Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 32
Euler-Bernoulli beam

by
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Hello, welcome to basics of finite element analysis, in the remaining part of this week starting today we will be discussing the beam theory and how it is analyzed using the finite element method, so what we are going to discuss in detail is the Euler-Bernoulli beam.

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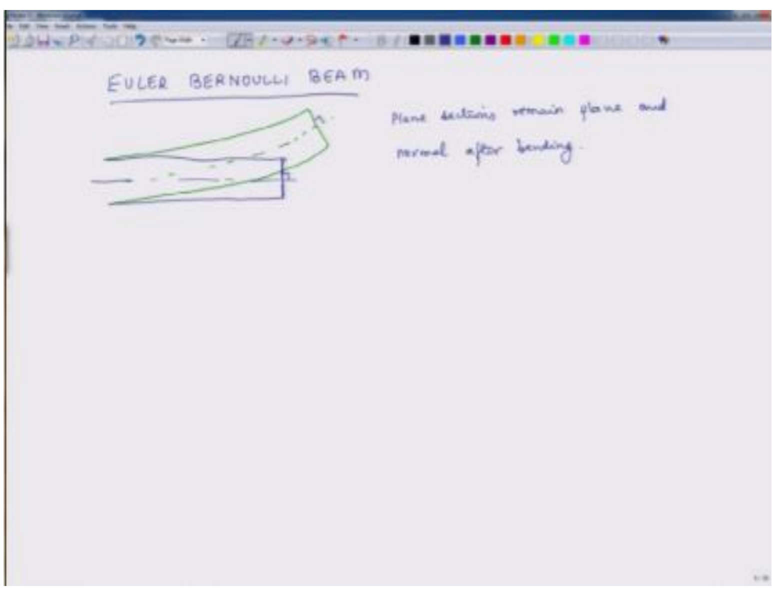

And in this particular beam theory we says that suppose there is a beam and when it deflects so it has a neutral axis and when it bends it bends in such a way, so that is my this is my un deformed thing and this is with the green is my deformed and it bends in such a way that the deformed, so in the original un deformed beam the neutral axis was perpendicular to the edge of the beam okay, and in the deformed beam if this, this is a vertical plane in the deformed beam also the deformed neutral axis is perpendicular it remains perpendicular to the edge of the beam.

If edge of the beam was perpendicular earlier it will continue to remain purple, perpendicular in the deformed state also, so a lot of times in literature you will see that they say that plane sections remain plane and normal after bending okay, so this is a plane section and originally it was at 90 degrees to the neutral axis. After bending

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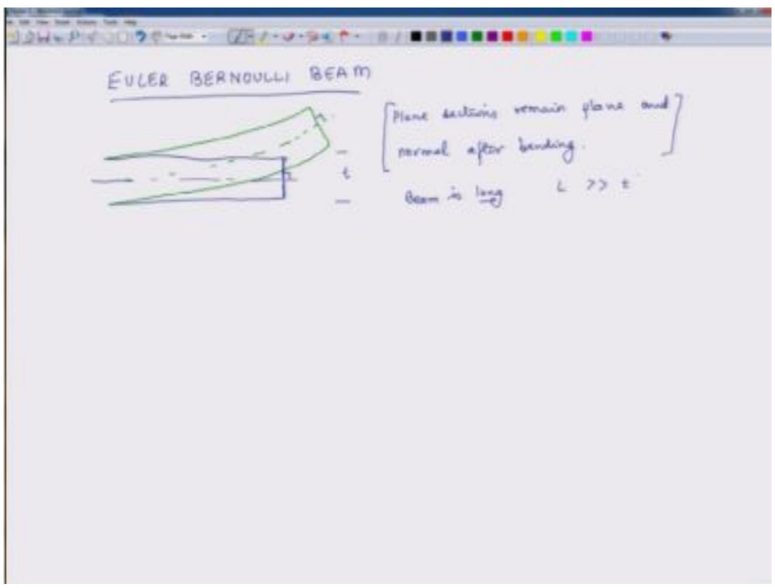

It continues to remain a straight plane section okay and it also continues to remain at 90 degrees of perpendicular to the neutral axis, so this type of a beam is known as a Euler Bernoulli beam and these, this type of deformation is there when the beam is long and what does long mean, that the length of the beam is very large compared to its thickness, so this is let us say over all thicknesses is b, I will call it some thickness T then this is very large compared to t for short beams or thicker beams this may not be true, so then Euler Bernoulli assumption that plane sections remain plane and also normal it may not necessarily be true. So in the last lecture we will discuss a little bit about those beams also.

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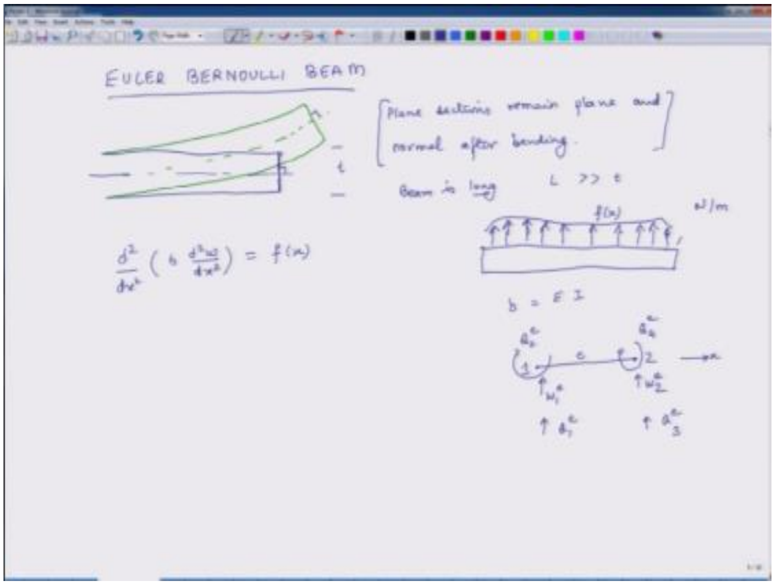

So the governing equation for this beam is d^2 dx^2 b time's d^2 w over dx^2 equals f (x). Let us look at what these parameters mean, so suppose I have a beam and it is having some vertical load which is distributed in nature then this distributed load I call it f(x), so its units are Newton per meter of f and b is a multiple of two things one is Young's modulus and I, I is the moment of inertia of the cross section at point x, so this I can vary with respect to x and so could E, so could E. Couple of other things because this is the convention we will use.

