Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 25
2nd order boundary value problem

by
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Hello, welcome to basics of finite element analysis, today is the beginning of the fifth week of this course and this week we will continue our discussion on the finite element method in context of second order boundary value problems, and what we will be doing this week is essentially continuing our discussion on element level equations and then we will use these element level equations to construct assembly level equations, impose the boundary conditions and arrive at a reduced number of equations which could be solved to find out the values of deflections, or the values of dependent variable at all the nodes in the problem

So we will start this by having a very brief recap of what we did in the last lecture, so the problem which we were trying to solve is

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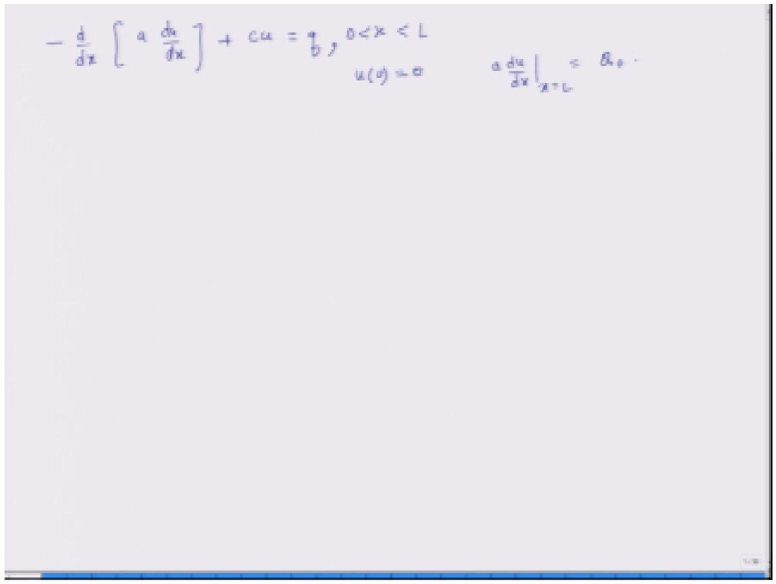

Represented by this governing equation, differential equation d over dx, a d over dx + cu equals q and this equation is valid when x is between the limit 0 to L and let us say that we have 2 specific boundary conditions, the first boundary condition is u is at x is equal to 0 is 0 and the second boundary condition is that a times d over dx at x = L is a sum constant and let us call this Q_0.

So what we had done was that we had constructed the weak form of this equation but in finite element method we developed a weak form on an element by element basis okay, and the weak form, the residual error and its weak form is equated to 0 on a element by element basis, so

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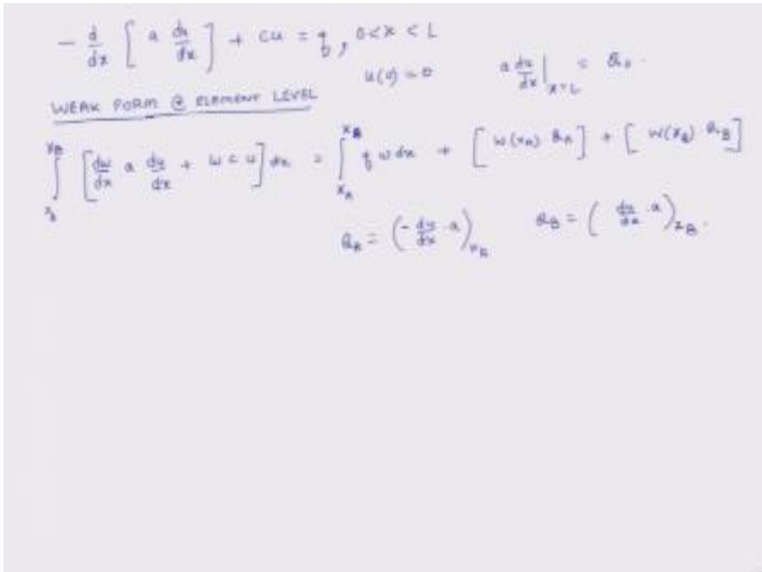

If the limits the, boundaries of the element are designated as x_A and x_B then the weak form is dw over dx, a du over dx + wcu, and then I am integrating it with respect to x equals, integrating it from x_A to x_B q times wdx and plus 2 boundary terms.

So the first boundary term is w which is evaluated at x_A times Q_A and the second boundary term is w which is evaluated at X_B times Q_B okay, and we had defined Q_A equals minus du over dx times a, this entire thing and it is evaluated at X_A, and Q_B is , it does not have this negative sign so it is du over dx times a evaluated at x_B.

So this is the weak form and particular, and the weak form at element level. So then what we had done was we had developed relations for interpolation functions, we had developed two relations for ψ_1 and ψ_2 for a linear element and we had also developed relations for a quadratic element. So

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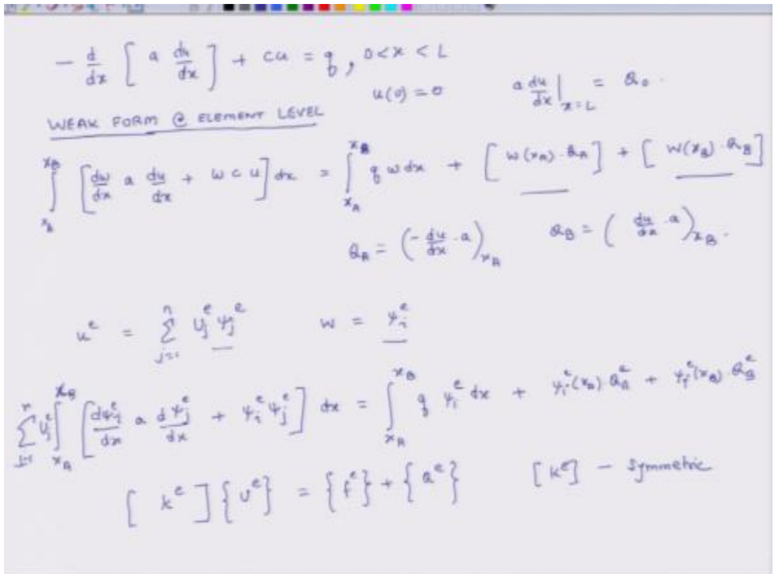

we assume at this stage that u for the e^{th} element is equal to u_j, ψ_j and I am summing it on j, so that is my approximation for u and I am assuming because I am going to use Rayleigh Ritz approach that my weight function is same as ψ_i so I put these 2 relations back into the weak form and what I get is this equation and the constants at node positions are u_j's so I have put it here and this equals, so that is the left side of the equation and the right side is q and times w which is ψ_i dx plus I have these two boundary times so I get ψ_i evaluated at x_A times Q_A + ψ_i evaluated at x_B times Q_B.

