Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 23
FEA formulation for 2nd order BVP
Part - II

by
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Hello, welcome to basics of finite element analysis, in the last lecture we have developed expressions for approximation functions corresponding to a two noded linear element ø1 and ø2. And these approximation functions look like these relations.

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Ø1 x equals $x_B – x / x_B - x_A$, where x_B and x_A are the coordinates of the for the e^{th} element.

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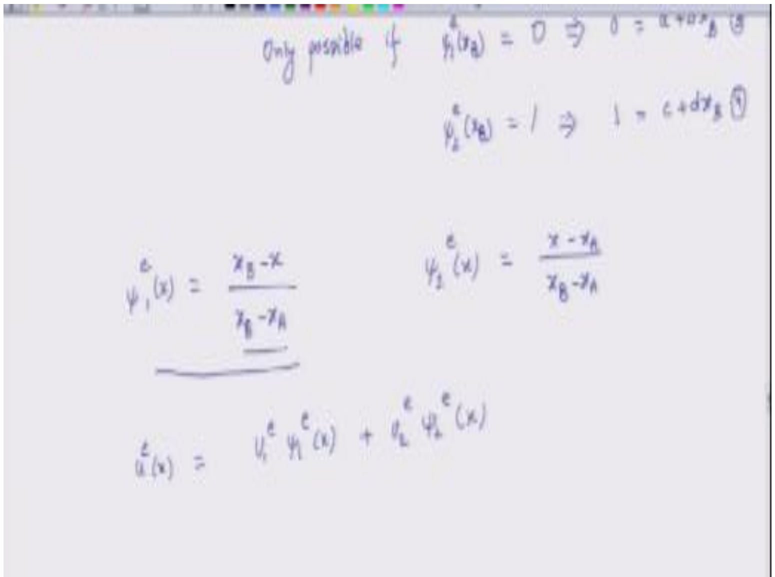

And ø2 is equal to x – XA/ xB – Xa, I would like to make couple of comments about these approximation functions. The first thing is that these functions.

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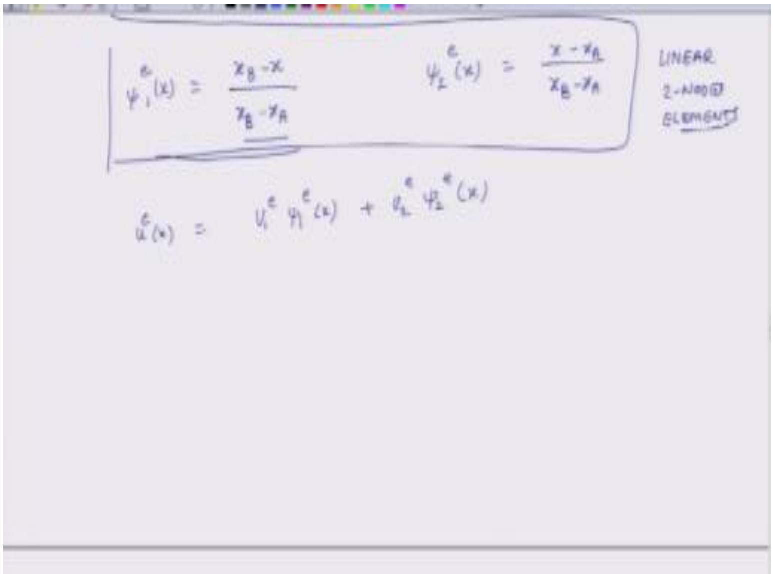

Are good for linear elements, two noded linear elements. If it is a quadratic function then our element has three points. So we will need a ø1, ø2, ø3, but using similar development process we can calculate those functions as well. If it is a cubic element then we will have four such functions. But the process is same, so that is one thing.

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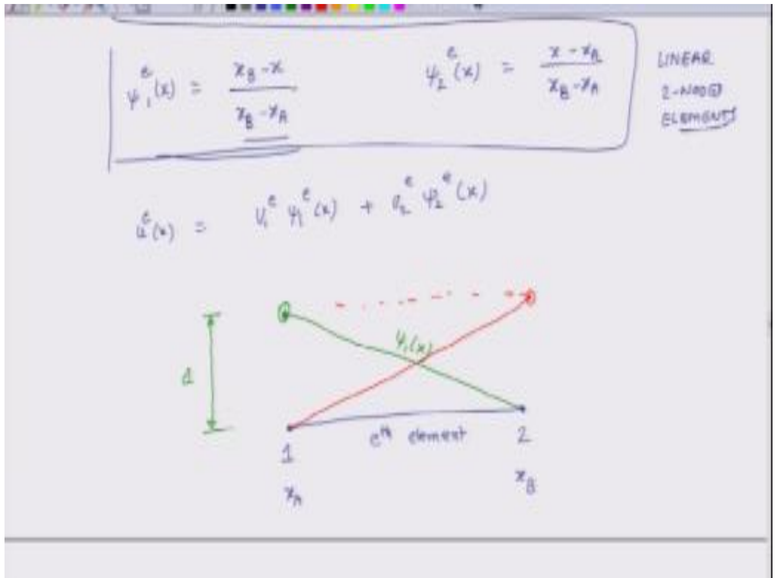

The second thing is let us consider this as my e[th] element, okay. The coordinator of the first node. So this is node one, this is node two, the first node, its coordinator is X_A, the second node its coordinator x_B. If I plot ø1, what you will see is at x = 0, Ah I am sorry, at X= X_A which is the first node the value of this ø function is one. So this is one. And at the second node x_B, this value it becomes zero, right?

