Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 22
FEA formulation for 2nd order BVP
Part - I

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Hello, welcome to basics of finite element analysis, yesterday we had concluded our discussion on weighted residual methods. And in the discussion on weighted residual methods, and also which we saw earlier in the last week on Rayleigh Ritz methods we found that the trick in getting a good solution lies in making a good guess of what the approximation function should be like. And the requirement for choosing the right approximation function, we had discussed was that it should meet all the essential boundary conditions in case of Rayleigh Ritz method and it should meet all the natural as well as essential boundary conditions.

In case of weighted residual methods, and it is not always easy to make these guesses, now in context of the problems which we have solved these are relatively simple problems one-dimensional problems, but if problem becomes two-dimensional and the domain is not just a nice rectangle, square or circle, but if it is a complicated two-dimensional shape then making such choices becomes extremely difficult.

So because of that so that is one thing and the second thing is that there is no standard method of making these choices, first thing is we have to make a guess and also in this guesswork there is no standard approach or something like that. So most of the real-life problems remain unsolved, because in terms of if, if we have to use these methods to find solutions for most of the real life problems. So with that context in mind people have developed the finite element method, and in context of finite element method a couple of weeks back we had said, that the process of developing a finite element method starts with first you discretized the whole domain into small, small elements, second you take small you take approximation functions which represent the displacement or the unknown dependent variable, how it varies over the element and then use either the residue or the weighted weakened residue, and you vacate it to zero over the element, and as a consequence you get element level equations, and then as a follow-up you assemble these assemble element level equations to get the overall assemble level equations, at that stage we apply the boundary conditions and we solve for the dependent variable.

Today’s class and next several lectures we will start this process and we will start understanding, how the, this finite element method works at a detailed level, and what we are going to do is that we are going to start in this direction by addressing.

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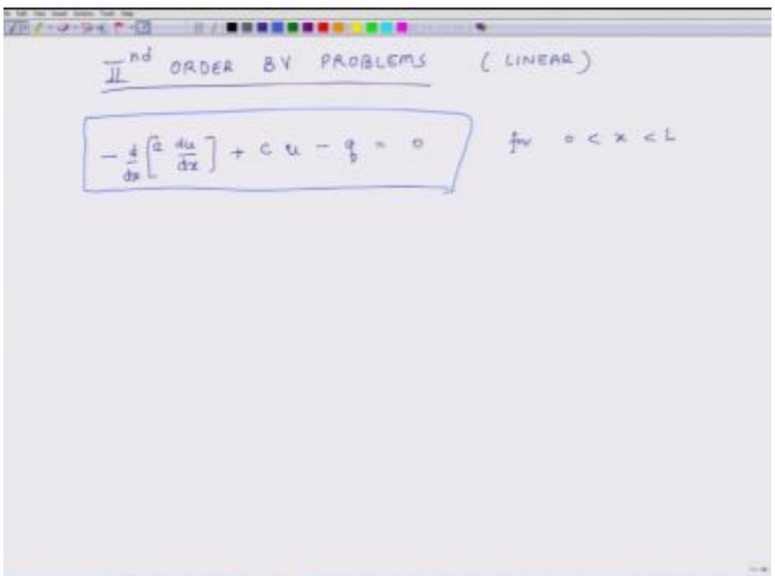

Second order boundary value problems, second order boundary value problems, and the problems which we will deal will be linear in nature. So one example of this boundary value problem and this is what we are going to solve for, is a problem represented by this differential equation, so this is the differential equation and it is valid for which is between zero and L. So our domain is zero to L, in one of our earlier lectures we had explained that this particular problem, this particular differential equation, it represents a large, large number of classes of problems, starting from a tension in a hanging string, electrostatics, bar under compression, one-dimensional heat conduction, diffusion in one dimension, and so on and so forth. So this is the problem we will learn FEA on as a first example, and what we will do is that the first step we will do.