And the superscripts on all these guys is e because this corresponds to e^{th} element, and then we had from this relation expressed, expressed the same equation in matrix forms so this is my k matrix for the e^{th} element multiplied by the vector for primary variable which is u^e, and this equals a force vector f^e + another force vector Q^e. And we had remarked earlier that in context of this particular differential equation because we are using the weak formulation our k matrix is symmetric, so next what I will do is I will actually calculate the values of different k's, k_1^1,k_2^2, k_2^1, k_2^2, and also the values of f_1 and f_2 okay. So

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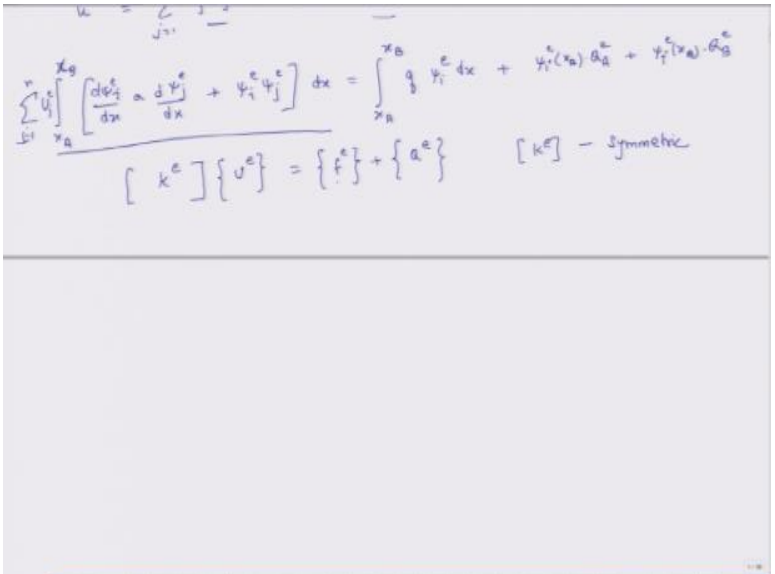

That is my overall equation and we are going to integrate, we will conduct the integrations and what we had discussed earlier was in the last week was, so this integral is using the global coordinate system because I am integrating from x is equal to x_A to x is equal to + x_B, but typically when we go in and conduct integrations in finite element we prefer to use local coordinate systems so this x_A will be replaced by zero and x_B will

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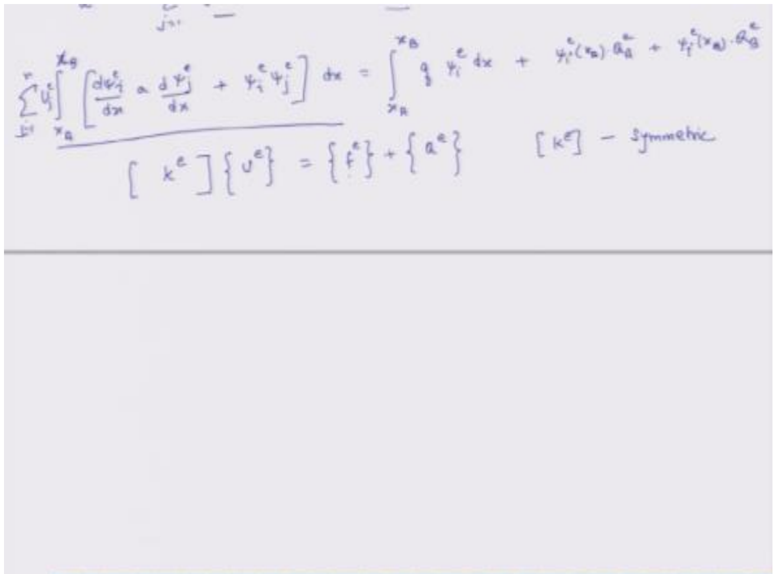

Be replaced by h_e which is the length of the element. Also we had said that x gets replaced by local coordinate which is $\bar{x}$.

So those are the transformations we will do and we will using those transformations we will find out the members of K matrix so

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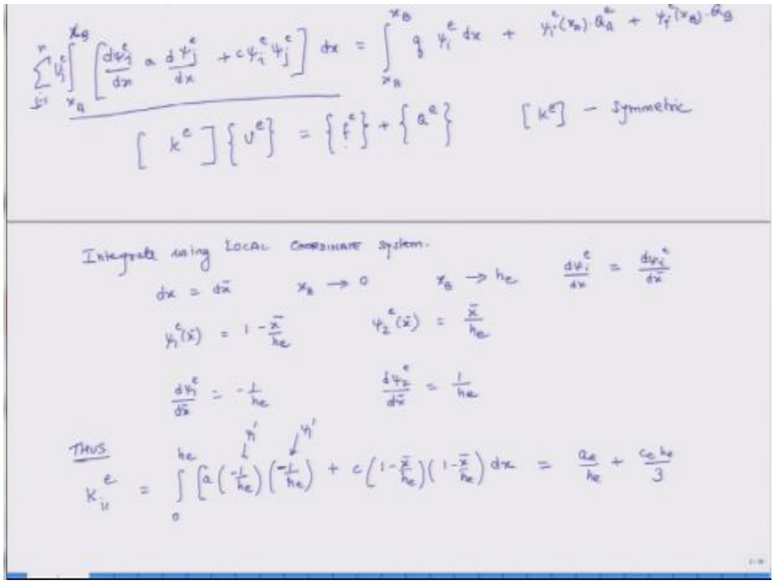

we are going to integrate using local coordinate systems, so the transformations will be that dx is same as $d\bar{x}$ okay because all I am doing is I am just offsetting it, the coordinate system by an amount x_A. x_A transforms to 0, x_B it transforms to h_e and because of this reason $d\psi_i$ for the e[th] element dx, it transforms to $d\psi_i$ for the e[th] element over $d\bar{x}$ okay.