When x is equal to xv, ø1 becomes zero. So this looks like this and this height of this function is one. So this is my ø1, the second function is similar but it moves in a different way. So it is zero at the first node and it is one at the second node, okay. And this feature is something you will see you again and again in quadratic elements you will have three functions ø1, ø2 and ø3, ø1 will be having a value of 1.

At the first node and it will have a value of zero at second and third nodes. Ø2 will have a value of 1, at the second node and it will have a value of zero at first and third nodes. And ø3 for a quadratic element will be one at the third node and it will have a value of zero at node number one and node number 2.We will actually see that later also in detail.

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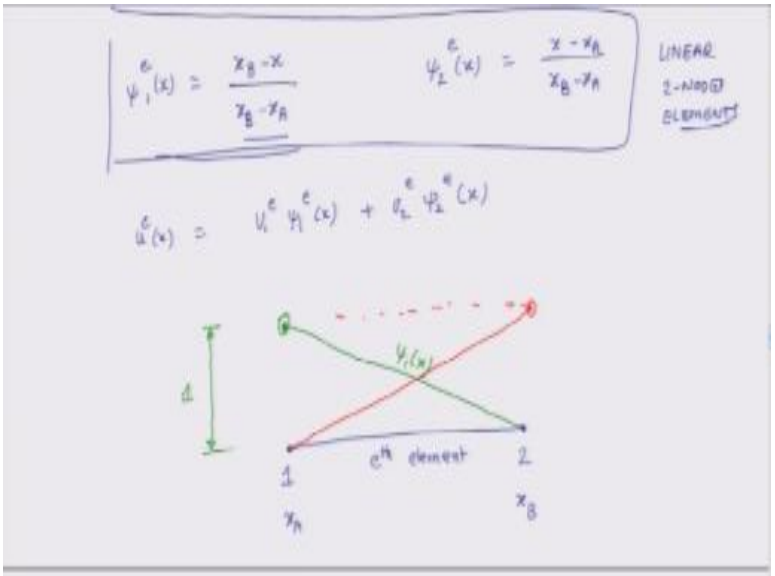

So, so this is important to note, at this point of time I will also introduce two different coordinate systems.

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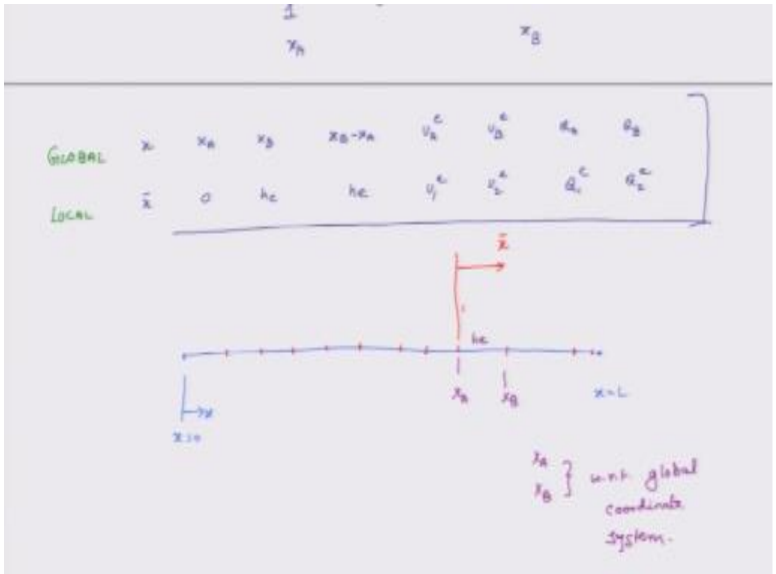

So one is global coordinate system and the other one is local coordinate system. So suppose this is the length 0 to l, okay. So here x is equal to 0 and I am breaking and this is X equals L and let us say I am breaking this domain into lot of elements and let us say the length of e^{th} element is h_e and it is at this location is x_A and let us this location is X_B. So these are x_A and X_B they are with reference to global coordinate system.

My local coordinate system it has a origin, my local coordinate system its origin is at the first node of the e^{th} element and I will call that coordinator as x- bar just to differentiate x –bar, okay. So my global coordinate system is x, my local coordinate system is x-bar, in my global coordinate system when x is equal to xA, in my local coordinate system x is equal to 0, okay.

In my global coordinate system when x is equal to x_B, locally it is $h_{e,}$ also XB – XA equals $h_{e,}$ in my global coordinate system the displacement of first node of e^{th} element is $u_1u_A^e$, right? If it is not there, here I just call it u_1^e this is u of point B for e^{th} element this is u_2^e. Same thing for q's, q_A, q_B here I call it q_1^e, q_2^e.

And so on and so forth, most of the competitions when we do at element level, we do not use the global coordinate system, we use the local coordinate system, okay.

