Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 33
Interpolation functions for Euler-Bernoulli beam

by
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Hello, welcome to basics of finite element analysis, in the last lecture we had started discussing the Euler-Bernoulli beam and we had covered basically its theory and the differential equation, and we had also discussed in somewhat detail about our conventions related to what is positive in the sense of the primary variable, and we also made a point to state that in case of Euler-Bernoulli beam there are at each point there are 2 degrees of freedom, the first degree of freedom is the deflection of the beam which is W, and the second degree of the freedom of degree of freedom is the slope of the beam which is negative of a partial derivative of our derivative of W with respect to X.

So today we will go one step further and our aim is that we want to develop ultimately element level and then assembly level equations for this beam but before we do that we have to develop interpolation functions for the beam. Now we cannot use the standard Lagrangian functions for beams because at each node there are 2 degrees of freedom and they are related to the same physical thing which is the displacement W, so if I use the Lagrangian formulation then it will not work out so that is why we have to develop our new set of interpolation functions and we will develop in context of a 2 noded linear element, 2 noded linear element.

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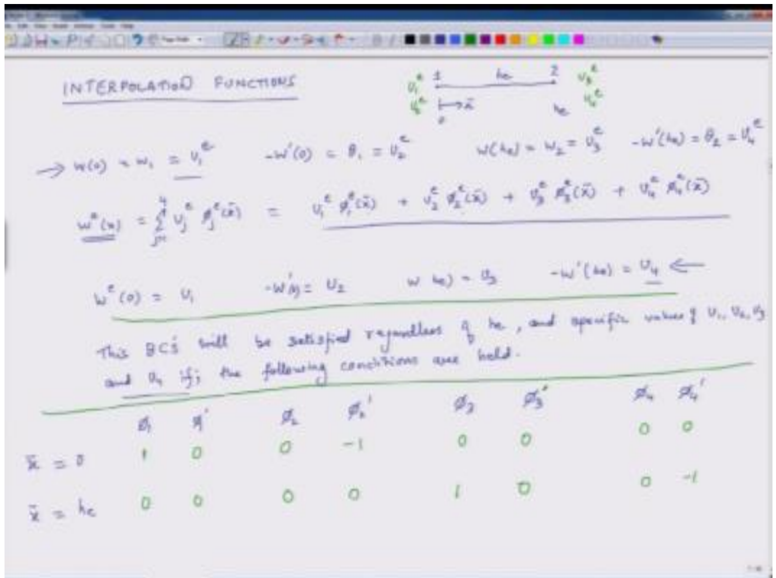

And whatever we develop in today's lecture we can extend it to a quadratic element or a cubic element and so on and so forth. So our aim is to develop interpolation functions okay, so before we start doing that we will again review that our degrees of freedom, so suppose I have an element this is node1, this is node2, let us say the length of the element is he the local coordinate system is such that this is X bar and the value of X bar is 0 here and h_e here, then my 4 degrees of freedom are W at 0 and this I call it as W_1 and I will use a generalized term for this and I will call it as U_1^e.

So u is a general notation for a degree of freedom, the second degree of freedom is slope at 0 and it is not just a slope it is negative of slope right and this is equal to some angle θ at location 1 and this I will call it as $U_2^{e,}$ so I have U_1 and U_2 located at node1, 2 degrees of freedom at node1 U_1 and $U_{2,}$ then my 3rd degree of freedom is w @ h_e which is the node2 and this is equal to physically it is deflection at point 2 and this I call as $U_{3,}$ and then the last one is slope at node2 of the element.

And this is physically some angle $\theta 2$ and this I call it is U_4^e. So at node1 I have $U_1^{\ e}$ and U_2^e and node2 I have U_3^e and U_4^e, U_1 and U_3 are translational degrees of freedom, U_2 and U_4 are rotational degrees of freedom, so now we say that W in the displays is the displacement it is a function of X and this equals U_J^e times some approximation functions j equals 12 and in this case there are total 4 degrees of freedom so we will say 4. So this is U_1^e times Ø1 of X+ U_2 Ø2 of X $+U_3$ Ø3 of X $+U_4$ Ø4 of X and because I am having a local coordinate system.