Now in local coordinate system we know that ψ_1 in local coordinate system is equal to 1 - $\bar{x}$ over h_e and so this is for the e[th] element and ψ_2 which is a function of $\bar{x}$, in local coordinate system is equal to $\bar{x}$ over h_e, so my derivatives $d\psi_1$ over $d\bar{x}$ equals -1 over h_e, and $d\psi_2$ over $d\bar{x}$ equals 1 over h_e okay.

Thus k_{ij} for the e[th] element is equal to I have to integrate 0 to h_e a times -1 over h_e which is okay so actually what I am doing is I am come computing k_{11} okay, so it is a times $d\psi_1$ over dx times d ψ_2 over dx which is 1 over h_e + so we see this relation here, d ψ_1, d ψ 1 over dx times $d\psi_2$, $d\psi_1$ over dx times a + oh there should be a c here $c\psi_1$, ψ_1 is 1- and again ψ_j is also ψ_1, so 1- $\bar{x}$ over h_e, and all this is going to be integrated with respect to x okay.

So this is k_{11} and this should be a negative sign because this is $\psi_1'$and this is also, so ψ_1'. So if I do this mathematics I get a_e over h_e + c_e, h_e over 3, and what is c_e it is the value of c on the e^{th} level, on the e^{th} element okay. Same thing for e, a it is the value of a for the e^{th} element, there is no reason to think that a is not changing with respect to x, so the only thing here when

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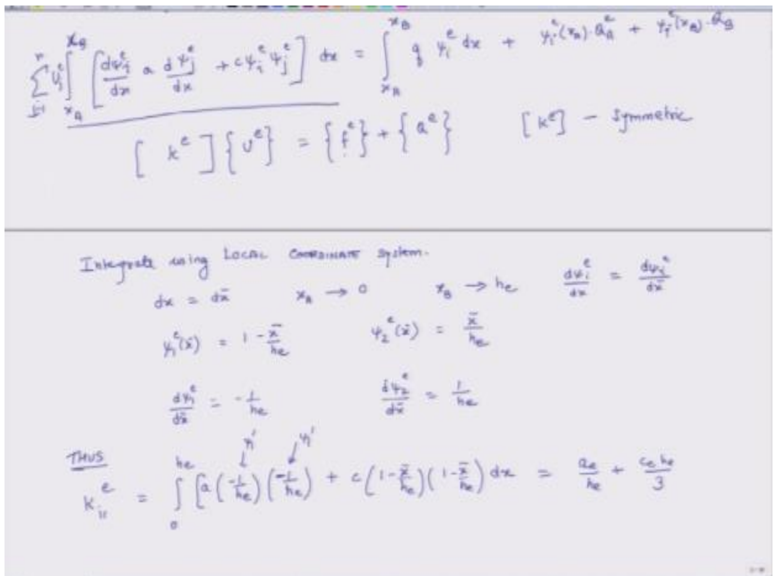

We are doing the integration we are assuming that a is constant over the length of the element, but in the next element it could be a different number.

I do not even have to make that assumption if I know the, how a varies as a function of x, if a, if

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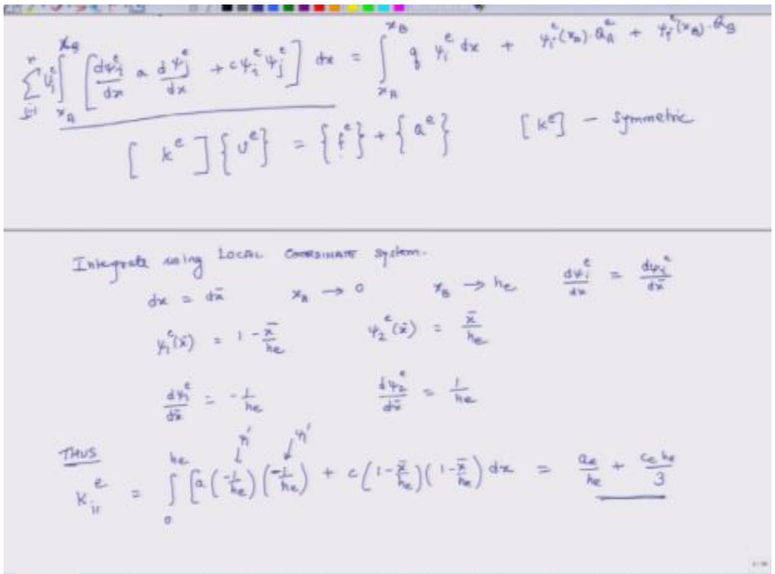

I know the value of a as a function of x then I can integrate it, then this relation will become somewhat modified, but here I am assuming that a is constant on an element level basis, but between one element and another element it changes, so that is why I put

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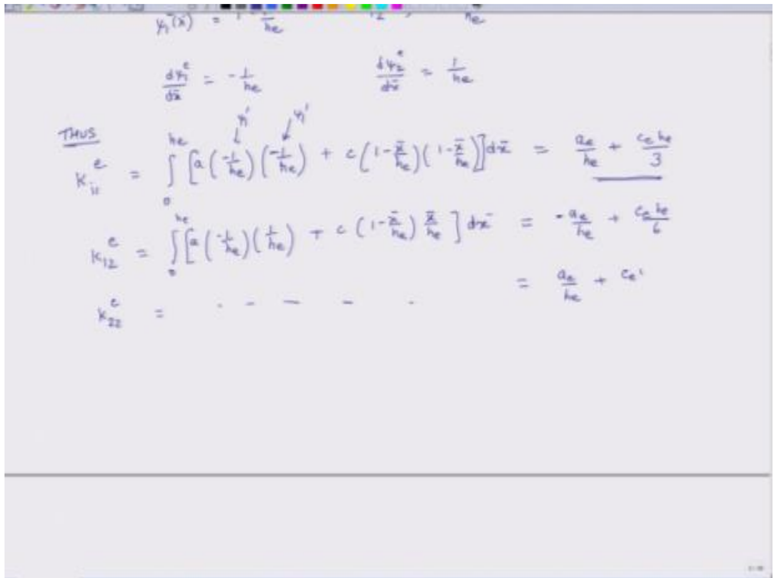

a_e and c_e. Similarly I get k_{12}^e if I do the same process my integration, integrant is -1 over h_e times 1 over h_e right + so yeah, so you have this is $d\bar{x}$ + c times 1- $\bar{x}$ over he and this will be $\bar{x}$ over h_e, d $\bar{x}$ and this comes to

a_e over h_e+ c_e h_e over 6 and there is a negative sign here and when I do calculation for k_{22}, I do all these things and the answer which I get is same as k_{11} which is a_e over h_e+ c_e h_e over 3. For the force vector, f vector this