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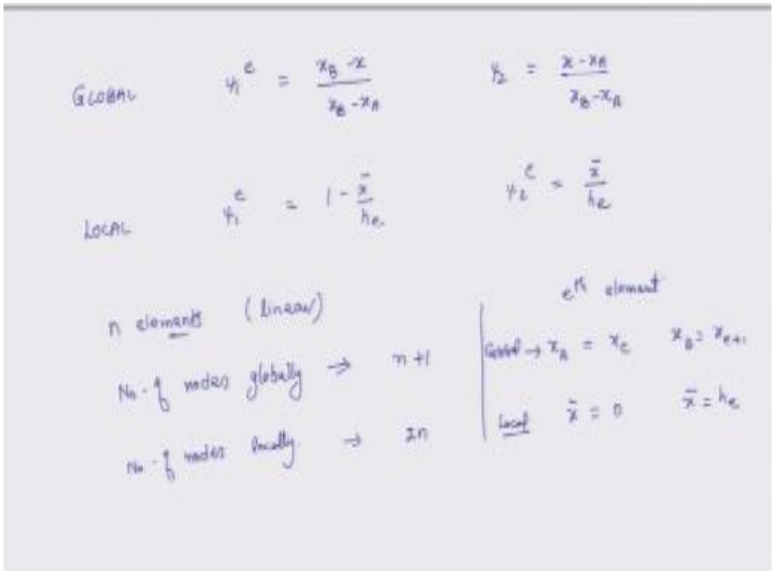

In a global coordinate system, ø1e we had calculated was $X_B – X/ X_B - X_{A,}$ in local its $ø_1^e$ which is the approximation function, this equals 1 minus x- bar over $h_{e,}$ in global my ø2 equals x - x_A/ $x_B – x_A$, in local it is equal to $ø_2^e$ equals x- bar over h_e, okay. And e is the e^{th} element, if there are total number of n elements overall and there are all of these are linear elements then total number of nodes globally, globally it will be how much? (n+1)

So I will make it short and small. But number of nodes locally there are n elements each element has two nodes and then there will be 2n elements, okay. Also for the e^{th} element, X_E which is the global coordinate it corresponds to X_E where E is the eth element and X $_B$ is equal to x_E + 1, right? Here this is global and the local you have x -bar is equal to 0; here x- bar is equal to $h_{e.}$ So you have a global coordinate system.

And this global coordinate system is comes into play when we are doing the assembly for the whole system and that element level we are restrict to local coordinate system because it is easier, we do not have to track coordinator of each element with respect to the origin that we can save that information in our crowd code somewhere else but we when we are doing integrations and all that the only thing we are worried about is the length of the element h_e, okay. So it makes mathematics, mathematics simpler and easier and faster.

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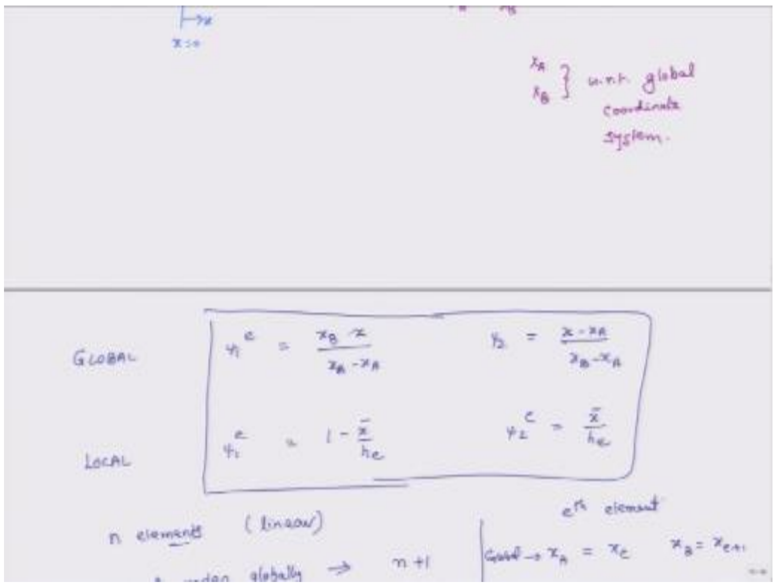

Couple of other points these elements, these functions whether they are in local or global we derive these functions based on what? Considerations related to displacements only, we did not derive them based on considerations related to derivatives of the displacement, we only consider displacements, right? We did not consider derivatives of displacement.

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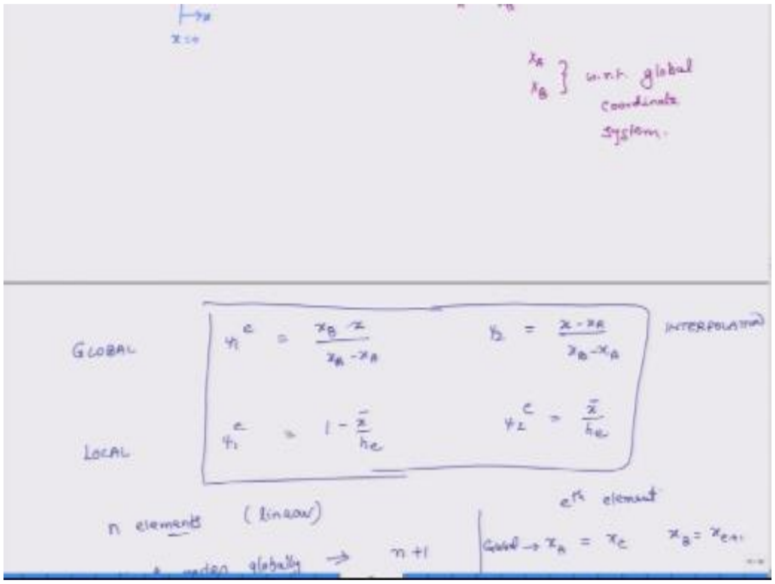