I will actually make them x bar and also because this is all related to et^h element I will have a superscript e for all the terms okay. So here Øs, Ø1, Ø2, Ø3 and Ø4 are the approximation functions and our aim is to develop these approximation or interpolation functions which we right now do not know. Now if you look at these approximation functions you will see that U_1 explicitly comes in Ø1, U_2 explicitly comes with Ø2 and so on and so forth right, and this w has to be the, this W represents the deflection over the length of the element okay, so the nature of W is such that W at 0 should be equal to U_1 right, so these are the conditions we have already written them w prime should be equal toU_2 so I am just rewriting those, W at he should be = U_3 and W′ at h_e should be=U_4 so we have just rewrite, written these conditions which we stated earlier. Now please note that these conditions these are the boundary conditions for the specific element right these are the boundary conditions, they have to be always true because U_1,U_2,U_3 represents their degrees of freedom they have to be and these excess the length of the element could be anything.

So for all values of X these conditions have to remain true and this will be true only if we have these following conditions satisfied, so this is these BCs and that this is U_4 so this - should be there so these BC's will be satisfied regardless of h_e and the specific values of U_1, U_2, U_3 and U_4 if the following conditions are held. What are those following conditions so there are 16 conditions and once I write them down you will, it will make sense to you, so this should be X bar is = 0 and X bar = h^e and ψ1 ψ1prime, ψ2 ψ2prime ψ3 ψ3prime, ψ4 and ψ4 prime so I am actually making a table okay.

So at X = 0 ψ1should be 1 and ψ1prime should be 0 everything else should be 0, then at X =0 ψ2 should be 0 its slope should be - 1 and ψ 2 and its slope should be 0 at $h^{e,}$ then at X =0 ψ3 should be 0 slope should be 0 but ψ should be 1 at X =h^e but its slope should be 0, and finally $\psi 4$ it should be 0 and it is slope should be 0 at X equal to 0 and also it should be 0 at h^e but its slope should be – 1, so if we had these conditions if we can somehow enforce these conditions while choosing $\psi 1$, $\psi 2$, $\psi 3$ then all these conditions will always be satisfied okay.

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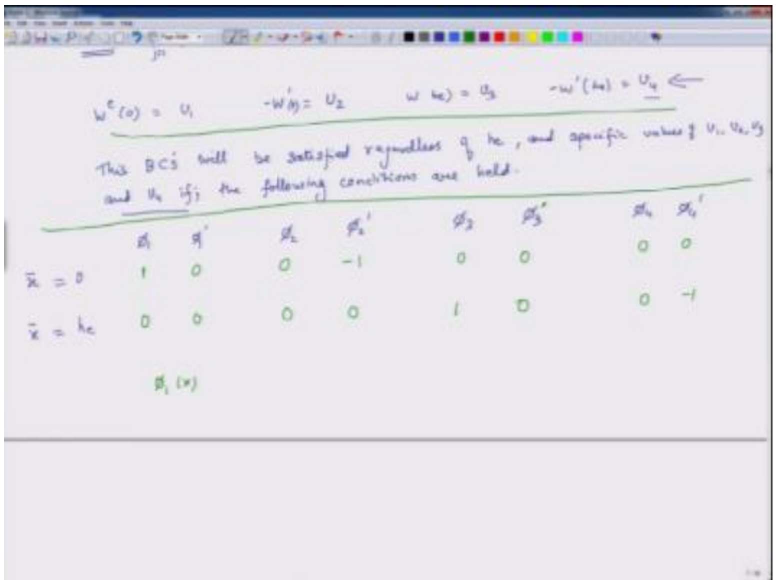

So now we will develop 4 different functions $\psi 1$, $\psi 2$, $\psi 3$ and $\psi 4$ so us let say $\psi 1$.

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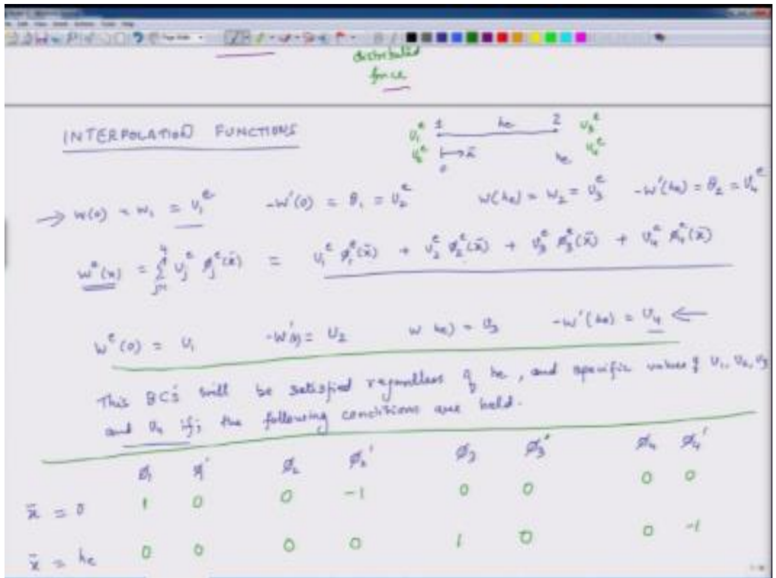

And because it involves 4 degrees of freedom, we have 4 degrees of freedom per element.