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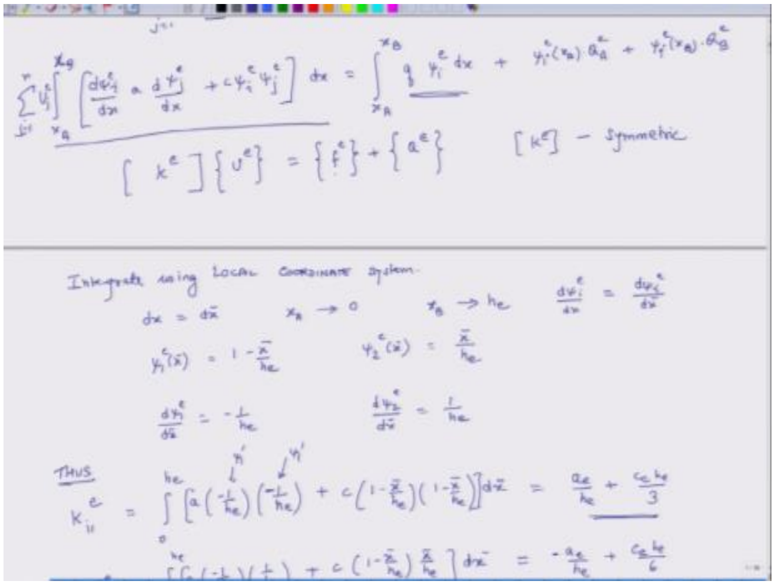

Is the definition for the f vector, so I have to integrate q, the product of qn and ψ_i over the domain so

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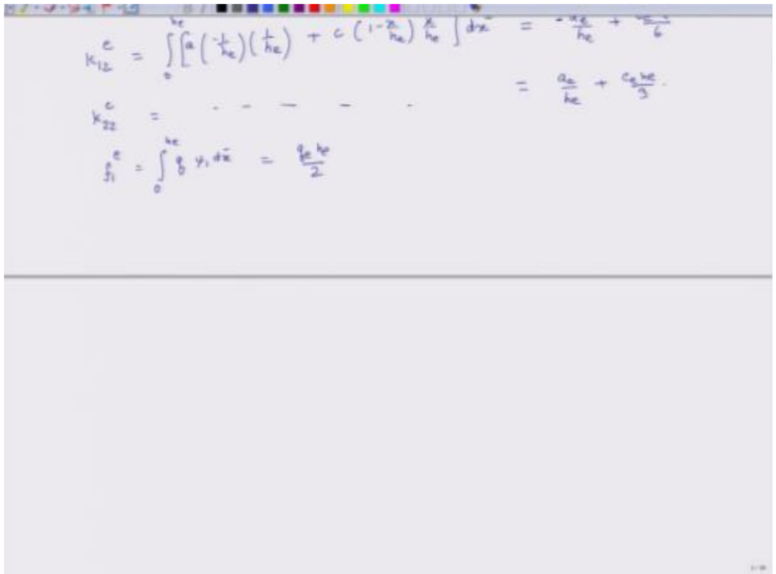

f_1^e is equal to integral of q times ψ_1 $d\bar{x}$ 0 to h_e and if I do the math what I get is q_e h_e over 2 and f_2^e is also q_e h_e over 2.

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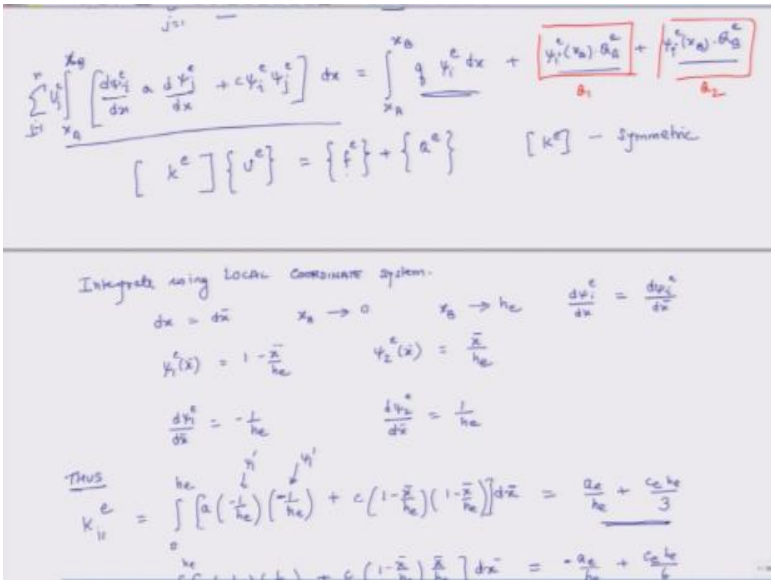

And finally we will compute these terms okay, we are going to compute these terms, so this term I will call Q_1, this will be associated with the first equation and this will be Q_2, this will be associated with the second equation right because based on the value of I that is my i^{th} equation, so when $\psi_i = \psi_1$ that is the value of q in the first equation. When $\psi_i = \psi_2$ that is the value of q in the second equation.