So such functions, so first thing is that these are known as interpolation functions. In general we have been using the term approximation functions, here we use these functions to interpolate the value of displacement between two nodes, that is why they are known as also inter we can use the term approximation function but in context of f_e^a at element level they are more popularly known as interpolation functions.

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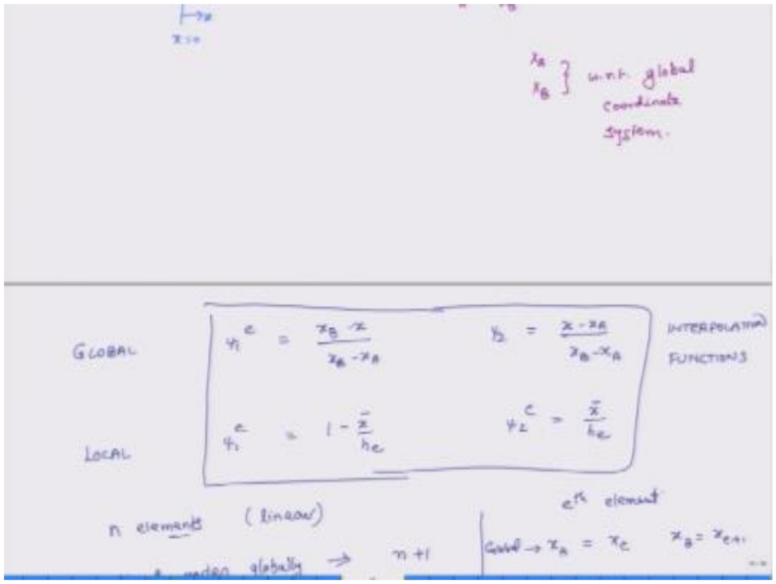

And because these functions where derived based on considerations of only the primary variable not its derivatives.

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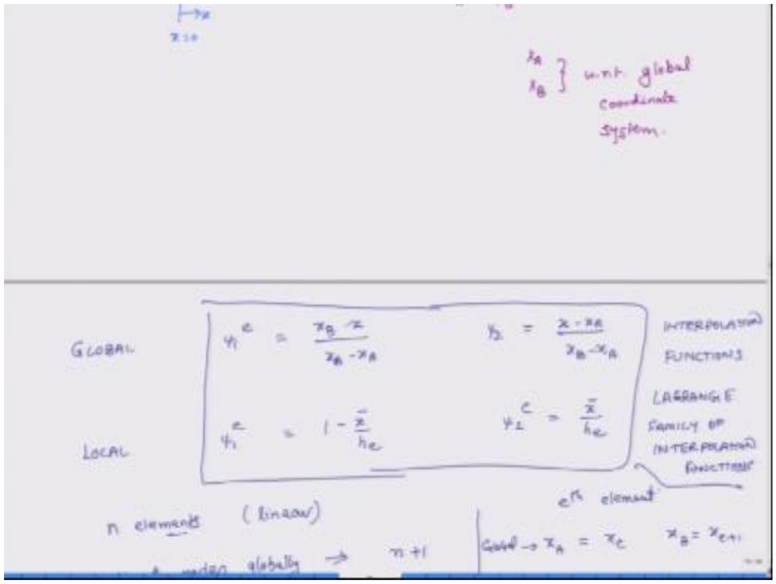

They are known as that they belong to Lagrange family, Lagrange, Lagrange family of interpolation functions, okay. Here the consideration was only primary this un variable not its derivatives, okay.

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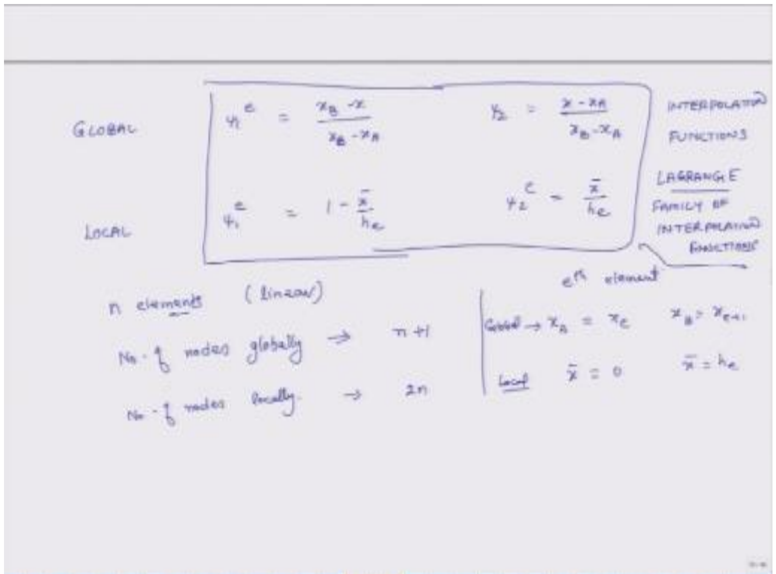

So these could be Lagrange family of interpolation functions they could be linear in nature, quadratic, cubic and so on and so forth. So these functions are very popular in second order differential equations, okay. Because in second order differential equations when we weakened the continuity we need only first-order derivatives c_0 continuity is only needed, okay.