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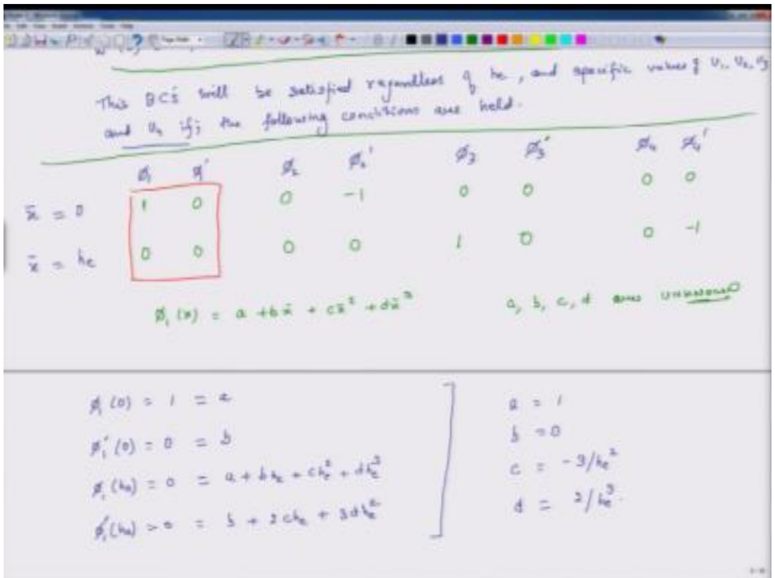

So it will have 4 terms and let us say this is = a+ b X bar + c x bar^2 + dx bar^3 and we do not know a, b, c, d are unknown, ABCD are unknown so to find ABCD we use these boundary these conditions related to ψ1, so we get 4 equations related to ABCD and we get 4 unknowns so we will write down those equations quickly, so ψ1 at 0 = 1 and this when I put X is = 0 I get a okay, the second condition is ψ1 its derivative at 0 = 0 and this is = b. If I differentiate ψ1 and put X = 0 this is what I get. Third equation isψ1 at h^e= 0 and this equals a + b times h^e + c he^2 +d $he^{3,}$ this is the third equation.

And the 4th equation is ψ1 h_e its slope = 0 and this is = b + 2 c h^e + 3d $h_e^{2,}$ so from these 4 equations these are all linear equations in ABCD I can very easily solve these for finding out values of a, b, c, d so what I get is a=1, b=0 is c = - 3 divided by he^2 and d =2/he^3.

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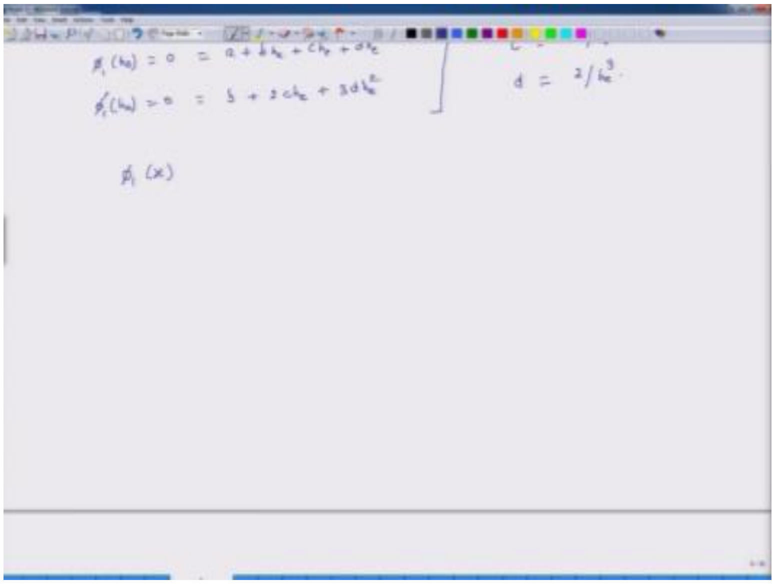

So ultimately what I get is that ψ1 of X so I put all these values of ABCD.