So when $\psi_i = \psi_1$ the value of ψ_i at x is equal to x_A is 1, the value of ψ_1 at the first node is 1 and at the second node it is 0, so Q_1 is, so this term is Q_A in the first equation and this term in the first equation is 0, in the second equation this term associated with Q_A is zero and this term associated with the second equation is 1 or q_e, so

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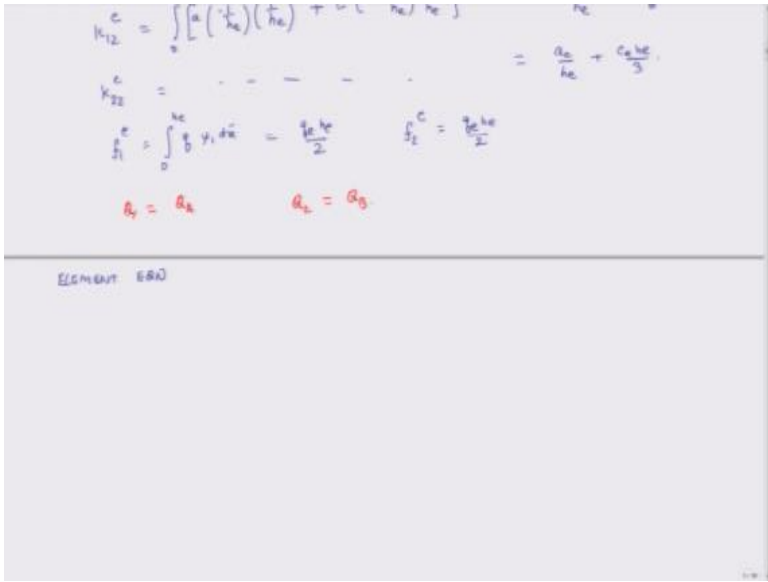

Q1 is equal to Q_A and Q2 is equal to Q_B.

So at this stage I write my overall element level equation so

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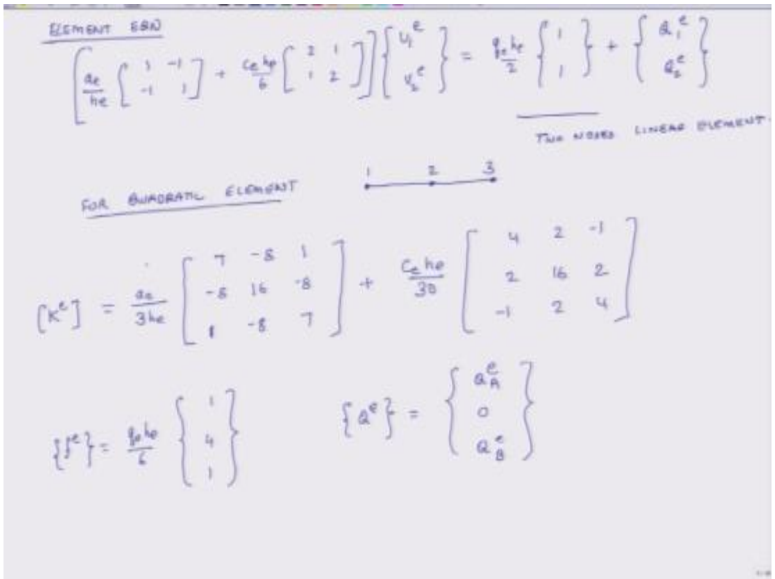

The equation is a_e over h_e, 1-1-11+ c_e h_e over 6, 2112 and these two matrices are added up and they are multiplied by the vector u_1^e, u_2^e and this equals the force vector q_e h_e over 2 11+Q_1^e and Q_2^e, where $Q_1^e = Q_A$ and $Q_2^e = q_e$ okay.

Now these, this equation is for 2 noded linear element, so if we had a quadratic element I would have element like this, so this is node 1, this is node 2, this is node 3 right and if I assume that the node 2 is located at halfway mark over the length of the node then my k matrix, this is actually I have to put a superscript, this is equal to a_e over 3h_e 7-8 1 16 -8 16-8 1-8 7+ $c_e h_e$ over 30 and in the matrix I have 4 2 -1 2 16 2 -1 2 4. Further my f^e matrix or vector = q_e h_e over 6 1 4 1and my Q matrix or Q vector is Q_1^e 0 Q, actually I will write to confusion Q_A and Q_B, and at this stage I wanted to make two important comments.

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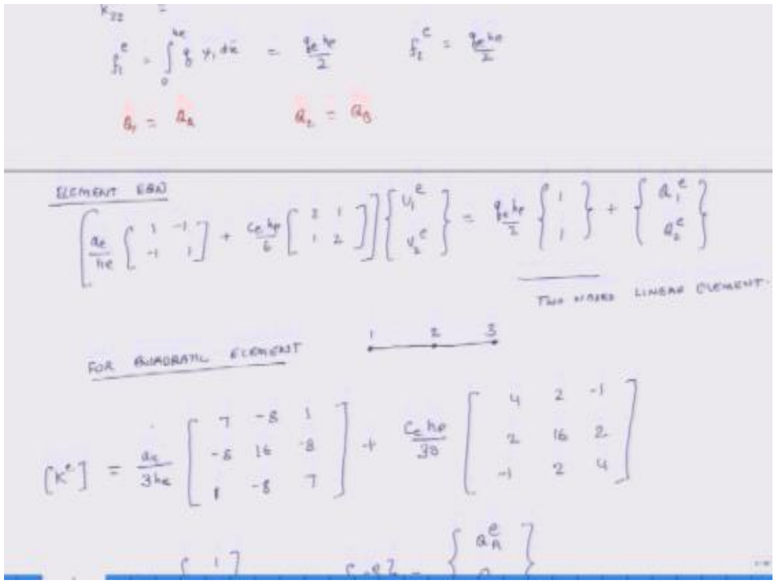