So if I have a small element of the beam which I just represent as a line and let us say its nodes are 1 and 2 this is my x-axis then W is positive is if it is going up, so this is w_1 and this is w_2 and if it is the e^{th} element I put superscript e okay. Similarly the shear forces is this is Q_1^e is the shear force at node 1 and this is at Q_2 this is Q_2^e at node two and the, oh excuse me I will actually call it Q_3 so you will wonder where did Q_2 go so Q_2 is the bending moment and Q_4 is moment at node 2. So Q_1 and Q_3 represent shear forces and Q_2 and Q_4 represent moments and these moments are positive if they are clock wise, they are negative if they are anti clockwise, so w is

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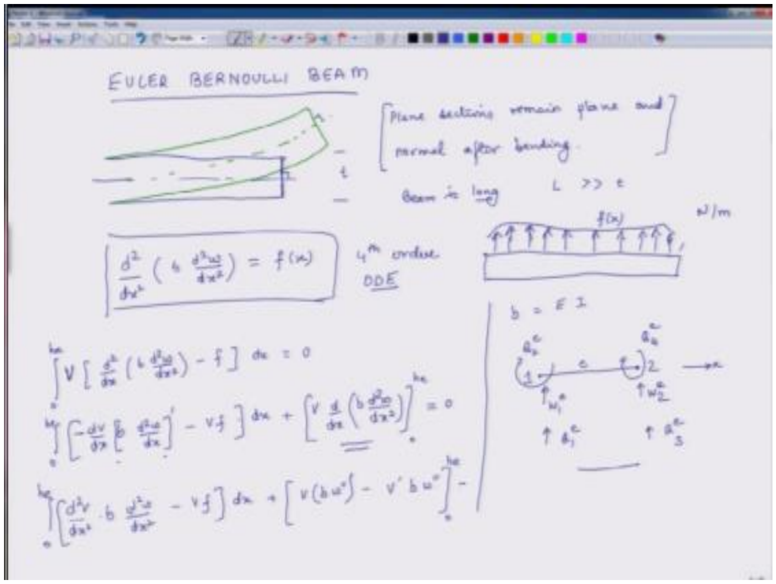

the deflection of the beam it is positive it is going up and if it is going down then it is negative and Q is our shear forces and moments and finally so this is there, I will also like to mention about slopes at both the ends but we will not discuss at this stage. So if this is the convention then if we have to develop a weak formulation using this governing differential equation please note that this is the fourth order, ordinary differential equation.

So if I have to develop a weak formulation then I will have to integrate in parts two times right so I am going to integrate it over the domain of the element h_e and the error, so this is the error if there is if the solution was exact this would be 0 but here we are looking at approximate solution, so this is the error and I multiply by a weight function and here I call my weight function as V.

So earlier we have called it w but in context of beam w makes more sense in terms of deflection so I call it V and then I integrate it over the domain and I equate it to zero. So I do my integration by parts once and I get 0 to h_e - dv over dx times b d_2w over dx and the entire thing is now differentiated only once – vf dx and then I get a boundary term + v times dy/dx b d^2w / $dx^{2,}$ I am evaluating it at two locations o and h_e and this entire thing becomes 0.

So this is integration by parts once and I will do the integration by part one, one more time so now I get d^2v over dx^2 times b times d^2 w over dx^2-vf, I integrate the whole entire thing and then the boundary terms now have two boundary terms one is V b w´´ so this is the same as the earlier term and then the other term is V´ b w´´evaluated at 0 and h_e and this equals 0

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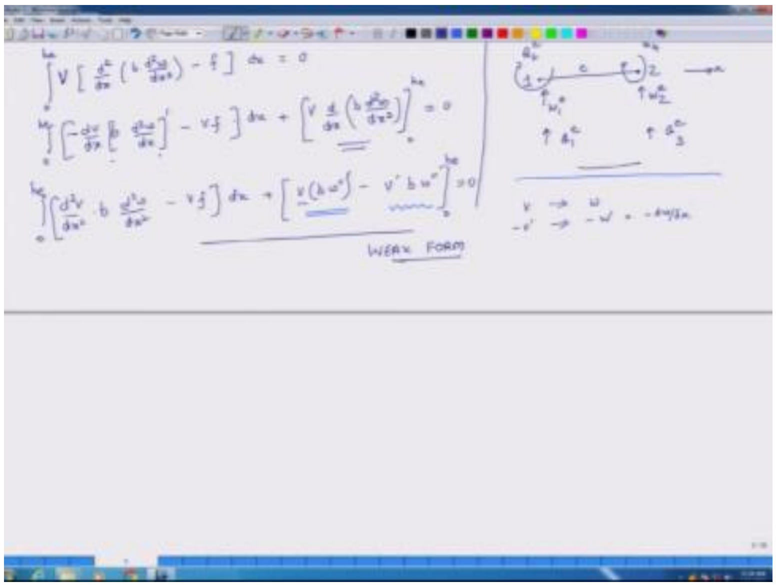

Now let us look at this equation in a little bit more of detail, so this is my weak form okay, if I look at the boundary terms I have now two boundary terms, one is v times the first derivative of b times w ´´and the second boundary term is V´´ prime b w´´ right, in our earlier lectures we had said that these boundary terms help us figure out what are the primary variables and what are the secondary variables.

So when I look at these terms v it helps us determine that one primary variable is w, and - V´ it helps us make a judgment that the other primary variable is – W´ or –dw over dx right, because we had said discussed in the last class and one of the earlier classes that whatever is the nature, nature of the weight function if you replace that weight function by the unknown then that will be a primary variable. So these are my PV’s, primary variables,

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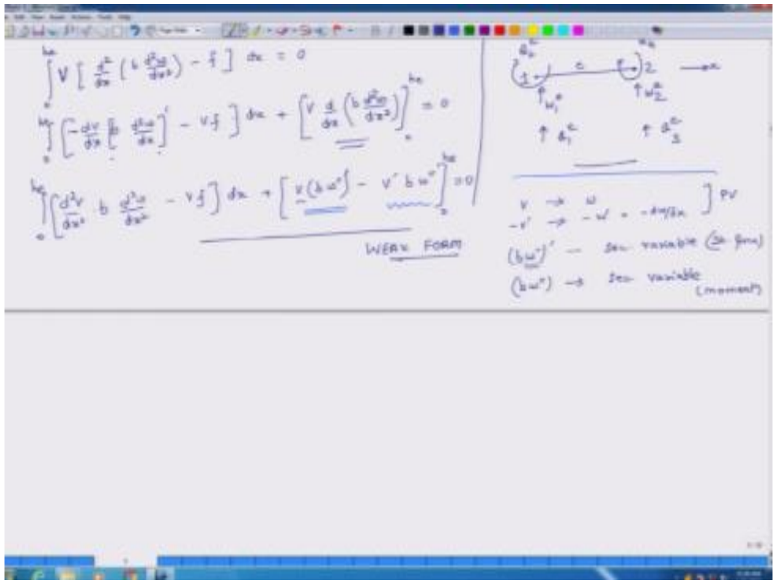

Then the coefficient of primary variable terms help us determine what are the secondary variables, so here I have V is the primary variable associated term, its coefficient is bw'' time, so this is my secondary variable, and physically this represents shear force right, it represents shear force. The second coefficient we are talking about is bw'' so this is another secondary variable and this represents moment in your solid mechanics or in lectures you would have seen that EI b represents EI times secondary weighted of w is moment so I have two in this problem, this is a fourth order differential equation.