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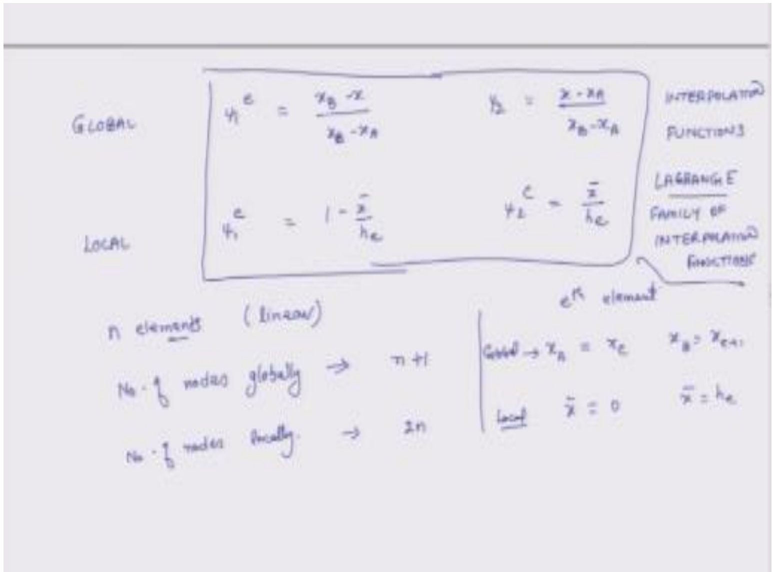

Then there are functions, there is another class of functions known as hermetic.

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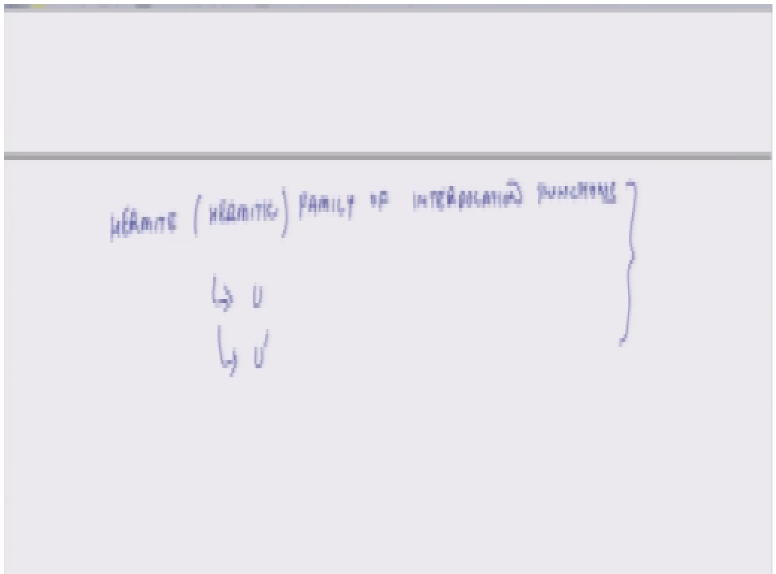

Hermit or hermitic. Family of Hermitic family of interpolation functions and these functions are developed not only on considerations of primary variable but also its derivatives. These types of functions are very popular when you are also trying to solve fourth order equations for instance b, where you are not only worried about the deflection but also about slopes for instance slope at x = 0 and a cantilever.

Fixed end is zero, right? So you are not only interested in ensuring that the function itself is continuous over the entire domain but you also want the slope should be continuous over the domain, okay. Slope has to be continuous over the entire domain.

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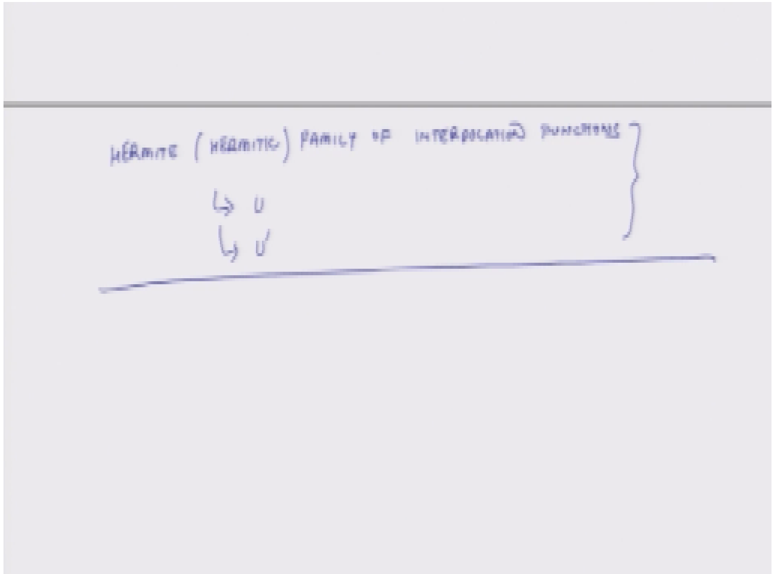

So those are hermetic family of interpolation functions, we will look at these functions maybe next week, but today and this week we will only work on Lagrange family of interpolation functions, okay.