When you

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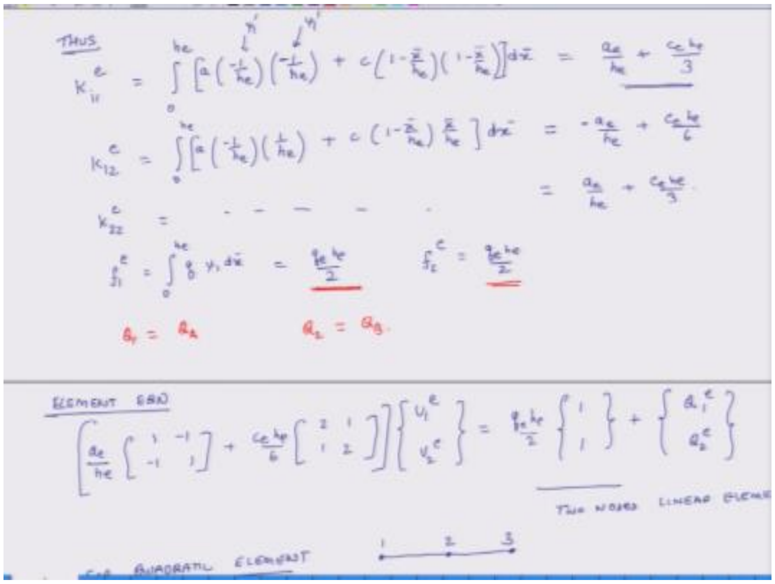

Look at this force term okay so now what we are going to do is we are going to interpret the differential equation in context of a bar problem so

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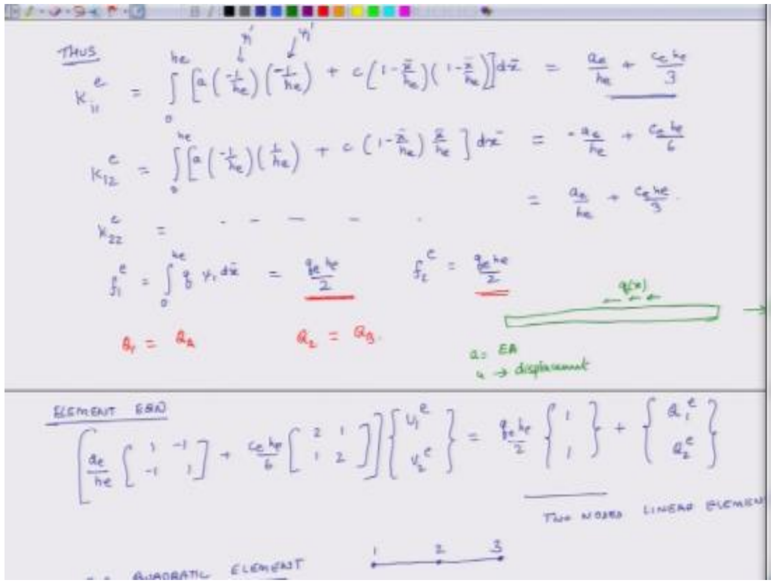

If I have a bar right its cross-sectional area so it is cross, so what is a corresponds to e times its cross-sectional area, this is my x coordinate and u corresponds to displacement, so this is a bar either under tension or compression.

So u corresponds to displacement then and q is some distributed load, external load q(x) so it is some distributed load per unit length, per unit length, it is distributed over the length of the bar, at some places it could be 0, at some places it could be non-zero and it could vary as a function of x, so q is the distributed load in the axial direction, in the axial direction and its unit is Newton square meter okay so if I have a, and suppose over an element suppose q is constant then the total force which is on the on the, on that element will be q_e times h_e right.

So this and if it

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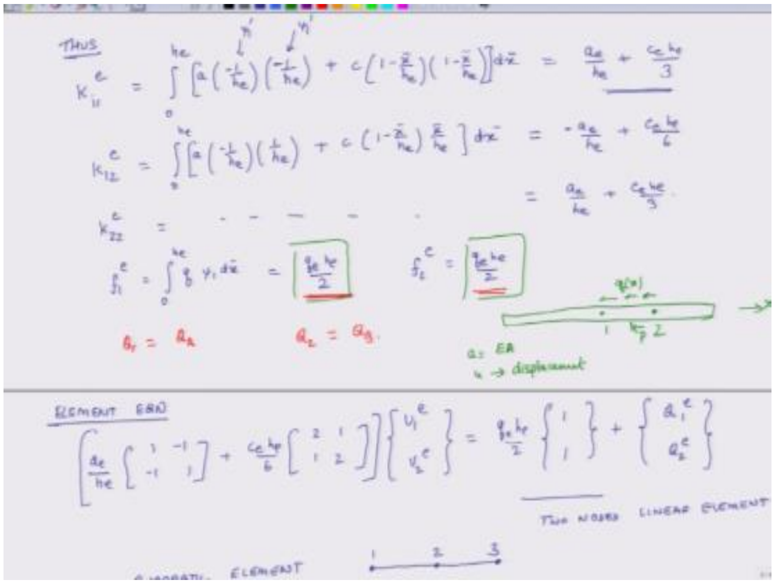

Is constant and that is what we have assumed here so half of that load is shared by node 1 and half of that load is shared by second load okay, so that is the meaning of this q_e times h_e over 2. This represents the total external load because of distributed load, because of distributed load suppose I also had a point load at this point, suppose there was a point load t then that t I have not factored into it because here all I am talking about is q okay. So f_1 and f_2 they represent total loads, consequential loads at nodes 1 and node 2 which are due to the presence of distributed load intensity in the direction of the bars axis and

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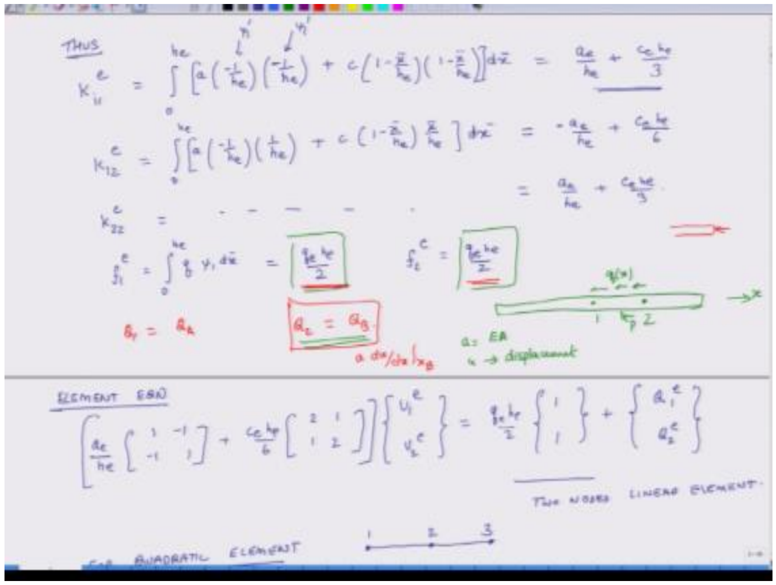

if that intensity is constant over the length of the element than half of that total load, the total load will be q_e times h_e, half of that will be taken by the first node, half of that will be taken by the second node, so it will be evenly divided. Second thing so this is one important statement, the second thing is this Q_2, Q_2 is what, it is defined as a times du over dx at x = b, $x = x_B$ right or at the second node element, second node of the element.