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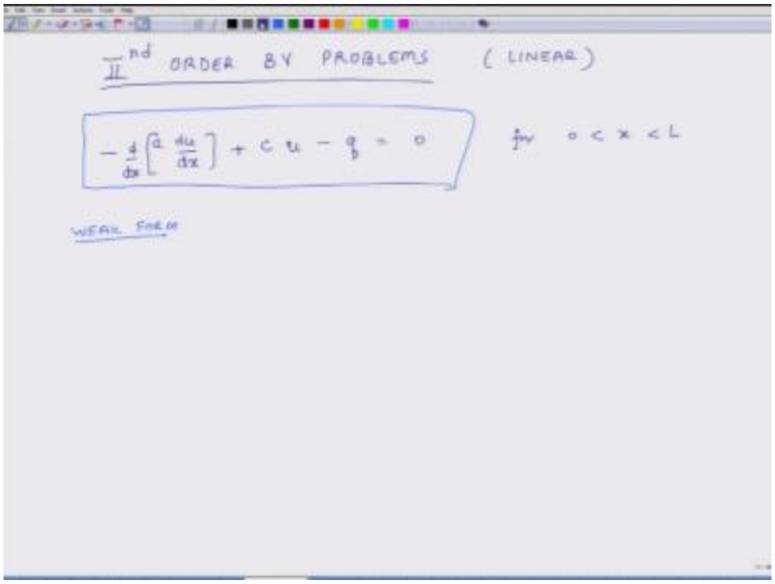

Is we will generate the weak form of this equation okay, and in this case we will generate the weak form on an element by element basis. So we are not going to generate the weak form for the whole domain we will, we will create a weak form which will be applicable to a small element. So suppose this is my entire domain.

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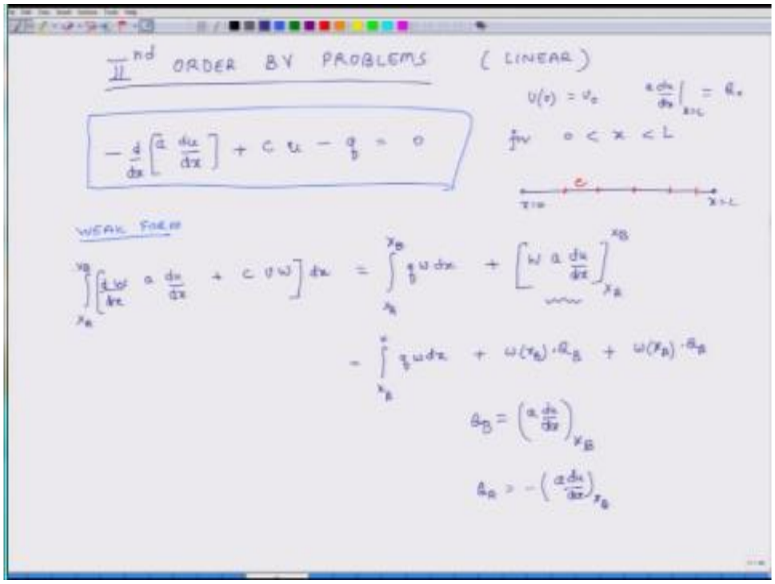

x is equal to zero, to x is equal to L, I will break this up into small elements and for the e^{th} element, I will write down the weak form okay, for the e^{th} element I am going to write down the weak form. So my domain is not zero to L in this case, but rather it is from X_A to X_B, where A X_A is the first coordinator of the e^{th} element, X_B is the second coordinator of the e^{th} element and w is the weight function, a du over dx + c uw i integrated, and this equals integral from X_A to X_B q of q times w times dx+ w a du over dx, X_A to $X_{B.}$

The other thing we will add here is that this is the governing differential equation and I will say that I know the boundary conditions for this problem and the boundary conditions are that at u at zero is equal to u_0 and the value of a du over dx at x=L is equal to sum or number q_0 okay. So these are the boundary conditions okay. So this term I will expand it, so this is my X_A to X_B, I am just rewriting the left, the right hand side of this equation again.