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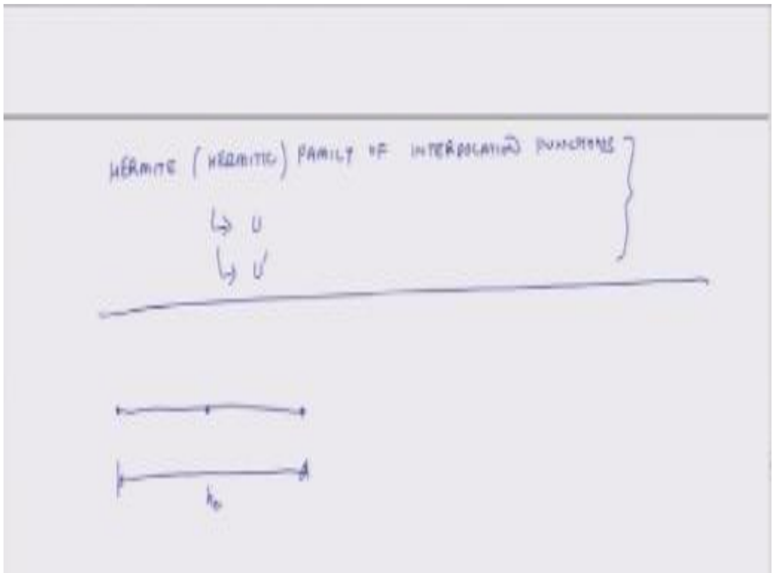

Suppose our element was not linear but it was a quadratic element. Let us say its length is h_e of e^{th} element. So we have looked at linear elements, right? Let us say now in and in other case we want that our interpolation function or shape function is quadratic in nature which means that u varies quadratically over the length of each element. Suppose we want to make that assumption.

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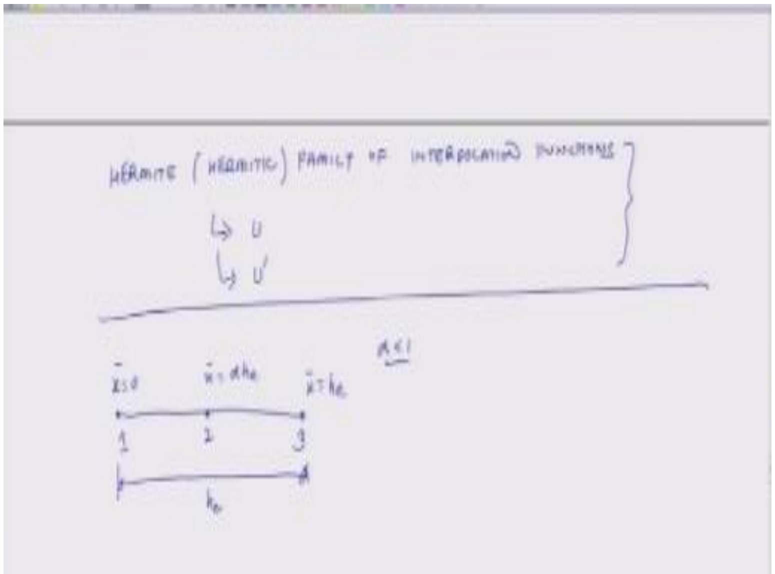

So my first node, so it will have three nodes, first node is location in global local coordinate system will be 0, x- bar equals zero. So this is node one, this is node three, this is two, overall length is $h_{e.}$ So this is x –bar is equal to h_e and let us say the location of second point is at x – bar equals αh_e where α is less than 1, some fraction, if it is perfectly located at the middle then α will be half.

So then how will you derive the interpolation functions? So we will develop these interpolation functions and because this is a quadratic element it has three nodes and the interpolation functions will be three number ø1, ø2, ø3 and we want to know what are the exact functions.

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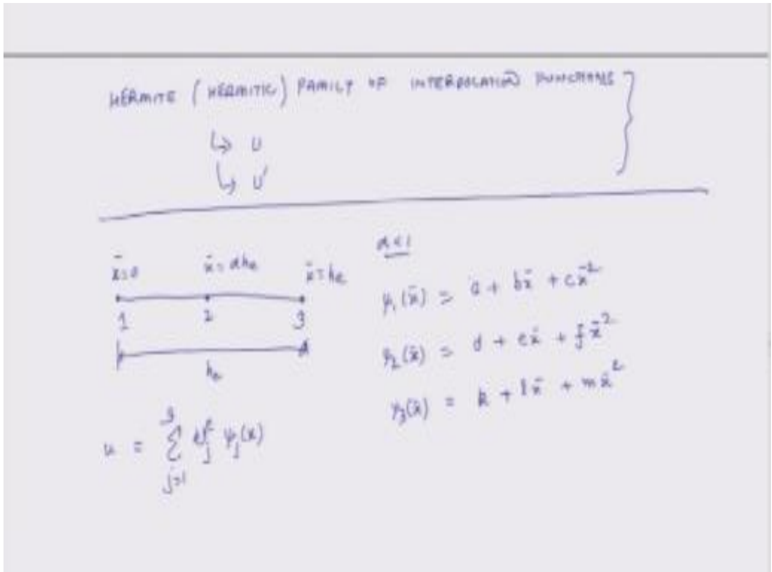