Now du over dx if u is displacement the du over dx is strain, du over dx is strain and when you multiply that strain by e because a is e times a so if you multiply that by e, e times strain is stress and that times a is the force, so what that means is that suppose I have an element and I am exerting a point load, q was distributed load this is a point load, I am putting some external point load here then that will be equal to a times d over dx at that particular location x is = x_B okay.

So these terms Q_A and Q_B they represent point load terms whereas f_1 and f_2 they represent the loads at the two locations which is attributable to the distributed load. This is important to understand because we will use this understanding when we are developing the assembly level equations okay.

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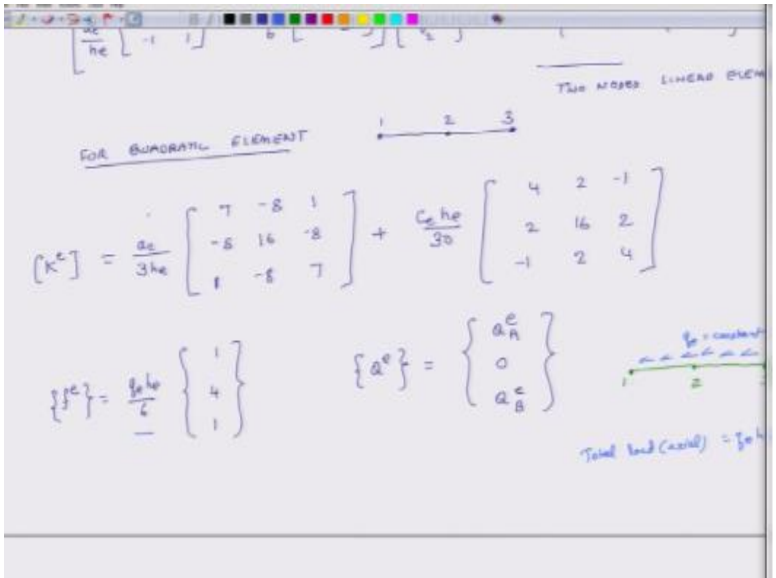

Now let us look at the quadratic element, so this is my quadratic element and I it has two nodes, node 1, node 2, and node 3 right and suppose that distributed load on this q_e and this is let us say constant okay, then what is the total distributed load on the length of the element, total load, total axial load will be q_e times h_e, will be q_e times h_e right.

$1/6^{th}$ of that load, so look at it, $1/6^{th}$ of that load q_e times h_e divided by 6 is bound by the first node, it is not equally distributed, this is what mathematics is telling us, so $1/6^{th}$ of that load is taken by node 1, $1/6^{th}$ of that load is taken by node 3 and $2/3^{rd}$'s of that load or $4/6^{th}$ of that load is taken by node 2. So that distribution or the bearing of these loads over a quadratic element is not 1 is to 1, it is 1 is to 4 is to 1 okay.

Similarly for a cubic element this distribution may be different okay

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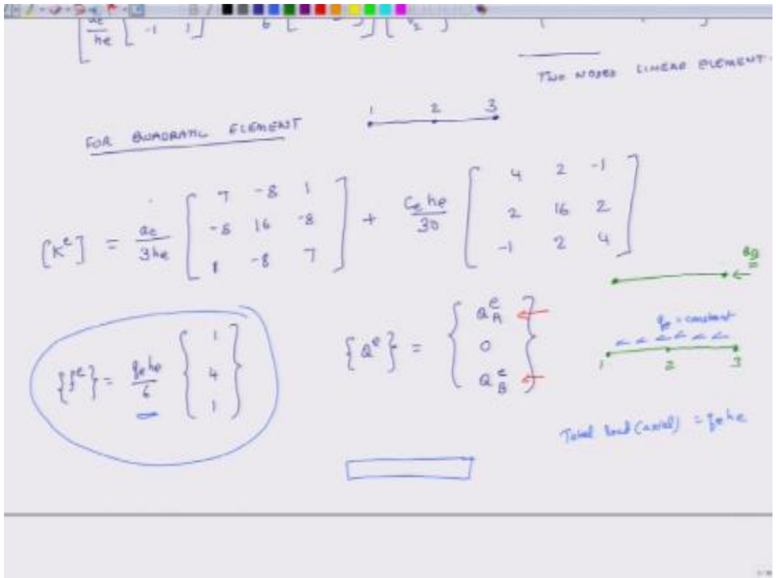

So that is the second thing, so that is second point so, so if we have, if you take, so why am I saying that, so suppose you are using a f^e software, if you are using an f^e software and you want to apply some distributed load over an element, if the software does not automatically do all this mathematics, if it does not automatically do all this mathematics then what you have to do is you have to distribute this load in the right proportion at different nodes.

If it is linear element then it is easy, you just divide it by 2. If it is quadratic element then you have to distribute it in 1 is to 4 is to 1 proportion okay, otherwise your answers will be wrong. And the last point or the third point is, now we will again look at this q vector okay. So at the, so when I have, so again so when I am looking at an element in isolation basically I have cut this element right and the force point load here will be Q_B which will be a times d over dx and why is this appearing, because I have cut it.