And then I have a term w which is valuated at location X_B times Q_B which is the value of a du over dx valuated at x=B and here note this, the other term also has a positive sign and I will explain that in a moment and that is w evaluated AT X_A times Q_A and the reason I am going to put a positive sign here is because I am defining Q_B as a du over dx at X_B, and I am defining Q_A

as minus of a du over dx X_A, if I had not defined it with this minus sign then this positive sign would have been replaced by a negative sign okay.

So this is something important to note, because what it does is it keeps our book keeping a little easier that is all. The other point to note here is that the domain here is not the length, the overall length which is zero to L, but it is.

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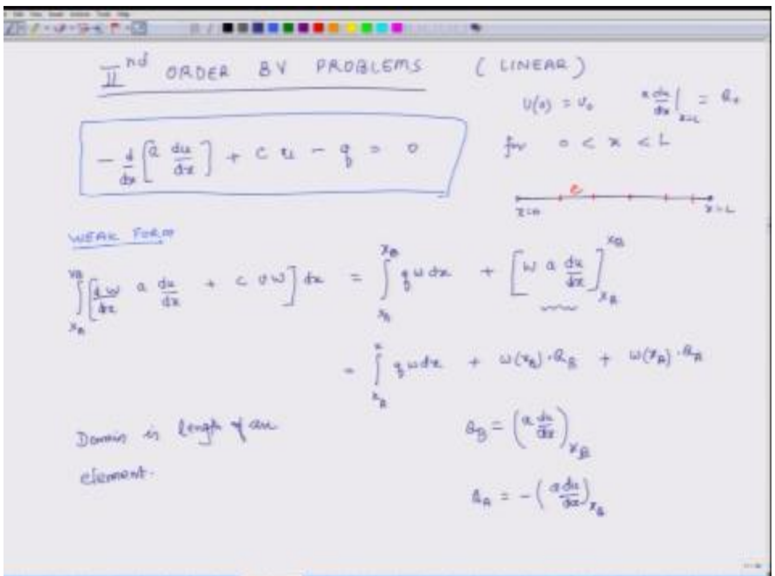

The domain of integral is the length of an element, so if there are 20 elements in the domain then I will write twenty different equations like these, the next step so what have I done I have started with a differential equation, I have discretized my system into different elements right, that is the first step which I did, I broke it up into all the elements.

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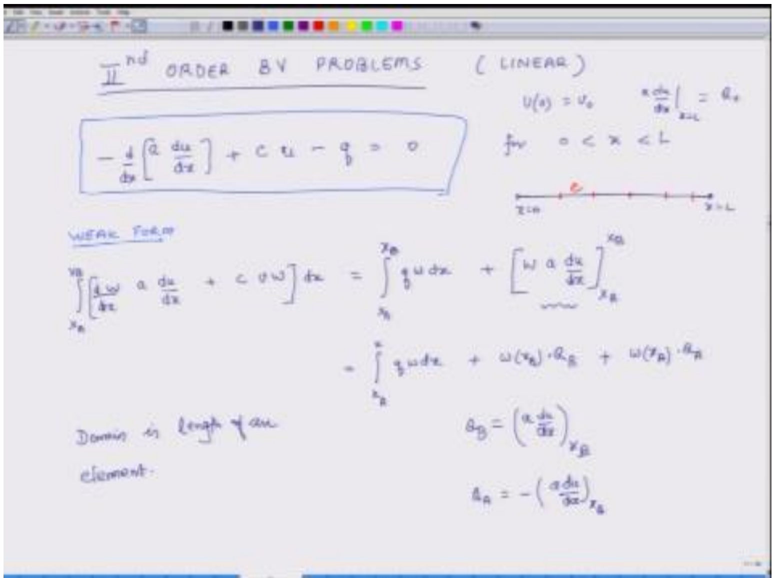

Then for each element I have written the weak form, for each element I have written the weak form. The next step is that for eth element, I will assume next step.