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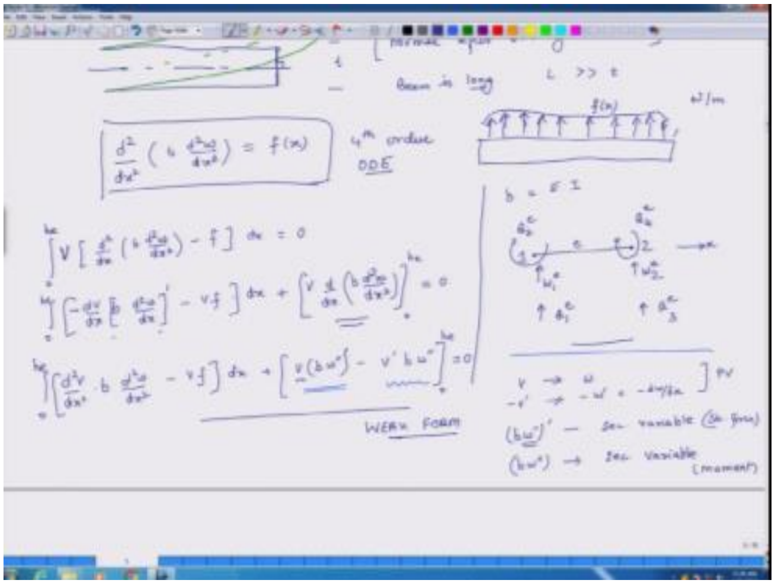

This is the fourth order differential ordinary differential equation.

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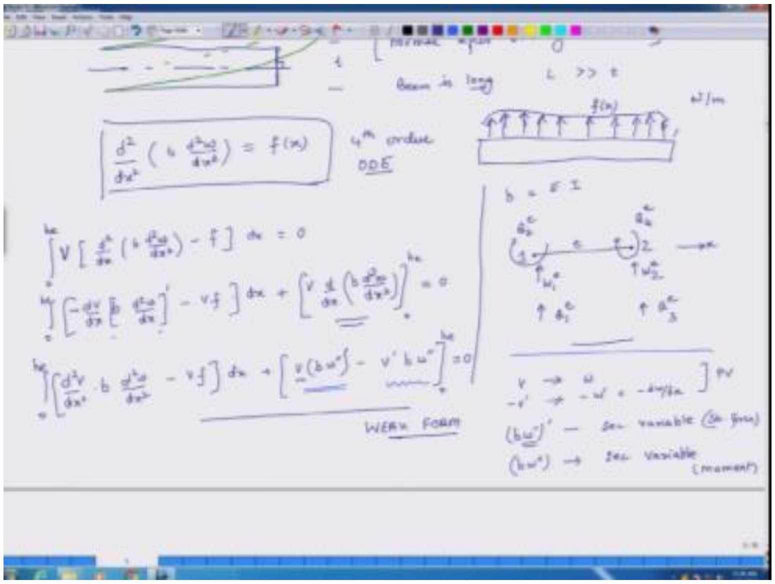

And because of this I have two primary variables w and its slope and I have two associated secondary variables shear force and the moment okay, when I integrate this differential equation because at fourth order I will get four constants, and those four constants can be determined if I know four conditions which can be, either I know v or v′ or bw″ or the shear force at four different you know I have to know four conditions using which I can calculate those constants.

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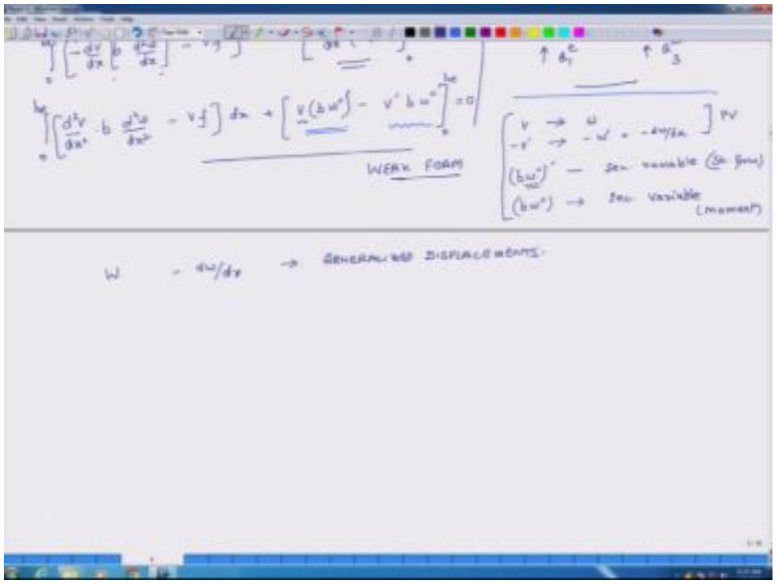

So I will call w and -dw by dx as generalized displacements, as generalized displacements. So it is not strictly a displacement because we think displacement as the movement of a point A to another point B. But this is a generalized displacement because w over dw over dx is also a degree of freedom, it is also a degree of freedom. So that is why and they are mutually to independent okay, so these are generalized displacements.

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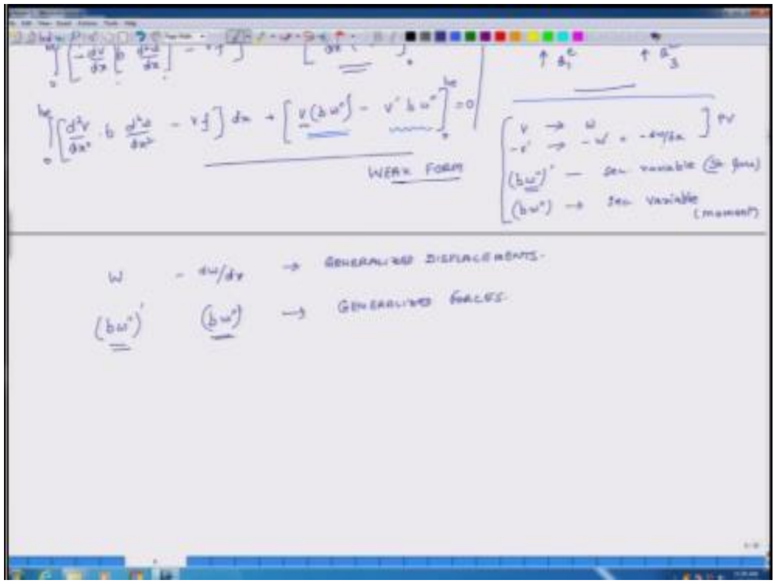

And then, my these secondary variables and bw″ these are called generalized forces, so this is a moment this is shear force but in a general sense we call them as generalized forces, so when I