So we just assume a general form ø1x equals a + bx + cx^2 this is the first approximation function,ø2x is equal to d + ex. So I have to put a bar here because now I am in local coordinate system plus Fx- bar square and ø3x. So again there is bar here, this equals f, g, h, I, I will use k+ l x –bar+ m x bar^2. So we do not know what are the values of A, B, C D, E, F, K , L ,M, right? And we have to figure it out.

But we also know that the value of u over the element it varies and it can be written as u^e_j ψj(x), j is equal to 1 to 3, right? Because there are three functions, so very much similar to what we did in the last case we will develop conditions.

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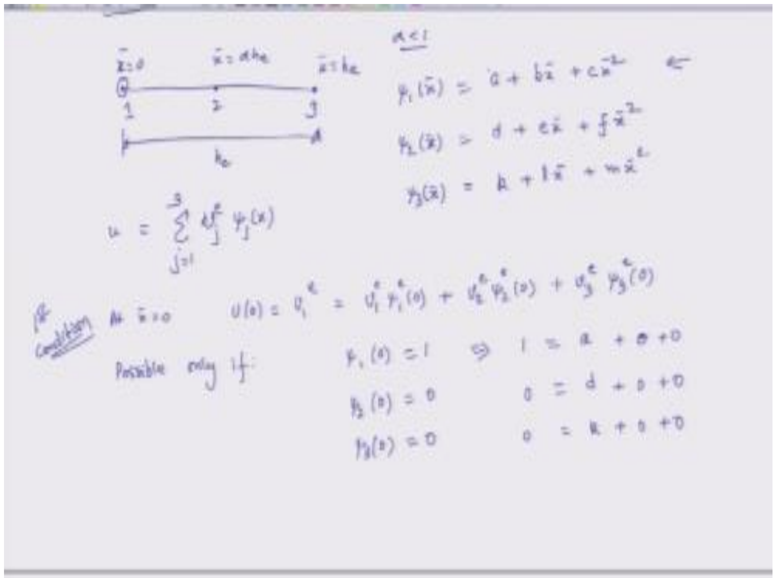

We have to develop nine different equations so that we can find these nine unknowns A, B, C, D, E, F, K, L, M, okay. So the first condition is that at x = 0 or x- bar equal zero which is this location node one, what is the displacement or the what is the value of primary variable ? u1, u_1^e, $u_0 = u_1^e$ and this is equal to u1 ψ (0) + u2ψ2(0) + u3ψ3 evaluate it at zero. And of course I have to put a superscript e.

For the e^{th} element suppose, okay? Now this relation is valid for all values off u’s it does not matter what values of u1, u2, u3 are but it is valid for all values of u’s. So I get three different relations, so this is possible only if ψ1 (0) equals one, ψ2(0) equals zero, ψ3(0) equals zero which means one equals A and in this case I am going to put in this equation ψ1 this is the relation for ψ1.

So this is equal to A+B x- bar, x- bar is, x- bar is local and the value of x - bar is zero. So 0+0, second equation is zero equals d + 0 + 0, third equation is 0 k+0 +0, okay. So I right away get three unknowns. Then I do the, so this is first condition. What is the second condition?

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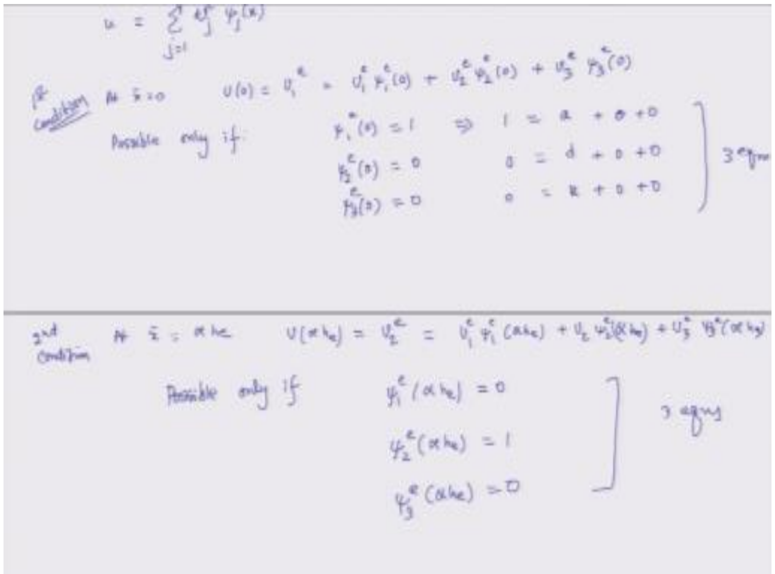

That at x is equal to or x- bar equals α time h_e , this is the second condition. $U(\alpha\ h_e) = u_2^e$ and this is equal to $u1\psi1(\alpha\ h_e) + u2\psi2(\alpha\ h_e) + u3\psi3(\alpha\ h_e)$, and again I have to put superscripts everywhere. So my second this is going to be possible only if $\psi1^e(\alpha\ h_e)$ when I calculate that is equal to 0right? So $\psi2^e(\alpha\ h_e)$ is equal to one and $\psi3^e(\alpha\ h_e)$ equals zero.

