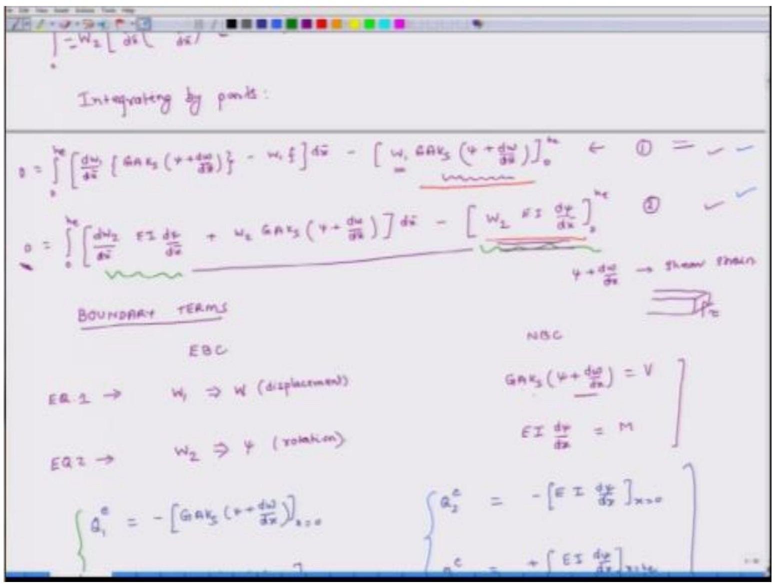

So this formulation creates a problem because of this numerical problem.

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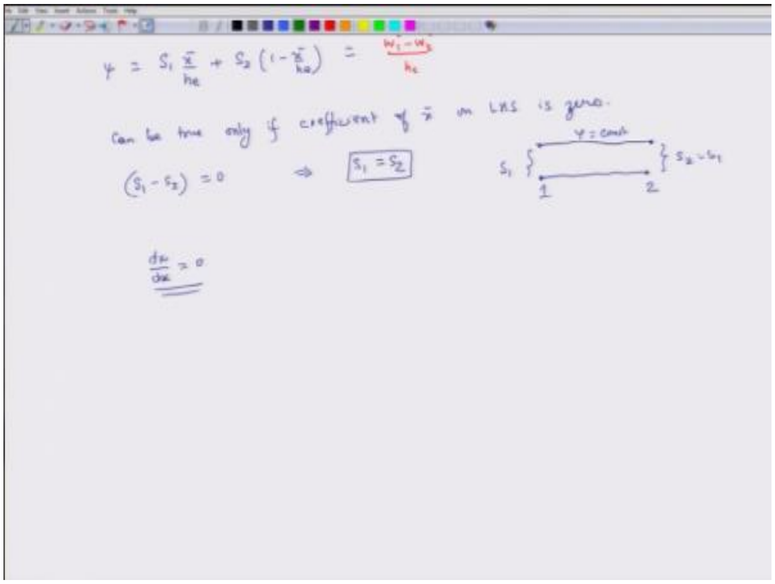

It is not a real problem but when we bend the beam it is bending and takes some energy.

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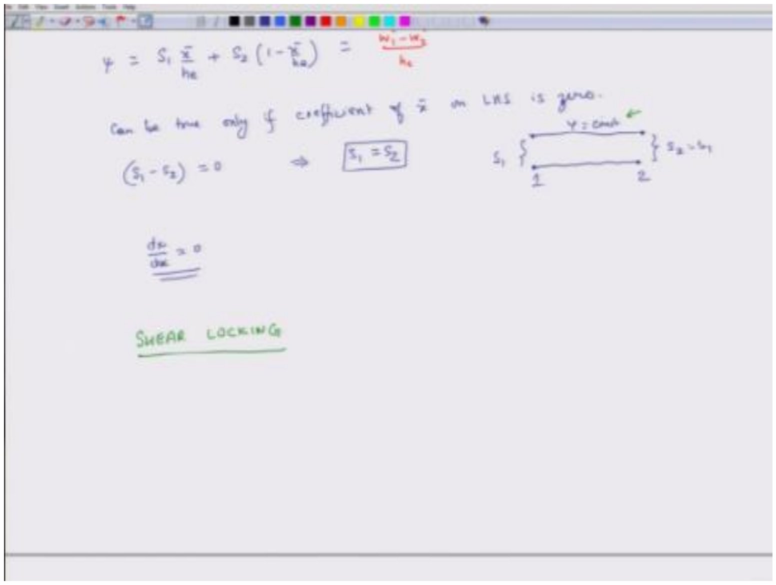

