

Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 26
Assembly of element equations

by
Prof. Nachiketa Tiwari
Dept. of Mechanical Engineering
IIT Kanpur

Hello, welcome to Basics of Finite Element Analysis, this is the fifth week of this particular course, and today is day two of this week. Yesterday we learned how to develop element level equations and today we will extend that discussion in context of the overall assembly. So what we will do today and possibly also tomorrow is figure out that how do we assemble all these different element level equations and bring them up to an assembly level. So with that intent in mind.

(Refer Slide Time: 00:50)

Diagram showing three elements (1, 2, 3) connected in a line between nodes 1 and 4.

Element 1:
$$\begin{bmatrix} K_{11}^1 & K_{12}^1 \\ K_{21}^1 & K_{22}^1 \end{bmatrix} \begin{Bmatrix} u_1^1 \\ u_2^1 \end{Bmatrix} = \begin{Bmatrix} F_1^1 + Q_{11}^1 \\ F_2^1 + Q_{21}^1 \end{Bmatrix}$$
 EL 1

Element 2:
$$\begin{bmatrix} K_{11}^2 & K_{12}^2 \\ K_{21}^2 & K_{22}^2 \end{bmatrix} \begin{Bmatrix} u_2^2 \\ u_3^2 \end{Bmatrix} = \begin{Bmatrix} F_1^2 + Q_{12}^2 \\ F_2^2 + Q_{22}^2 \end{Bmatrix}$$
 EL 2

Element 3:
$$\begin{bmatrix} K_{11}^3 & K_{12}^3 \\ K_{21}^3 & K_{22}^3 \end{bmatrix} \begin{Bmatrix} u_3^3 \\ u_4^3 \end{Bmatrix} = \begin{Bmatrix} F_1^3 + Q_{13}^3 \\ F_2^3 + Q_{23}^3 \end{Bmatrix}$$
 EL 3

3 elements linear

We will first define the domain. So let us say so this entire discussion is going to happen in context of the bar under x scale compression or tension problem. So let us say this is my domain and I am breaking this domain into three elements, element number one, element number two, element number three okay. Now for, so the global node numbers are node one, node two, node three and node four, the local element numbers are one unknown numbers are u_1, u_2 for the first element, then this is one and two for the second element, and this is one and two for the third element okay.

Now at this stage when we have to, so what I will do is we will first write down the element level equations, so for the first element it is $k_{11}, k_{12}, k_{21}, k_{22}$, and we realize that these k matrices are symmetric atleast in context of our problem, u_1, u_2 is equal to F_1+Q_1, F_2+Q_2 and this is for element one, so everywhere I have this superscript one. So this is for element one, for element two I have similar equations but the superscript simply changes equations, so the superscript simply changes. So this is these are the equations for element, the second element and finally I have the equations for the third element so our aim is, so these are six equations.

Why do I have six equations, because I have three elements and each element has two nodes, if it was a quadratic element it would have three nodes then we would have nine equations okay? So if there are three linear elements, then we get six equations. If there are three quadratic then we get nine equations, and so on and so forth. But ultimately we want only four equations because there are four actual independent nodes okay.

So how do we do that, so we do that by understanding two important principles. The first principle is that because in element one and in element two these nodes are physically connected, it means that the displacement of u_2 is same as displacement of node two of first elements and node one of second element okay. So this is based on, now suppose there was a crack here then we could not have made this assumption okay, but because these are physically connected it means that u_2 for the first element equals u_1 of the second element. If this was a case of temperature heat conduction problem, there are u corresponds to temperature.

(Refer Slide Time: 05:49)

Diagram showing a 1D bar with nodes 1, 2, 3, 4 and elements 1, 2, 3. The global degrees of freedom are U_1, U_2, U_3, U_4 .