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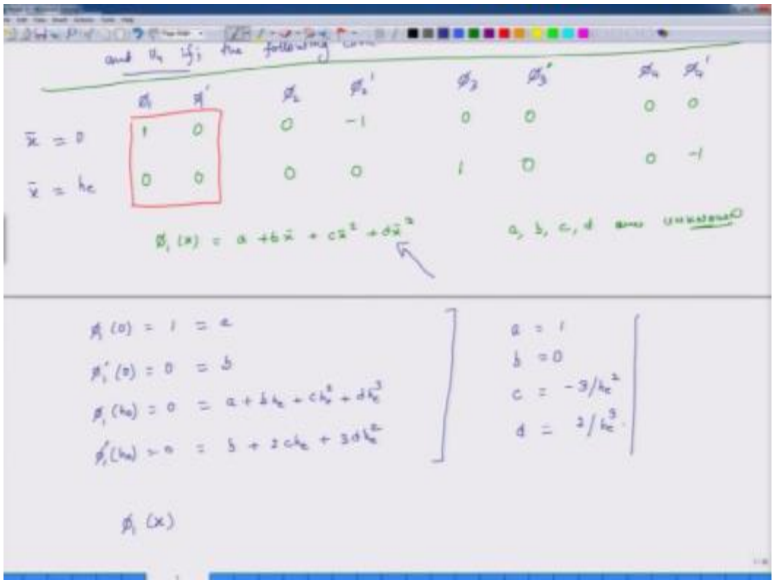

These values of ABCD I put it in this equation.

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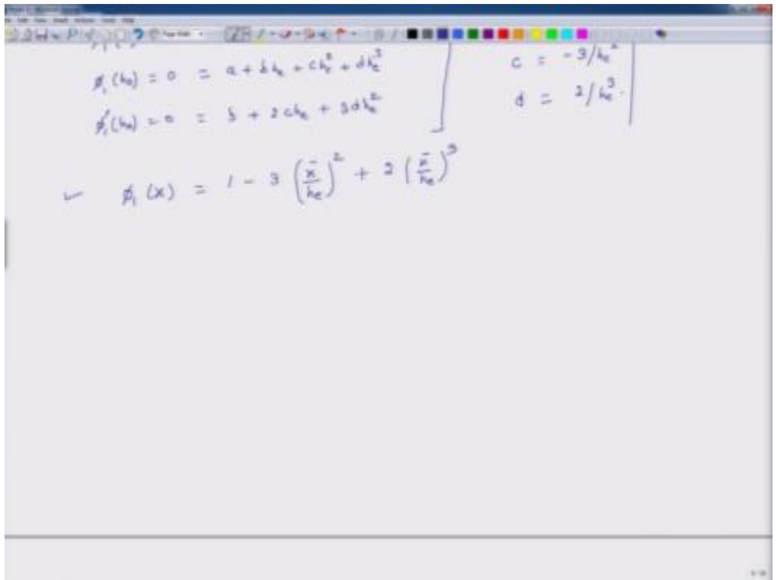

And I get my expression for $\psi1$ x so my $\psi1$ x is = 1 -3(X bar/ $h^{e)2}$ + 2 times X bar /h_e^3 yeah this is how I get my $\psi1$.

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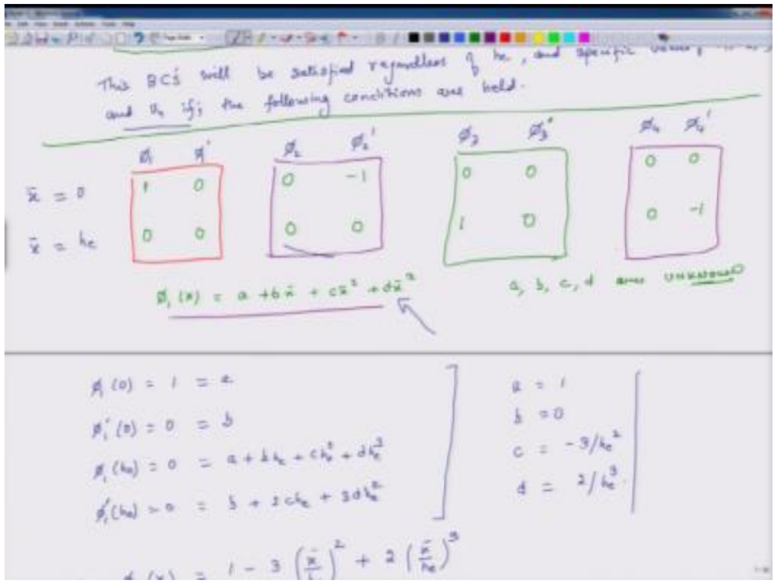

Similarly I can calculate the nature of ψ2 from these 4 conditions, I just write the same expression for ψ2 use these 4 conditions I can get new values of a, b, c, d and I will get ψ, then I can get another set of conditions to get the expression for ψ4 and then a fourth set of conditions to get expression of ψ3 and ψ4 okay.