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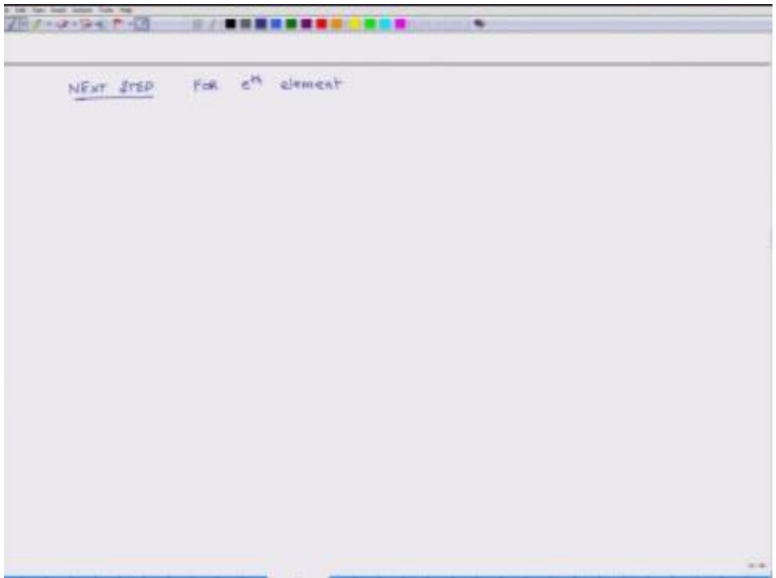

For e^{th} element I have to do what?

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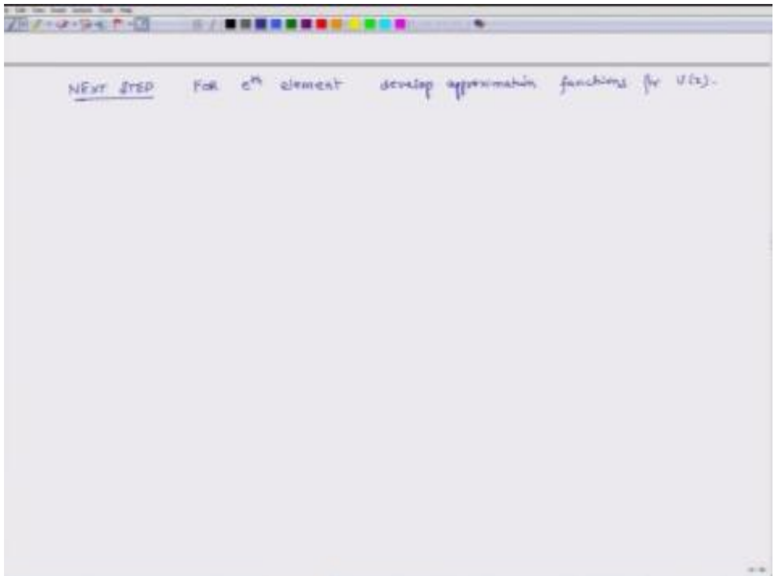

Develop approximation functions, develop approximation functions for u(x) okay, if it is a linear element the element has two nodes then I will develop two approximation functions, if it is a cubic element u is going to vary, if it is a quadratic element u is going to vary in a quadratic way over the, over the length of the element and in that case I will have three nodes on the element, if it is a cubic element u will vary cubically over the length of element and I will have four nodes.

So what we are going to develop?