Element 1 equations:

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 \\ K_{21}^1 & K_{22}^1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} f_1^1 + Q_1^1 \\ f_2^1 + Q_2^1 \end{Bmatrix} \quad \text{EL. 1}$$

Element 2 equations:

$$\begin{bmatrix} K_{22}^2 & K_{23}^2 \\ K_{32}^2 & K_{33}^2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} f_2^2 + Q_2^2 \\ f_3^2 + Q_3^2 \end{Bmatrix} \quad \text{EL. 2}$$

Element 3 equations:

$$\begin{bmatrix} K_{33}^3 & K_{34}^3 \\ K_{43}^3 & K_{44}^3 \end{bmatrix} \begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} f_3^3 + Q_3^3 \\ f_4^3 + Q_4^3 \end{Bmatrix} \quad \text{EL. 3}$$

Continuity conditions for PV:

$$U_2^1 = U_1^2 = U_2 \quad U_3^2 = U_2^3 = U_3$$

Balance forces:

$$f_2 = f_2^1 + f_2^2 \quad f_3 = f_3^2 + f_3^3$$

Then the same thing would hold, the temperature of node two for the first element would be same as temperature of node one of the second element okay. So first thing is that we write down these equalities, so these are known as continuity conditions. Continuity conditions for primary variable, we assume that throughout the domain of the problem the primary variable which in this case is u is continuous.

So this is u_2 for first element is equal to u_1 of second element and that is this global u okay, global u . So this is the first continuity condition right, likewise I have also another continuity condition for at node three, so the equation for that is u_2 of second element equals u_1 of third element, and that equals global degree of freedom which is u_3 okay. So what I do is, the first step so, so this is my first step so what I do is that I make these modifications to my element level equations and I replace them by u_2 , and this is also u_2 agreed.

Similarly, because u_{22} and u_{13} are same I replace this by u_3 , and then of course I realize, I realized that u_{11} is same as u_1 and u_{23} is same as u_4 , agreed. So I replace this by u_1 , and I replace u_{23} by u_4 . So what I have done is, so at this stage that I have replaced element degrees of freedom with global degrees of freedom. See all these individual displacements had nodes, they are

degrees of freedom so I have replaced element level degrees of freedom with global level, global degrees of freedom.

This is the first thing, the second thing I do is balance forces, second thing is that I balance forces, what does that mean okay. Now I had said that this is what distributed force which is Q , and this distributed force which is Q it appears in my element level equation says F right. So what I know is that the total force which is at let us say node two is partially due to forces on elements one and also due to forces on element two.

It is not due to forces on element three right, so total f , f at node two, total f and this f is only due to this distributed force, I am not talking about point loads at this time okay. So total f at node two is equal to f due to element one, and f due to element two, this is again because node two is connected, if node two was not connected if we had split, then this would not be the case right. Then this will not be the case, if there was a crack then this will not be the case. So what that means is that the overall force at node two, f_2 is equal to f_{21} plus f_{22} okay. Similarly I can say that f at node three is equal to its contribution from the second element which is f_{32} plus contribution from the third element, which is f_{33} , agreed. So this is the second thing which we note, the third thing which we note okay.

\

(Refer Slide Time: 11:29)

Diagram of a 2D truss structure with 4 nodes (1, 2, 3, 4) and 3 elements (1, 2, 3). A horizontal force f is applied at node 2.

Element matrices and equations:

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 \\ K_{21}^1 & K_{22}^1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} f_1^1 + Q_1^1 \\ f_2^1 + Q_2^1 \end{Bmatrix} \quad \text{EL. 1}$$

$$\begin{bmatrix} K_{11}^2 & K_{12}^2 \\ K_{21}^2 & K_{22}^2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} f_1^2 + Q_1^2 \\ f_2^2 + Q_2^2 \end{Bmatrix} \quad \text{EL. 2}$$

$$\begin{bmatrix} K_{11}^3 & K_{12}^3 \\ K_{21}^3 & K_{22}^3 \end{bmatrix} \begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} f_1^3 + Q_1^3 \\ f_2^3 + Q_2^3 \end{Bmatrix} \quad \text{EL. 3}$$