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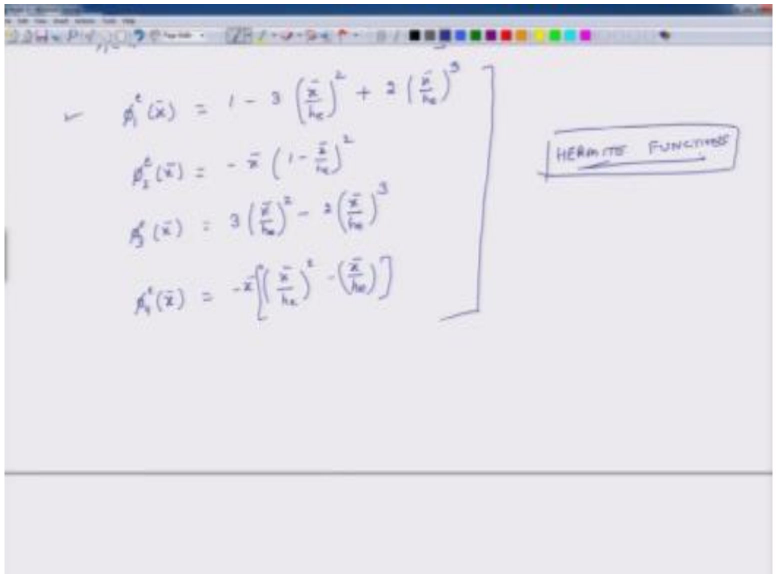

So my 4 approximation functions using this method I will write them directly so this is for e^{th} element is - X bar 1 - the whole thing square ψ3 X bar=3. So these are my 4 approximation functions and I call them these are not Langarian functions, they are significantly different and I call them hermit, HERMITE functions, so I call them HERMITE functions okay, and if we verify we will see that ψ1.

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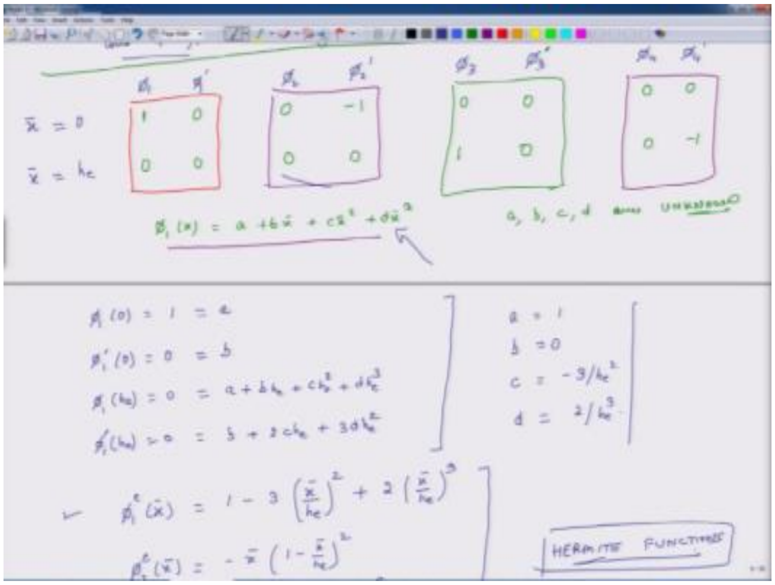

Is such that its values are like this and same thing for ψ2, ψ3, ψ4 okay.