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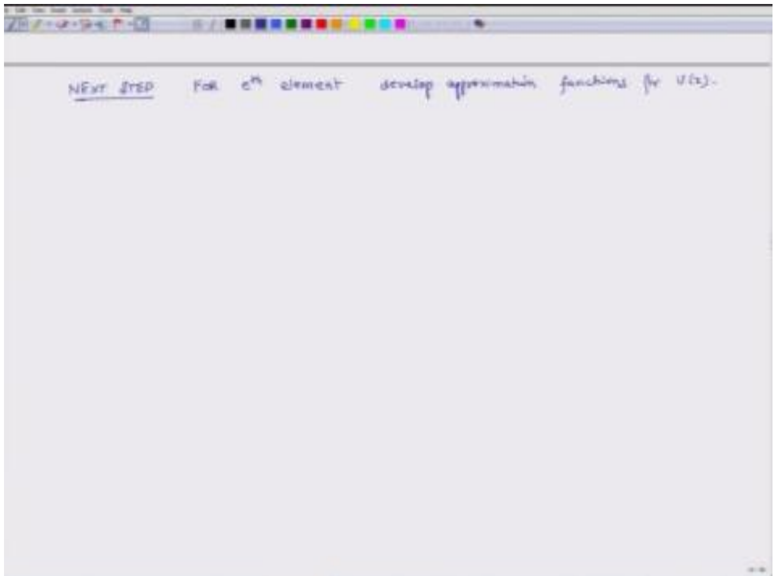

Is an approximation function, for a linear element for illustration purposes, then we will generalize it for other situations also.

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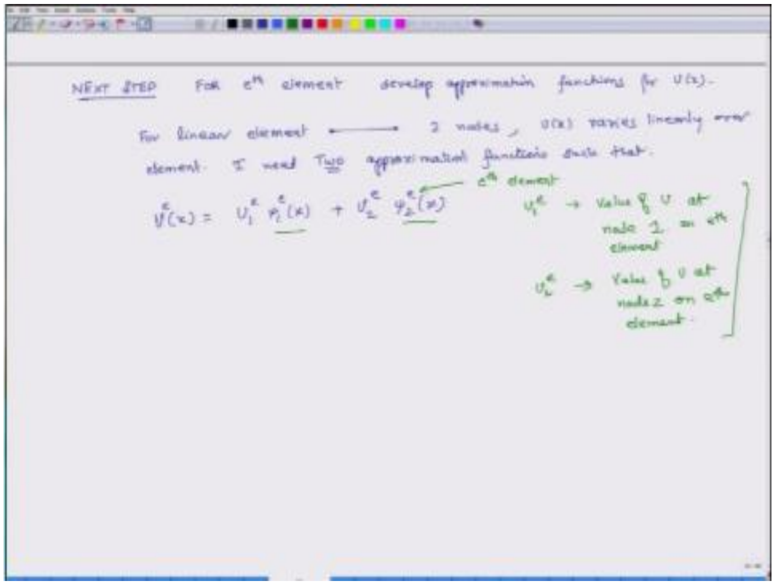

So for linear element okay, I have two nodes and u(x) varies linearly over element, so I and then because it is there are two nodes I need two approximation functions, such that u(x) is equal to u_1^e ψ_1(x)+$u_2^e\psi_2$(x), so this super script represents eth element, super script represents eth element, u_1^e is the value of u at node 1, u_2^e on eth element, and u_2^e is value of u at node 2 on eth element okay. At this stage we do not know what is the nature of,

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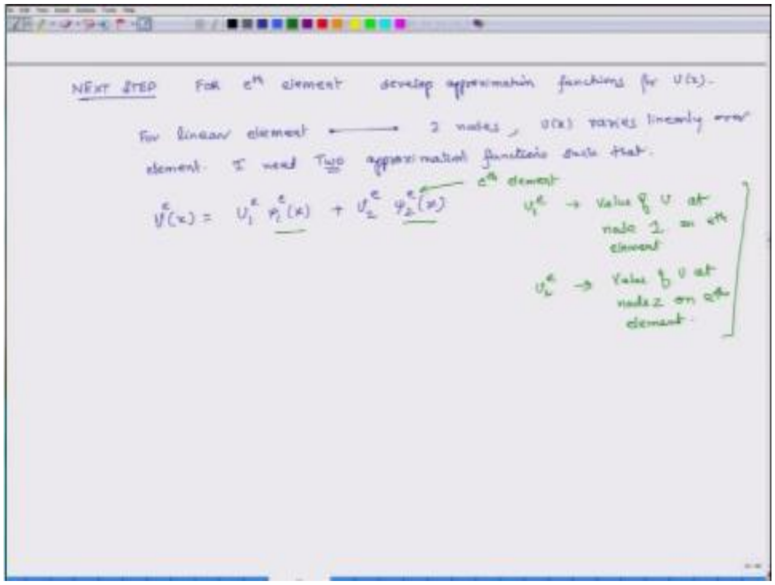