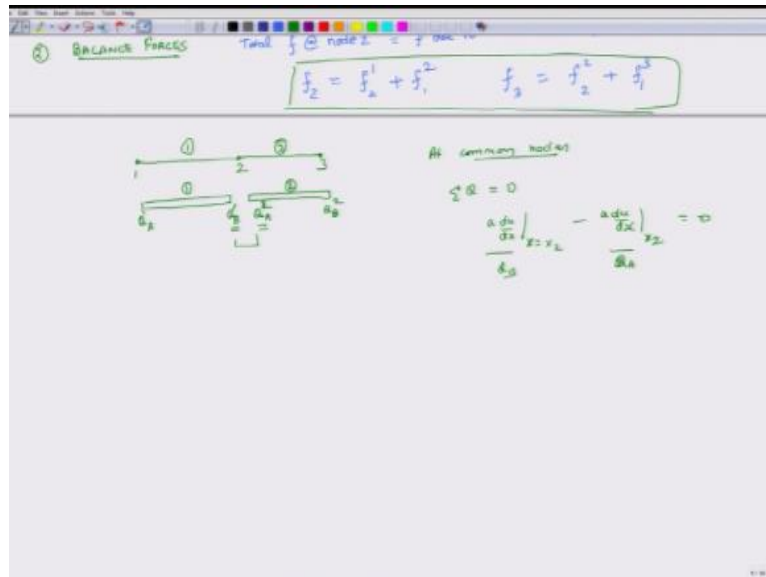
6 EQUATIONS
3 elements
unknown

Continuity conditions for PV: $U_2^1 = U_1^2 = U_2$ $U_2^2 = U_1^3 = U_3$

BALANCE FORCES: Total f @ node 2 = f due to element 1 + f due to element 2.

The third thing which we note is, that so this Q, these Q's are because of a times du over dx right, so from the first element.

(Refer Slide Time: 11:46)



Suppose this is element 1 and this is 3, third node, then when I look at this element by itself I have Q_A for the first element, and Q_B for the first element and for this node, for this element this is, this is element one, this is element two, this is Q_A for the first element, and this is Q_B for the, oh for the second element and this is Q_B for the second element right, I have written two okay. And these forces Q 's, if there is no external force they only appear because I have cut it here.

And in reality there is no partition here so these two they have to cancel out and they have to become zero, so the other condition for balancing forces is that at common nodes sum of $\sum Q$ is 0 okay, and that is what we also see through the definition, because what does this mean, a $\frac{du}{dx}$ at x is equal to you know x_2 so this is what Q_B minus a $\frac{du}{dx}$ at x is equal to x_2 , this is Q_A and I am computing both the things there and this will be 0, okay so mathematically also this is there so what

(Refer Slide Time: 14:01)

Diagram showing a 1D bar with 4 nodes (1, 2, 3, 4) and 3 elements (1, 2, 3). Element 1 connects nodes 1 and 2, Element 2 connects nodes 2 and 3, and Element 3 connects nodes 3 and 4.

Global degrees of freedom: U_1, U_2, U_3, U_4

Local degrees of freedom for each element: u_1, u_2, u_3

Element equations:

- EL. 1: $\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} f_1 + Q_1 \\ f_2 + Q_2 \end{Bmatrix}$
- EL. 2: $\begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} f_2 + Q_2 \\ f_3 + Q_3 \end{Bmatrix}$
- EL. 3: $\begin{bmatrix} K_{33} \end{bmatrix} \begin{Bmatrix} U_3 \end{Bmatrix} = \begin{Bmatrix} f_3 + Q_3 \end{Bmatrix}$

Continuity conditions for u : $U_1 = U_1 = U_2$, $U_2 = U_2 = U_3$

Balance forces: Total f @ node 2 = f due to element 1 + f due to element 2.