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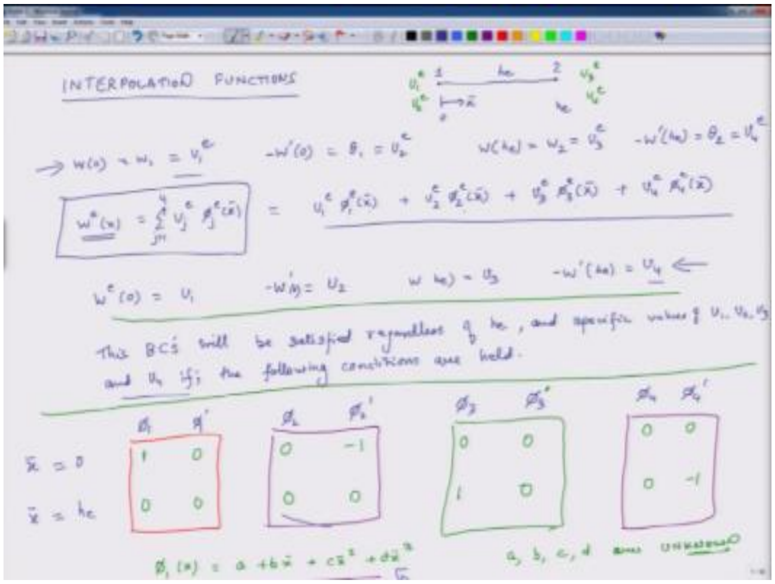

So if I use these functions 4 functions in my expression for W then I get what I wanted to have okay.

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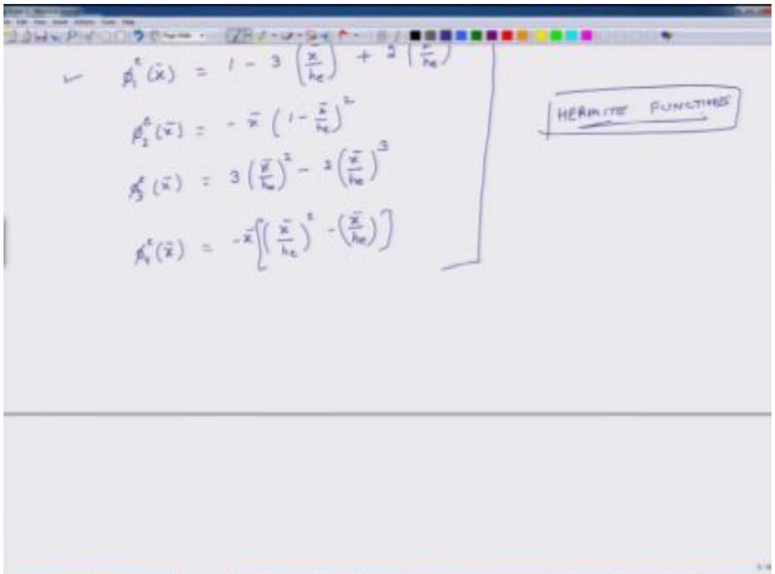

So here instead of using Lagrangian functions interpolation functions which we had used in context of a second order ordinary differential equation for 4th order because at each location we have w and w prime as primary variables I have to use a different set of functions known as HERMITE functions.

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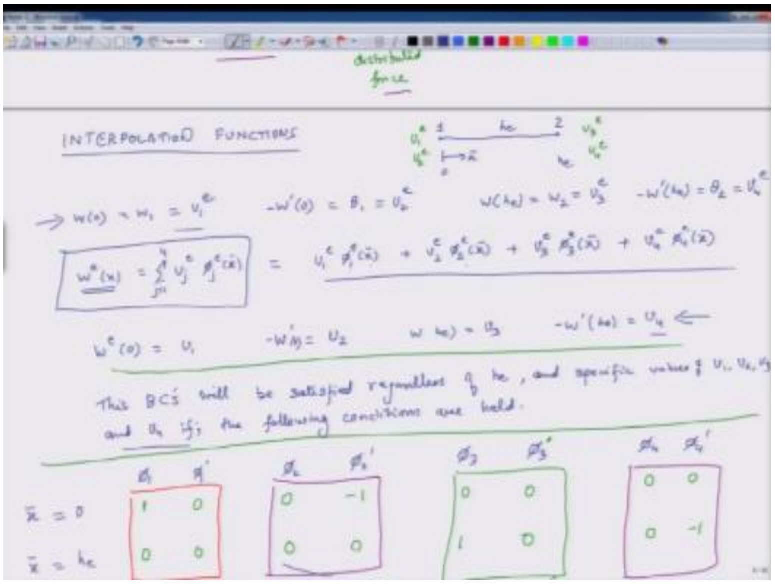

Now this is these functions are valid for a 2 noded element but I can construct a 3 noded element using same theory and I can have a new set of HERMITE functions in a 3 noded element I will have because I have 3 nodes at each node there at 2 degrees of freedom so total degrees of freedom is 6 so because of that.