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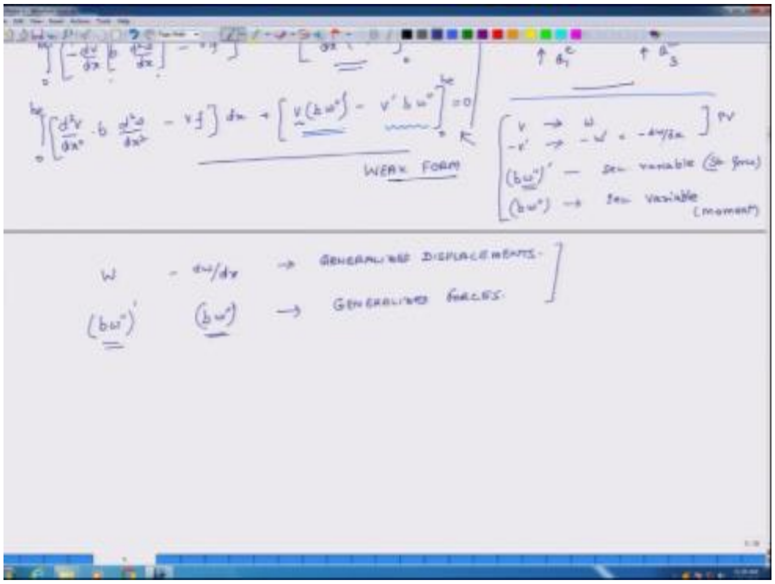

Develop finite element equations using the weak form I will have a vector, I will have a vector and in that vector for generalized displacements and this vector will be on the left side and on the right side I will have a vector for generalized forces okay.

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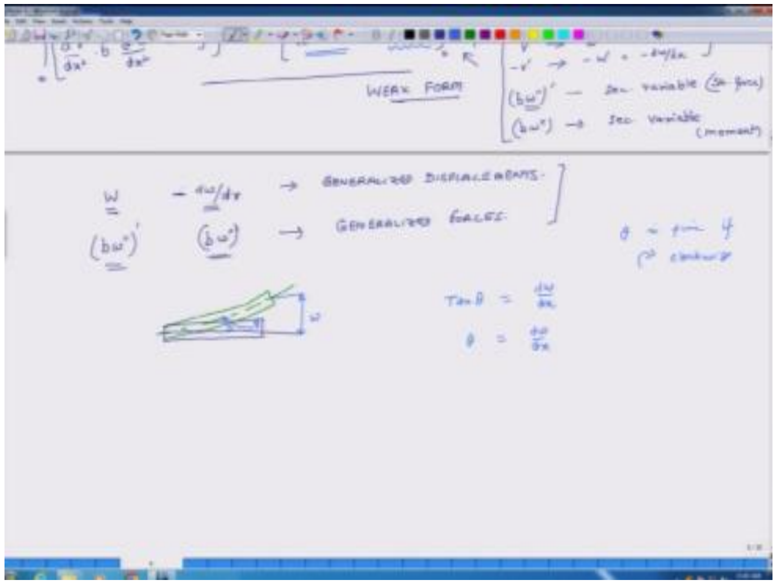

So these are my degrees of freedoms, other thing I wanted you to remember is that this dw dx the negative of dw dx is something I am calling it as a primary variable, I am not calling the negative sign is important okay, because then that makes things more consistent. Why is that important because I have said that my Q_2 which is the moment is positive when it is moving clockwise okay, same thing about Q_4 so I will elaborate on this a little bit further so suppose I have a beam, and then this beam bends okay, then the angle this angle let us say this is Θ, then Tan Θ is what, suppose this is the deflection right, then Tan Θ will be basically dw over dx agreed.

Tan Θ will be dw over dx, but this Θ we are considering that Θ is positive if clockwise right, now because the Θ is very small I am taking a very small element it is a differential element because the Θ is very small I can call also this thing as Θ is equal to dw over dx. But Θ is positive if it is clockwise now in this case it is negative. So that is why I say that my degree of freedom is negative dw over dx. So it is important otherwise.

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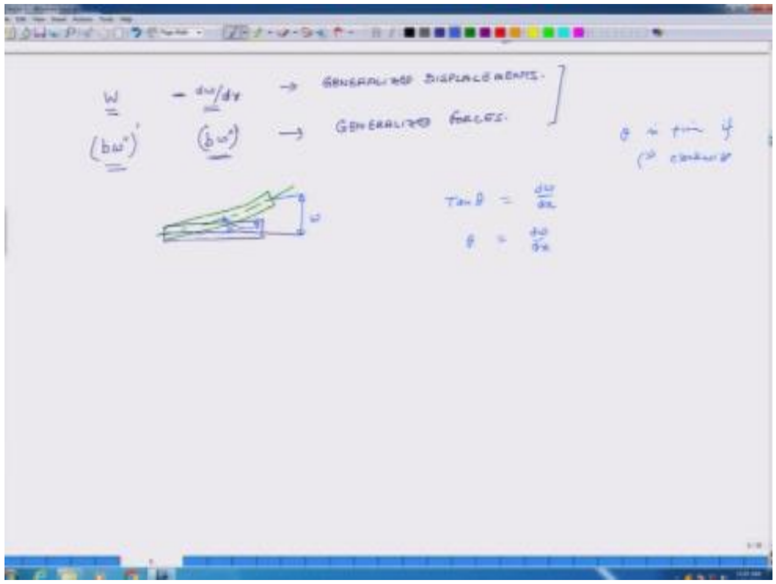

We will have some errors in our equations based on our convention.

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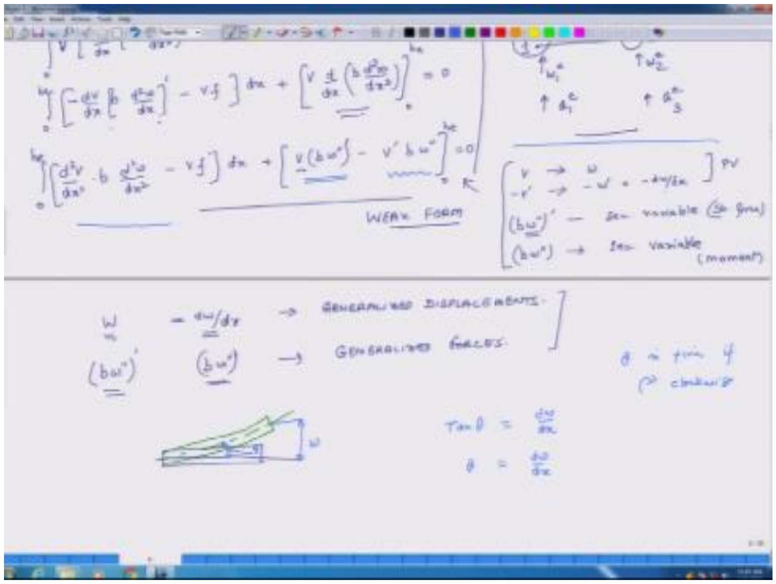

So what we will do is we will rewrite this equation.