Assembly equations:

- $f_2 = f_2 + f_1$
- $f_3 = f_3 + f_2$

6 FBS (3 elements) linear

$U_1 = U_1$
 $U_2 = U_2$

It means is that when I do my assembly level equations I have to do two things, one is that I have to replace local degrees of freedom with global degrees of freedom and also I simultaneously ensure the continuity conditions for u , the second thing is that at interfaces I have to add up the forces right so I have to add up F , and I also have to add up Q 's so the way I add up these F and Q 's is basically this is, these two equations I add them up these two equations right, when I add them up what do I get, I get $\sum f_1^2$ and f_1 and Q_2^1 and Q_1^2 right so I add them up. Similarly I add them these two I add these equations okay, so first stage is I employ the continuity conditions and replace local degrees of freedom by global dus and then second I add up forces at nodes which are common

(Refer Slide Time: 15:20)

Handwritten notes on a whiteboard showing a finite element assembly process. At the top, a diagram shows two elements, 1 and 2, connected at a node. Element 1 has nodes 1, 2, and 3, with degrees of freedom u_1, u_2, u_3 . Element 2 has nodes 2, 3, and 4, with degrees of freedom u_2, u_3, u_4 . Below the diagram, the global stiffness matrix K is assembled. The matrix is 4x4, with rows and columns corresponding to nodes 1, 2, 3, and 4. The entries are: $K_{11} = k_{12}$, $K_{12} = k_{12}$, $K_{13} = 0$, $K_{14} = 0$; $K_{21} = k_{12}$, $K_{22} = k_{12} + k_{23}$, $K_{23} = k_{23}$, $K_{24} = 0$; $K_{31} = 0$, $K_{32} = k_{23}$, $K_{33} = k_{23} + k_{34}$, $K_{34} = k_{34}$; $K_{41} = 0$, $K_{42} = 0$, $K_{43} = k_{34}$, $K_{44} = k_{34}$. The global displacement vector U is $[u_1, u_2, u_3, u_4]^T$. The global force vector F is $[F_1 + F_2, F_2, F_3, F_3 + F_4]^T$. The assembly equation is $K U = F$. To the right, the text "ASSEMBLY LEVEL EQUATION" is written. At the bottom, the text " $K_{11} = k_{12}$, $K_{12} = k_{12}$ " is written.

Okay so when I do that what do I get my first equation is still same, so this is my assembly level equation and I know that when I add up Q 's at the interfaces their sums are 0 so this is gone, similarly Q_3 's are gone so and the definition of, so what is given, so this is global, this is my global K matrix right global K matrix so what is k_{11} it is equal to k_1^1 , what is k_{12} it is k_{12}^1 .

(Refer Slide Time: 17:28)

Handwritten notes on a whiteboard showing the assembly of a global stiffness matrix for a 3-bar system.

Element Matrices:

$$\begin{aligned}
 \left[\begin{matrix} k_{11}^1 & k_{12}^1 \\ k_{21}^1 & k_{22}^1 \end{matrix} \right] \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} &= \begin{Bmatrix} f_1^1 + q_1^1 \\ f_2^1 + q_2^1 \end{Bmatrix} \quad \text{2L-1} \\
 \left[\begin{matrix} k_{11}^2 & k_{12}^2 \\ k_{21}^2 & k_{22}^2 \end{matrix} \right] \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} &= \begin{Bmatrix} f_1^2 + q_1^2 \\ f_2^2 + q_2^2 \end{Bmatrix} \quad \text{2L-2} \\
 \left[\begin{matrix} k_{11}^3 & k_{12}^3 \\ k_{21}^3 & k_{22}^3 \end{matrix} \right] \begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} &= \begin{Bmatrix} f_1^3 + q_1^3 \\ f_2^3 + q_2^3 \end{Bmatrix} \quad \text{2L-3}
 \end{aligned}$$