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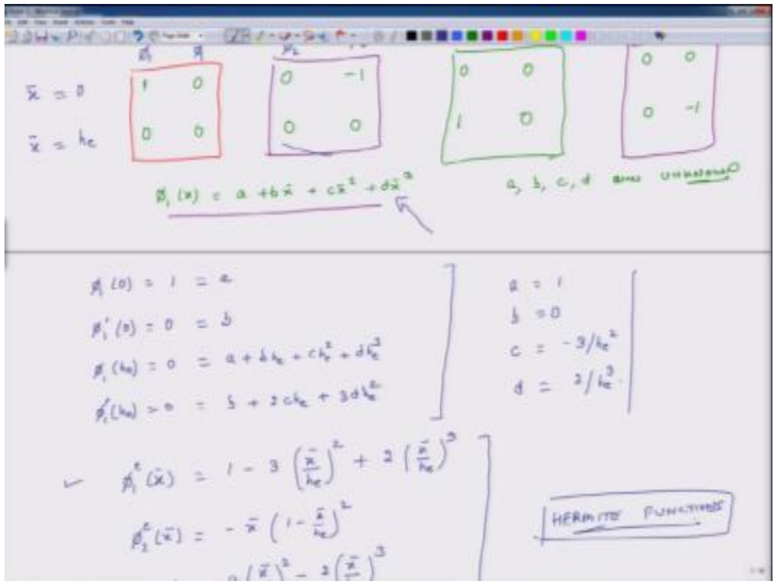

I will have 6 her HERMITE functions not 4 and so on and so forth.

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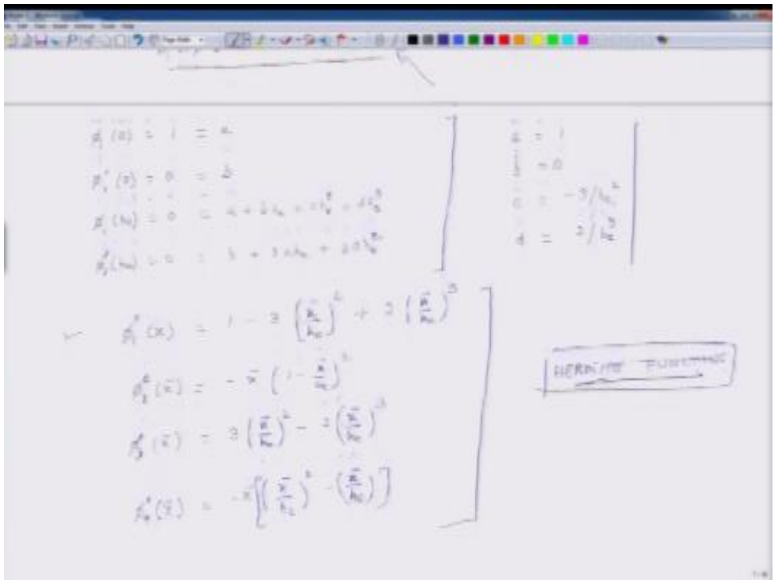

So in that case it will be I will have two more terms a quadratic a quatic term that is X^4 and a pentic term X^5 also.

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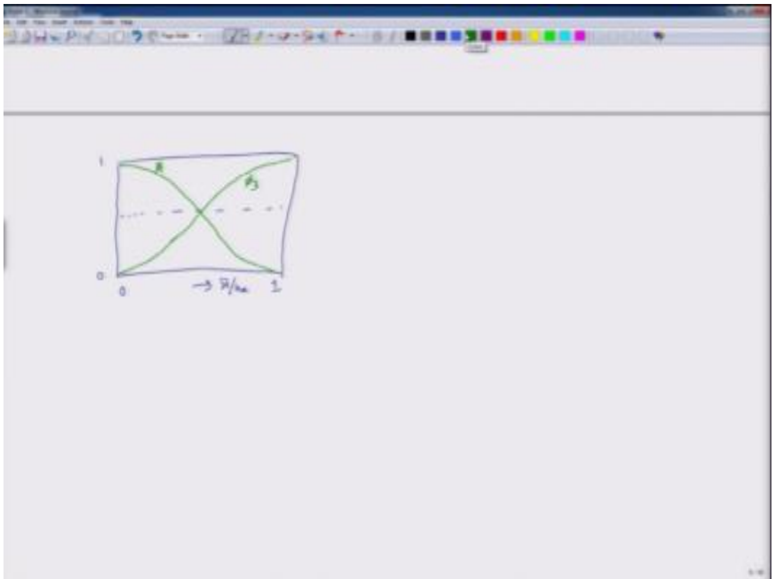

If I plot these functions then these plots look something like this, so let us say this is my normalized X so minimum value is 0 and maximum value is 1and on the vertical axis I am plotting the value so this is 0 and 1 so my ψ1 is 1 at X = 0 and it becomes 0 at X=1 so that is my ψ1, and ψ3 which is the other which is the deflection at the second node its value is one at here and 0 here and this function looks something like this so this is ψ3. If I plot the slopes of ψ1 and ψ3 they will be 0 at both the ends because that is what our.