ψ_1 and ψ_2 so we have to figure out, now because it is a linear element ψ_1 can only be a linear function, and because it is a linear element ψ_2 only also can be only a linear function okay.

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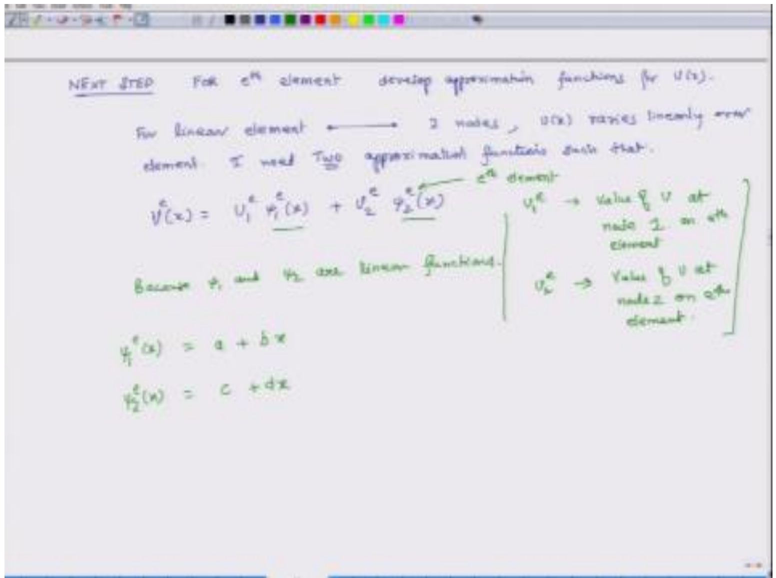

So because, ψ_1 and ψ_2 are linear functions I can in general write down, $\psi_1^e(x)$ as a+ bx right, it is a linear function and similarly $\psi_2(x)$ I can write down as c + dx, so what is it that we are doing right now? We are trying to figure out what kind of approximation functions we should have for the e^{th} element, for the e^{th} element, and we do not know the values so here.

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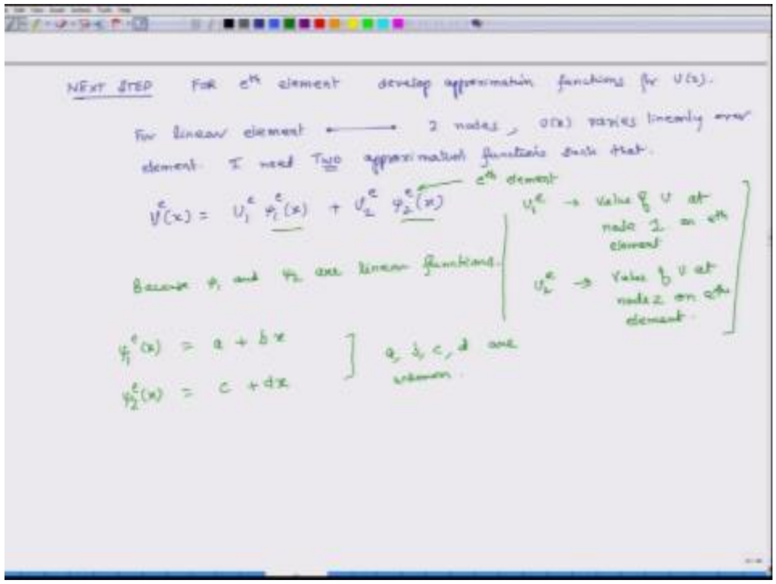

a, b, c, d are unknown, so our aim is to figure out the values of a, b, c, d okay, we know that the value of u at first node is u_1^e, and we know that the value of u at the second node is u_2^e it may be, I mean we do not exactly know, but some general number u_1^e and $u2^e$, so in terms of u_1^e and $u2^e$