Global Assembly:

$$\left[\begin{matrix} k_{11} & k_{12} & 0 & 0 \\ k_{21} & k_{22} & k_{23} & 0 \\ 0 & k_{32} & k_{33} & 0 \\ 0 & 0 & 0 & k_{44} \end{matrix} \right] \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} f_1 + q_1 \\ f_2 + q_2 \\ f_3 + q_3 \\ f_4 + q_4 \end{Bmatrix}$$

Boundary Conditions:

① CONTINUITY CONDITIONS FOR PD: $U_2^1 = U_2^2 = U_2$ $U_3^2 = U_3^3 = U_3$

② GLOBAL FORCES: Total f_i at node i = f_i due to element 1 + f_i due to element 2 + f_i due to element 3.

$$\begin{aligned}
 f_2 &= f_2^1 + f_2^2 & f_3 &= f_3^2 + f_3^3
 \end{aligned}$$

Diagram:

A horizontal bar with nodes 1, 2, 3, 4. Elements 1, 2, and 3 are shown between nodes 1-2, 2-3, and 3-4 respectively.

At common nodes:

$U_2^1 = U_2^2 = U_2$ $U_3^2 = U_3^3 = U_3$

Basically I am just right, this is my equation 1, the sum of these two, the sum of these two, second equation and third equation is my second equation global assembly question right, so this gives my second equation, this gives my third equation and this is my fourth equation, and this is my first equation okay, any questions?

(Refer Slide Time: 17:58)

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 \\ 0 & K_{32} & K_{33} & K_{34} \\ 0 & 0 & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_1 + B_1 \\ f_2 \\ f_3 \\ f_4 + B_4 \end{Bmatrix}$$

ASSEMBLY LEVEL EQUATION

$K_{11} = K_1$ $K_{22} = K_2$

So this is how I construct my global K metrics and global f and q vectors, we will talk about that later yeah. So what I have explained since he asked that is that suppose

Refer Slide Time: 18:19

① CONTINUITY CONDITIONS ARE PV : $u_1 = u_2 = u_2$ $u_2 = u_3 = u_3$

② BALANCE FORCES Total f @ nodes = f due to element 1 + f due to element 2

$$f_2 = f_2^1 + f_2^2 \quad f_3 = f_3^2 + f_3^3$$

ASSEMBLY LEVEL EQUATION

There is a point load so in this case we have assumed, not assumed we know that Q's, qv and the interface they add up and become 0, they add up and then they become 0, but if there is an external point force then what do we do, so we will discuss that question a little later, maybe in the next lecture or something like that but this is how we construct the global K metrics.

(Refer Slide Time: 18:42)

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 \\ 0 & K_{32} & K_{33} & K_{34} \\ 0 & 0 & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 + Q_1 \\ F_2 \\ F_3 \\ F_4 + Q_4 \end{Bmatrix}$$

ASSEMBLY LEVEL EQUATION

$K_{12} = K_{21}$
 $K_{34} = K_{43}$

If there are no point loads, if there are only distributed loads applied on the system, if there is an n node, suppose there is an n node here then the value of this Q_4 will be that number, but suppose there is something in between that is something we will discuss later okay. So this completes our discussion for the overall assembly process, we will now continue this discussion and now we will start discussing about boundary conditions and how do we apply boundary conditions in the next class, thanks.

Acknowledgement

Ministry of Human Resource & Development

Prof. Satyaki Roy

Co-ordinator, NPTEL IIT Kanpur

NPTEL Team

Sanjay Pal

Ashish Singh

**Badal Pradhan
Tapobrata Das
Ram Chandra
Dilip Tripathi
Manoj Shrivastava
Padam Shukla
Sanjay Mishra
Shubham Rawat
Shikha Gupta
K. K. Mishra
Aradhana Singh
Sweta
Ashutosh Gairola
Dilip Katiyar
Sharwan
Hari Ram
Bhadra Rao
Puneet Kumar Bajpai
Lalty Dutta
Ajay Kanaujia
Shivendra Kumar Tiwari**

an IIT Kanpur Production

©copyright reserved