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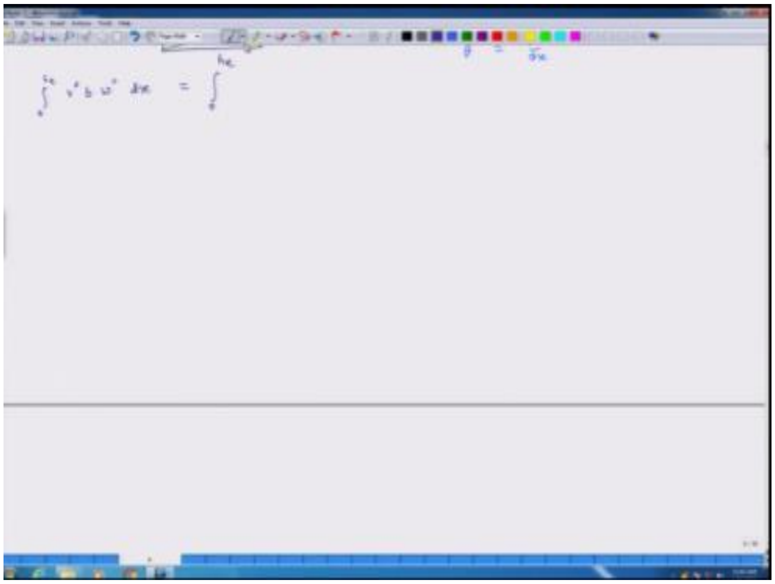

So my weak form equation is v″ bw″ dx and all other terms I am taking to the right side, zero to h_e and one thing I did not do which I should have done because I am integrating it over.

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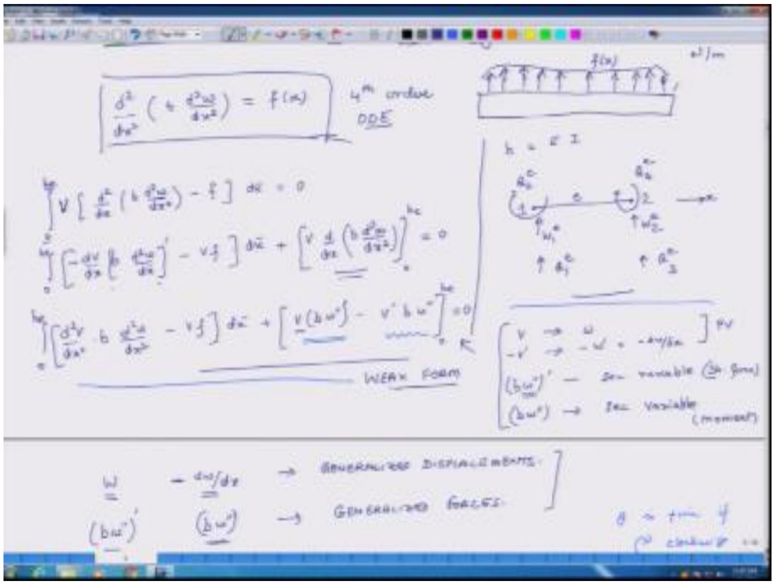

An element and this is the local coordinate system so this x should have been $\bar{x}$, so every where you may want to replace this x by $\bar{x}$ okay, this is important.

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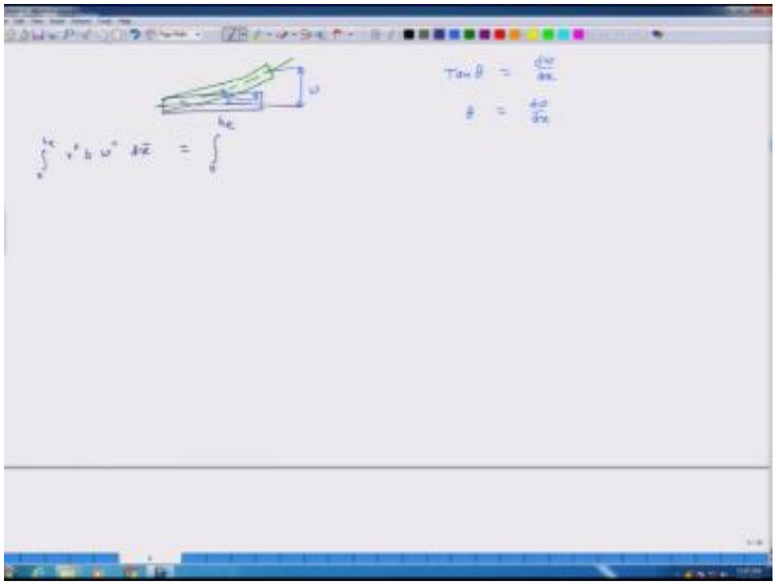

So 0 to h_e, so going back, I am going to rewrite this equation.

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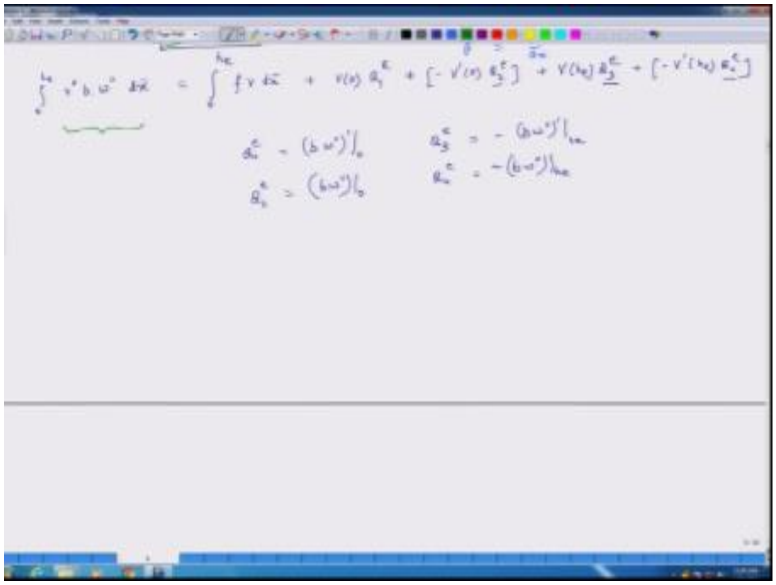

So this equals fv times d $\bar{x}$ plus, and what I am going to do is all these boundary terms I am going to open this bracket and I will explicitly write down all the boundary terms. So this is v evaluated at 0 times Q_1^e plus minus v evaluated at v′ evaluated at 0 times Q_2^e + v evaluated at h_e times Q_3^e plus negative of v′ evaluated at h_e times Q_4^e okay. So these are this is how we are defining Q’s, so Q_1^e equals bw″ its derivative evaluated at 0, Q_3^e is negative of the bw″ the whole thing differentiated at h_e and then Q_2 and Q_4 are shear forces, no these are moments evaluated at 0 and Q_4 is the moment evaluated at like that.