So these are the second three conditions. So I have three equations here, I have three equations here and similarly I will have a third condition, that at the third node I will get three more equations. So in this way I get 9 equations. 9 unknowns I can solve them and I eventually what I get is.

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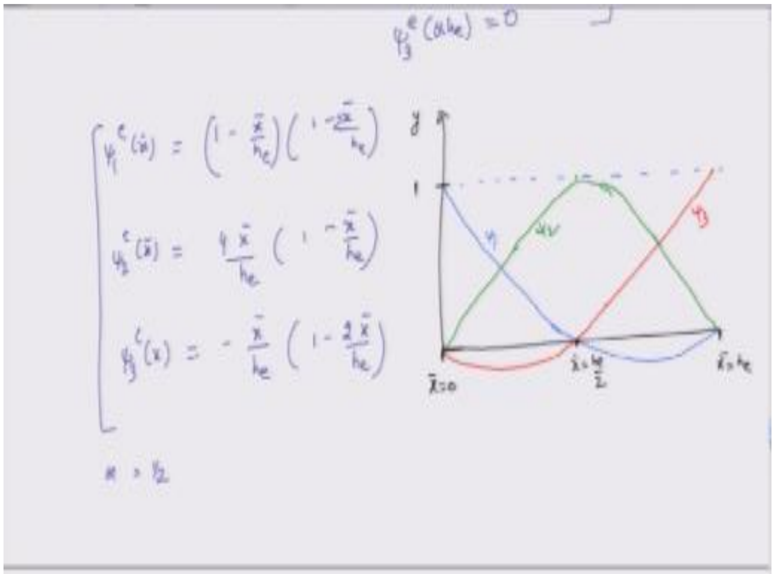

That $\psi 1^{e}$ which is a function of x- bar equals 1 minus x - bar over h times 1 minus x – bar over (αh_e) you have to check that, okay. $\psi 2^{e}$ of x- bar equals four x- bar over h_e 1 minus x- bar over h_e and $\psi 3(x)$ is equal to minus x- bar over h_e into 1 minus 2 x- bar by $h_{e,}$ and I think these relations are valid. So there is a 2 here when $\alpha = 1/2$ if the midpoint is located at the centre, okay. And if you plot these, suppose here this is x- bar equals zero.

x- bar equals h_e here x- bar equals h_e over 2 and your y – axis let us say y- axis and this is one, then your first function $\psi 1$ varies like this, $\psi 2$ it varies like this, and $\psi 3$ will vary like that. So that is my $\psi 3$, this is my $\psi 2$. So once again you see that these shape functions, these shape functions are having a value of unity at specific nodes and at other nodes they are zero, at other nodes they are zero.

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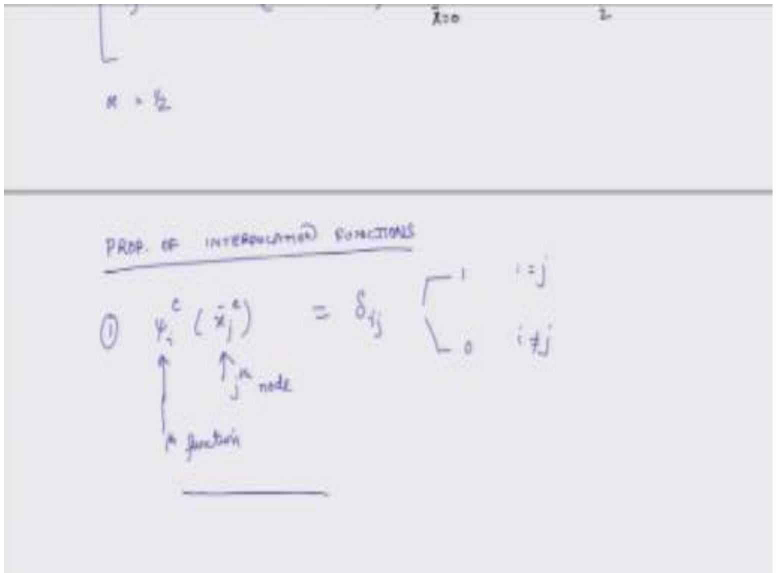

So very quickly we will cover these properties. The first thing is that the value of ψi at location j, j^{th} node. So this is the coordinate of jth node. This is ith function, okay. So the value of ith function at jth node is equal to δ ij where δ equals1if i = j and it is zero if i does not equal j. This is the important thing. So this concludes our discussion of interpolation functions and in the next class what we will do is we will use these interpolation functions.

To develop element level equations and once we have developed those element level equations then we will actually assemble those equations. So now we will start looking at all the nitty gritty details of $F_{A.}$ So thank you very much, and we look forward to meeting tomorrow.

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