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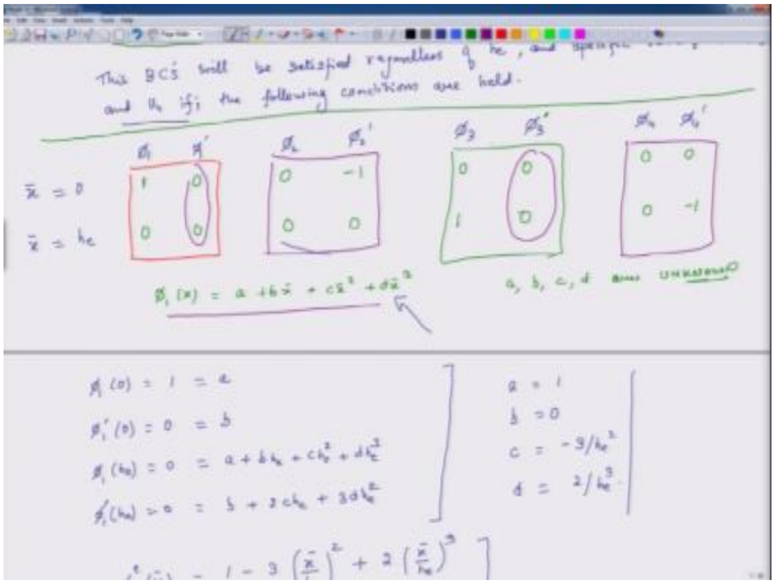

Condition was, the slope of ψ1at both ends is 0 and slope of ψ3 is also at both ends it is 0.

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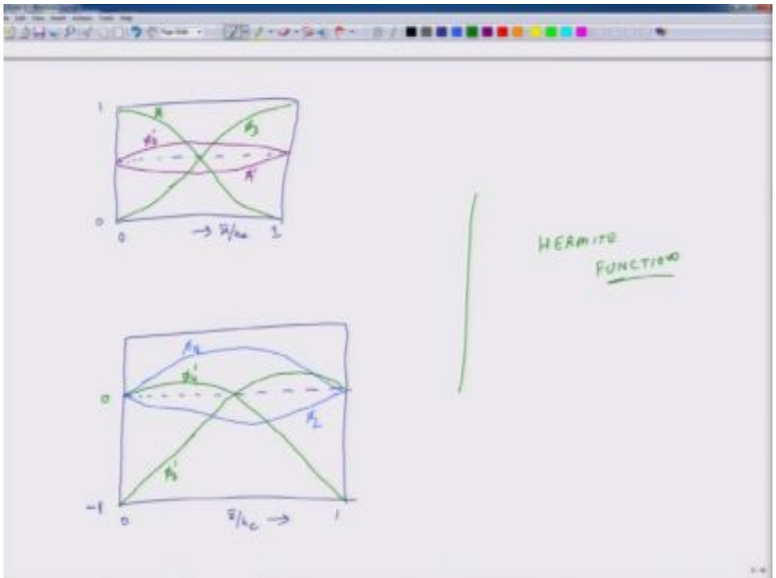

So if I differentiate these functions and I plot the slopes the slopes look something like this and that is ψ3 prime and this is ψ1 prime, so the contribution of the slopes at the nodes is again 0 and then we will draw another set of course for ψ2 and ψ4 so that is my normalized x-axis maximum value is one minimum is 0 and if I so this is - 1 and the slope of ψ 2 is - 1

So this is 0 and so this is ψ2 and this is ψ4 prime and the actual value of these functions ψ2 not their slopes is something like this so this is ψ_4 and this is ψ2 okay. so this is the nature of HERMITE functions and so this is ½ 4 prime. So these are again these are known as HERMITE functions and these are the functions which we will use for developing finite element equations for an Euler Bernoulli beam, so that is the exercise which we will do it in our next lecture and this concludes our discussion for today, thank you very much.

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