Now we will look at this, on the left side this term is what, it is bilinear right, it is bilinear and also symmetric in u and w, so I will call it as.

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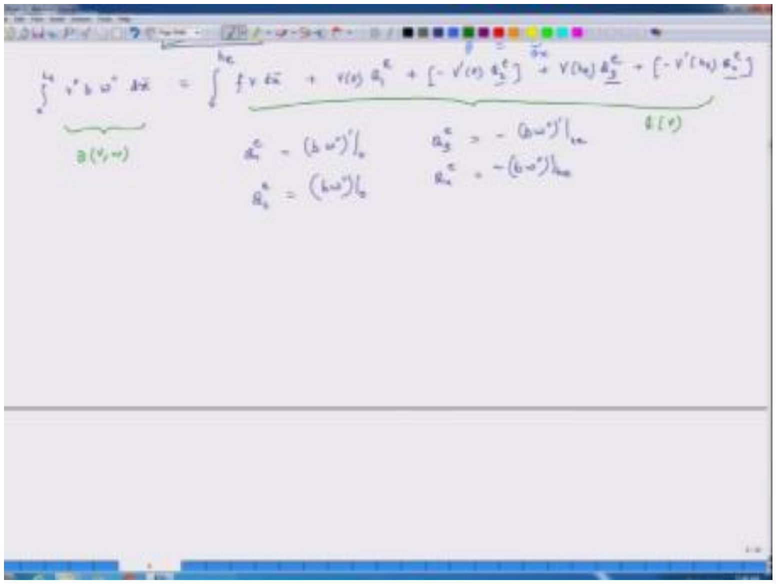

B (L, w) is bilinear and symmetric, B of v and w this term, these all these terms this is a linear functional and it is dependent on v. So this again brings us to our standard form which we had developed earlier where B of a bilinear symmetric functional was equal to a linear functional.

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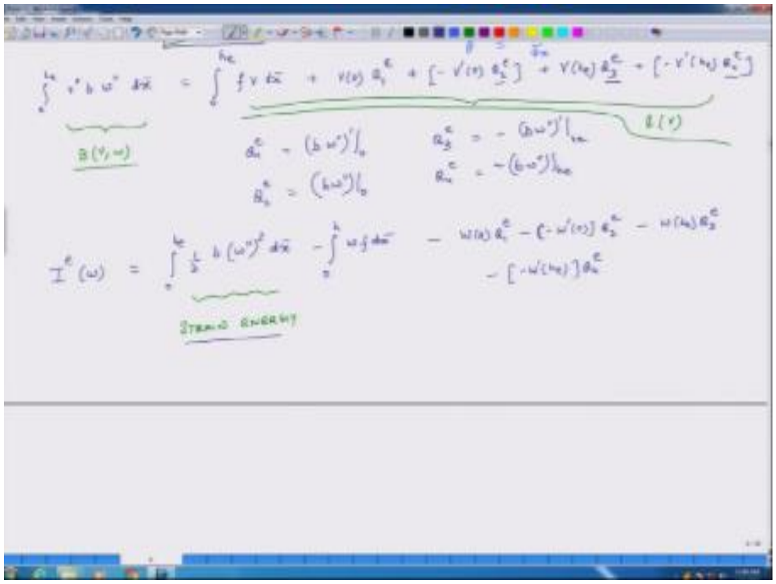

And lastly we will show that this term is nothing, this entire thing is nothing but the minima of a overall term we will call it I, and in this case it represents the overall potential energy of the system. So just consider the term I and this is for the e^{th} element, so I have the superscript e and let us say that this is dependent on w and I will define it as 0 to h_e ½ bw″ whole square dx minus 0 to h_e wf dx minus w evaluated at 0 Q_1^e minus, minus w′ evaluated 0 Q_2^e minus w evaluated h_e Q_3^e and then minus, -w′ h_e Q_4^e so I just written this functional but look at it.

This term is EI times square of slope right, EI is EI times the second derivative of w is moment right, and then when you multiply it by w″ which is the curvature that gives you the strain energy, u_2 bending of a, excuse me yeah, so this is the strain energy of a beam which has been bent okay, this is the work done.

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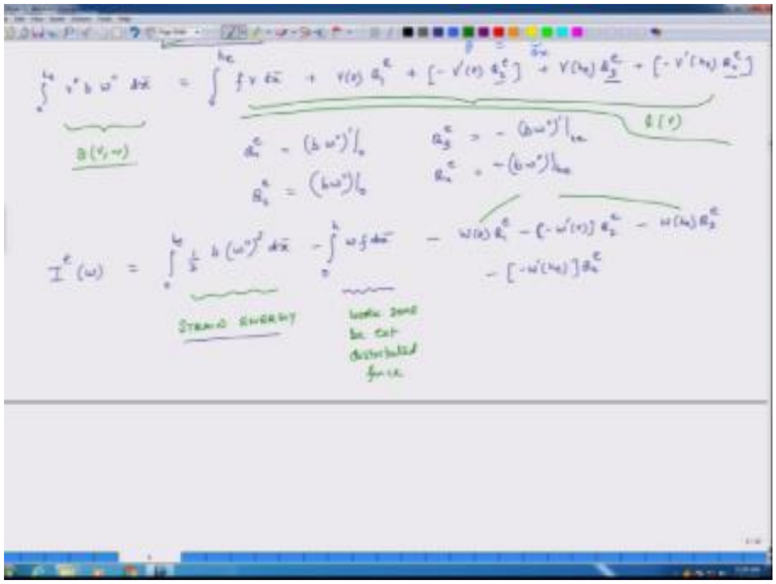

By an external force which is distributed right, the distributed force intense is what over a small element is f times dx, and at that location deflection is w than this wf is the work done by on that small element, and if I integrate it over the entire length that will be the work done by external force which is uniformly distributed. So this is work done by external distributed force okay. These terms which relate to wn at 0n w at h_e what are these, Q_1 is the shear force right, Q_1 we had explained earlier $Q_1$is the shear force, if you look at.