So when I take the free body diagram that

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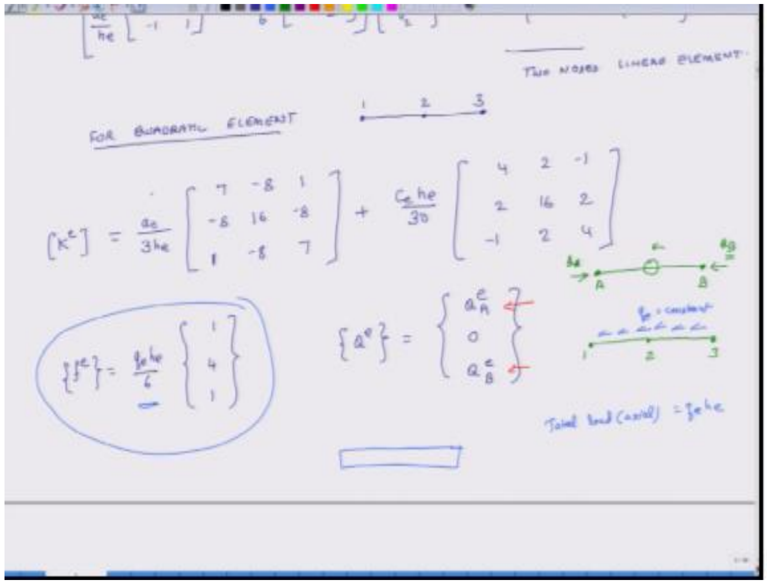

Q_B will appear right, similarly they will be if I take the free body diagram there will be Q_A on this end, if this is point a and this is point b but there will be no force at the middle point because of my free body considerations okay. There may be a load in the middle point only if I apply some external point load at the middle point, but how do we deal with that, we will talk about it later.

But if there are no external loads and if I cut an element what I will see is that there will be Q_A, Q_A at one end and Q_B at other end okay, and these forces emerge because I have taken body a body in equilibrium and cut it so for the body to remain in equilibrium these things have to show up okay. You can, so this is in context of the differential equation as we see it in

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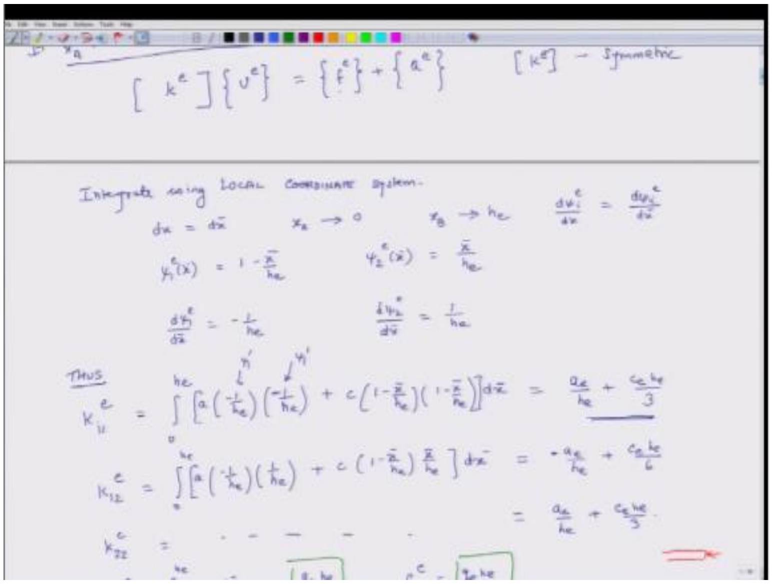

Context of an axial bar problem

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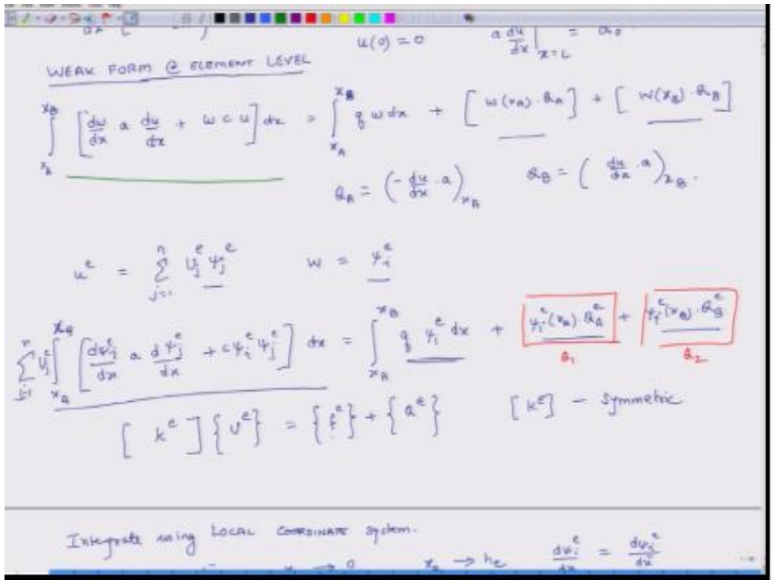

But we can make similar conclusions for other types of problems because this equation it not only represents that bar thing but also for heat transfer and all that, but we can make similar conclusions in context of the nature of the problem for other categories of problems as well. So

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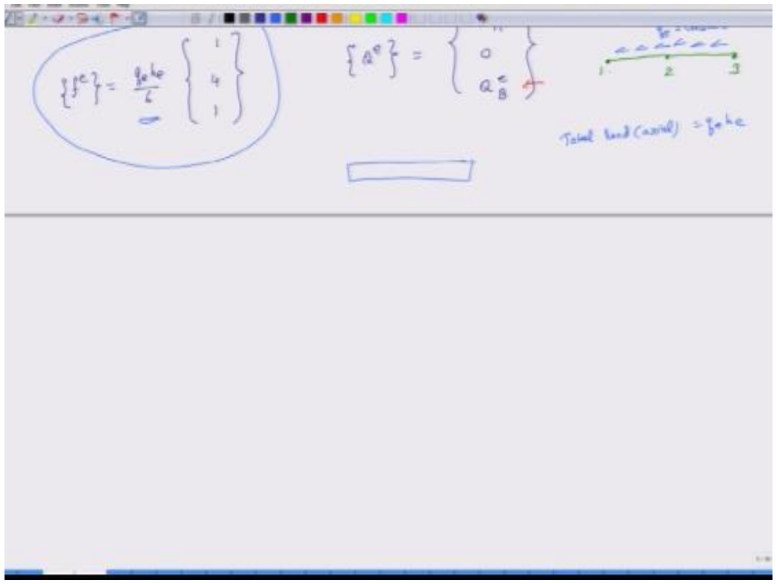

This concludes our discussion today and we will continue this discussion in the next class.

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