But this problem the way we have formulated it creates this problem because it forces ψ to become constant over the element which causes the energy related to bending vanish in the equilibrium formula in the weak formulation and this problem is known as shear locking, it is known as shear locking okay, and it is known as shear locking and the reason for this is that ψ because of all our assumptions it comes as constant at the end of the day over the domain of the element so to solve this to avoid this numerical problem, so this is a numerical problem.

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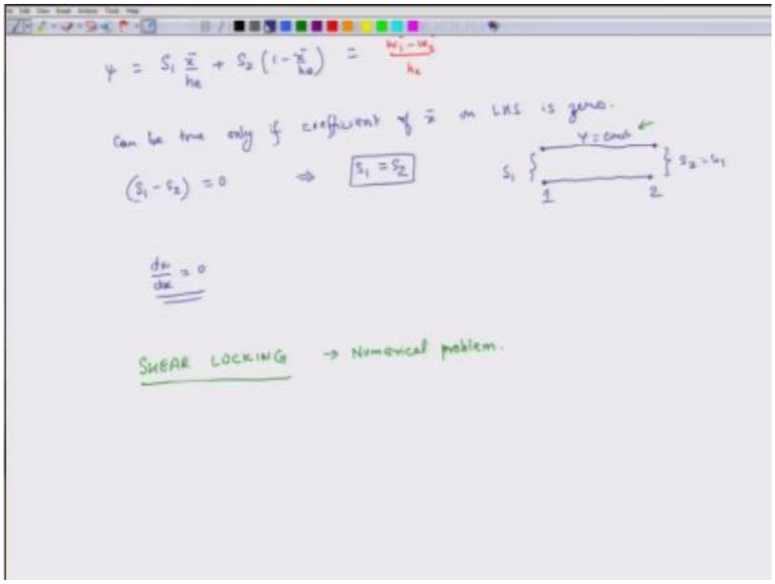

To avoid this numerical problem we take we take two methods and we will discuss about these two approaches in our next lecture so all what we have discussed in today's lecture is the cause of shear locking which causes this bending energy term to vanish because the way we have formulated it forces.

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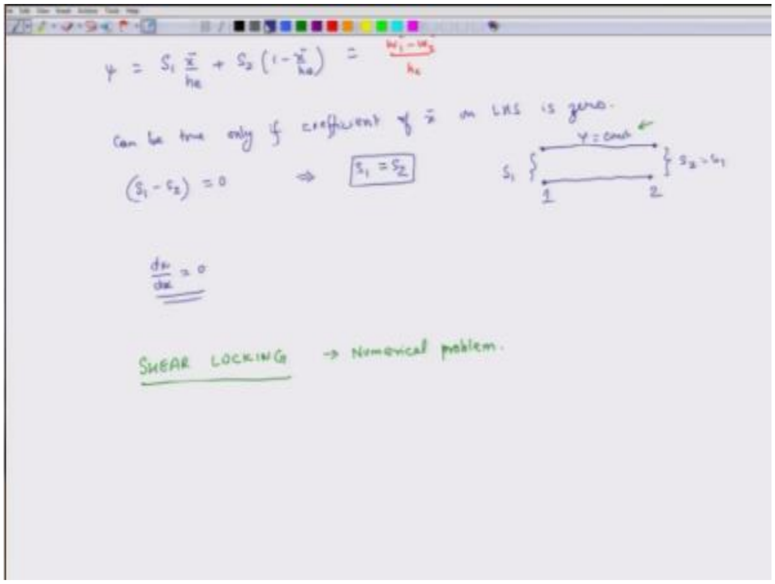

ψ to become constant over the element and to avoid this we have to adopt some methods and what we do about those methods is something we will discuss in our next class, thank you.

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