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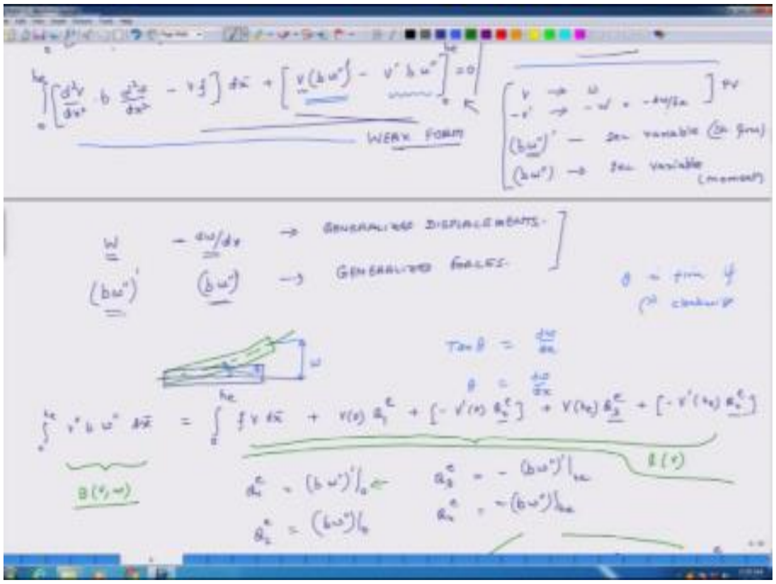

The definition of Q_1 this is my Q_1 it is B times third derivative of w and when you go back to third derivative bw″ and then the whole thing.

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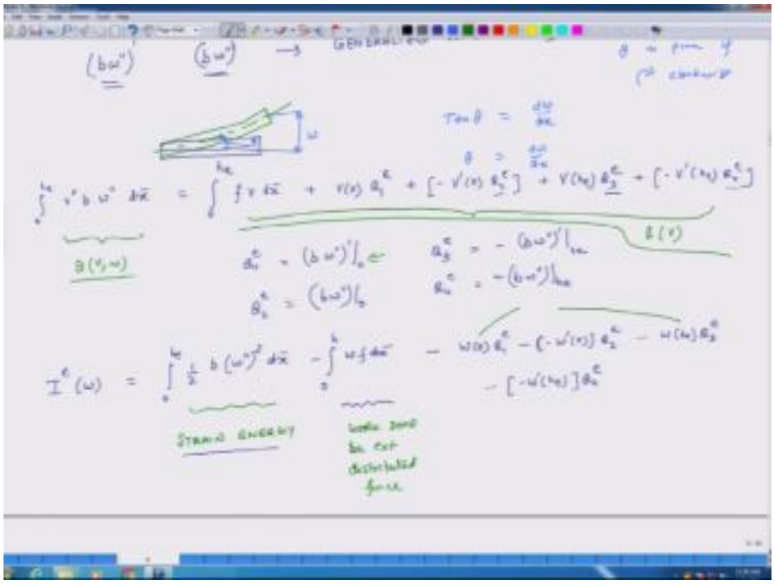

So, so this is the shear force and if I have an external point force in the shear force and at that point of the deflection is w at 0 then that is the work done because of that particular shear force right. So this is work done.

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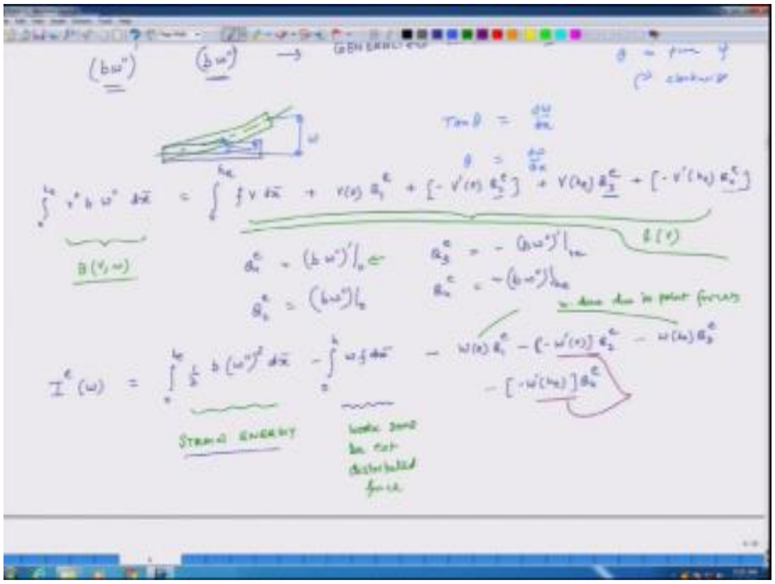

Due to point forces and finally you look at these terms, Q_2 and Q_4 are moments applied on the.

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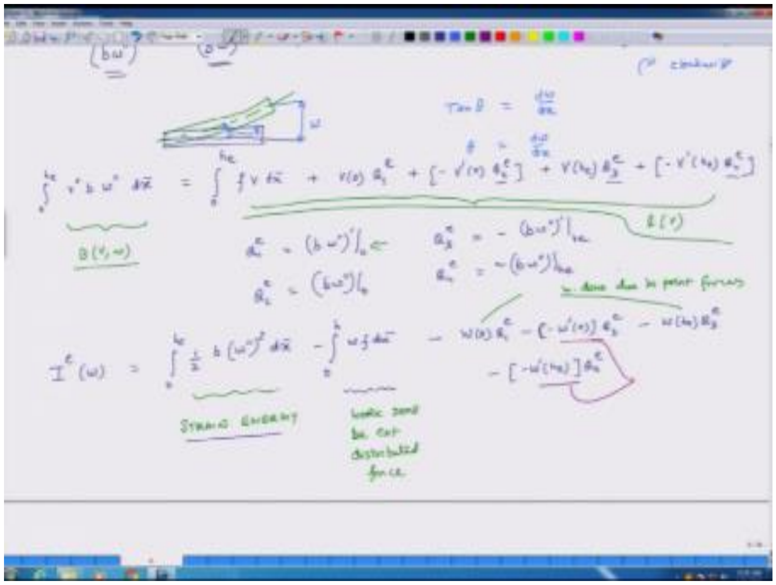

At the ends of the element and what is w′, w′ represents.

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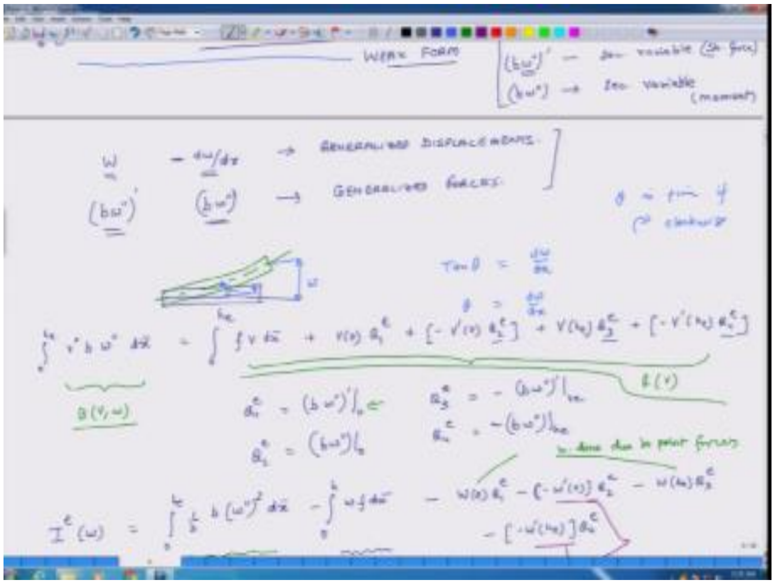

The rotation because of that the moment times rotation is again work done, moment times rotation is again the work done.

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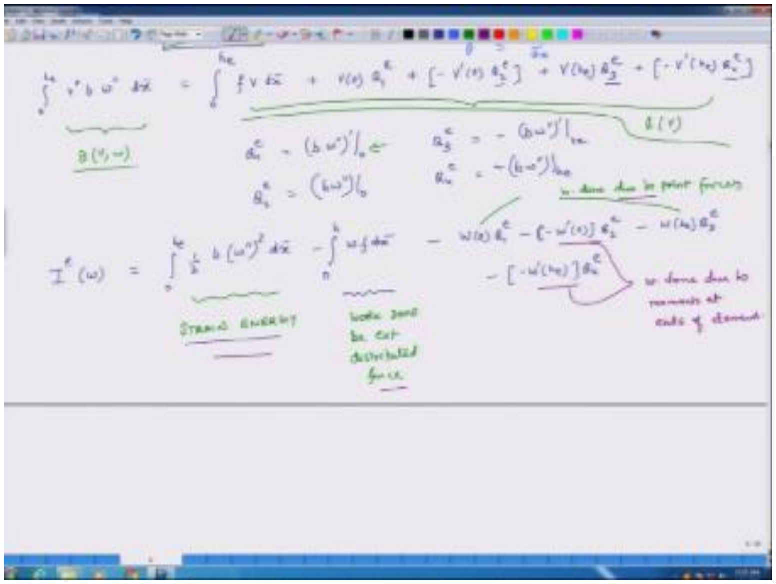

So this is work done due to moments at ends of element agreed. So this is a strain energy this is then work stored in the system and these are all work done by external forces, so this if you add them up this is the total potential energy of the system.

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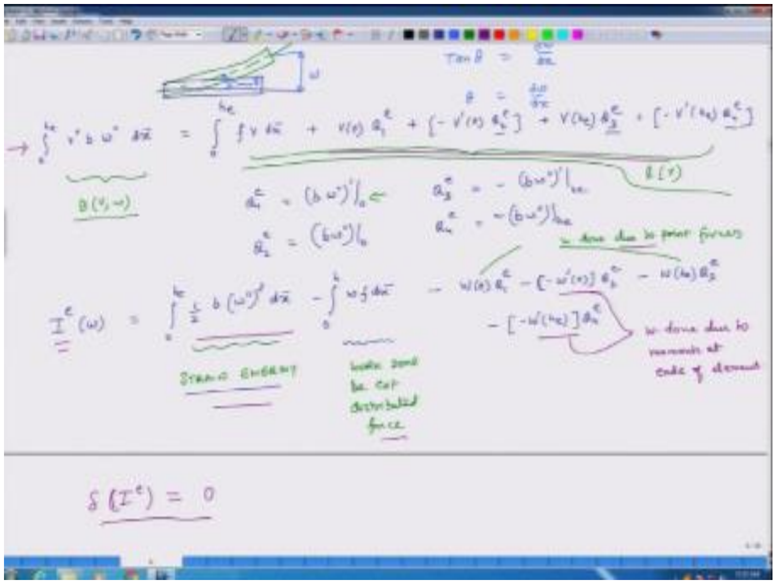

And then if I take its variation, and if the system is in equilibrium then the variation of this has to be equal to 0, and if I take the variation of this thing if you go back to our earlier lectures on the variational operator if you take the variation of this entire thing you will get this entire expression, this equilibrium, so what this equation represents is the first variation of I when it is equal to 0.

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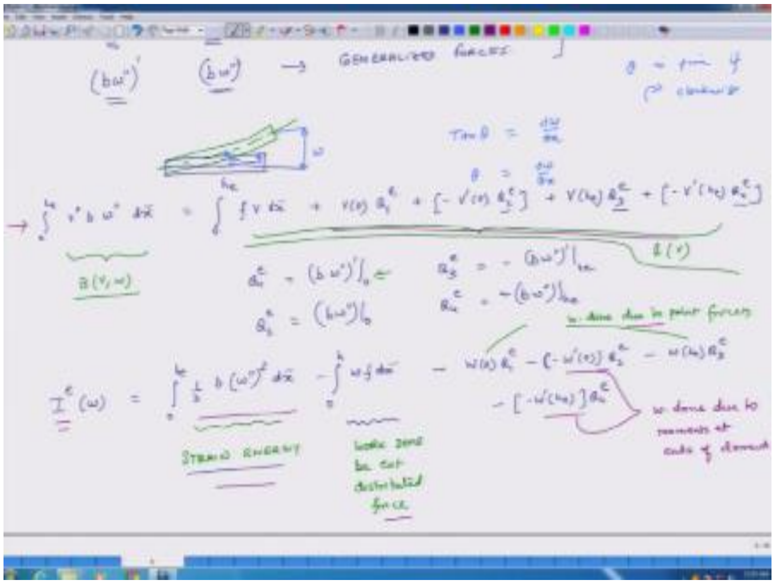

So this is the overall equation and what we have shown today is the basic details of the equation and how we formulate it in context of finite element method.

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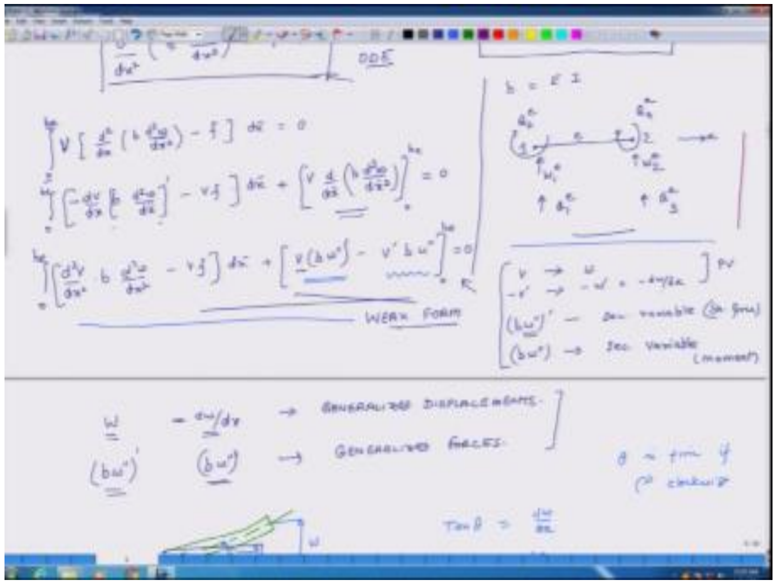

And in this particular context our conventions are very important which are listed down here, so now what we will do in the next class is will now develop element level equations for a beam specifically the Euler-Bernoulli beam. So thank you very much, and we will meet once again tomorrow. Bye.

<u>Acknowledgement</u>
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