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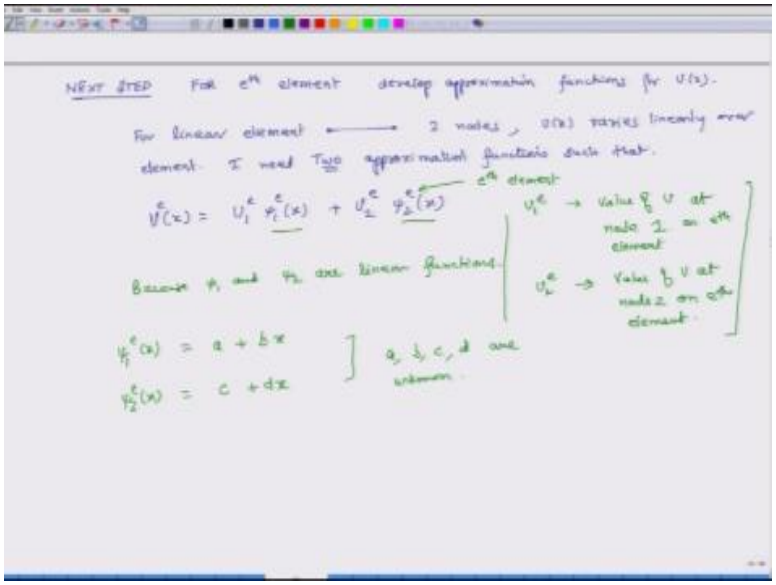

We will try to infer, using this information we will try to infer what are the values of these constants a, b, c, and d, okay.

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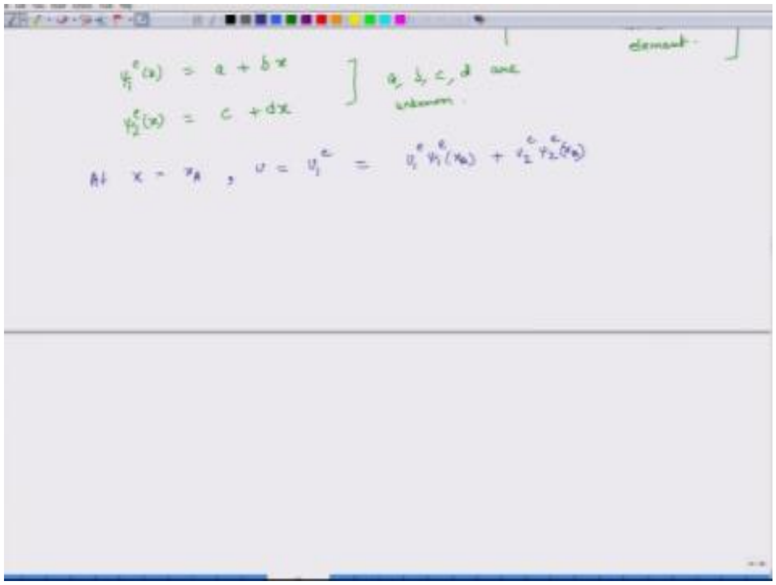

So at $x = x_A$, x_A is the first node of the element, u is equal to u_1^e right, and we know that u is what $u_{1,}$ $\psi_1(X_A)$ e $+u_2$ $\psi_2(X_A)$ e agreed, u is equal to u_1^e this is, this is how we have defined u_1^e and we have also defined u as, u as i summation of that, so if I have to calculate the value of u at X_A, I am using that.

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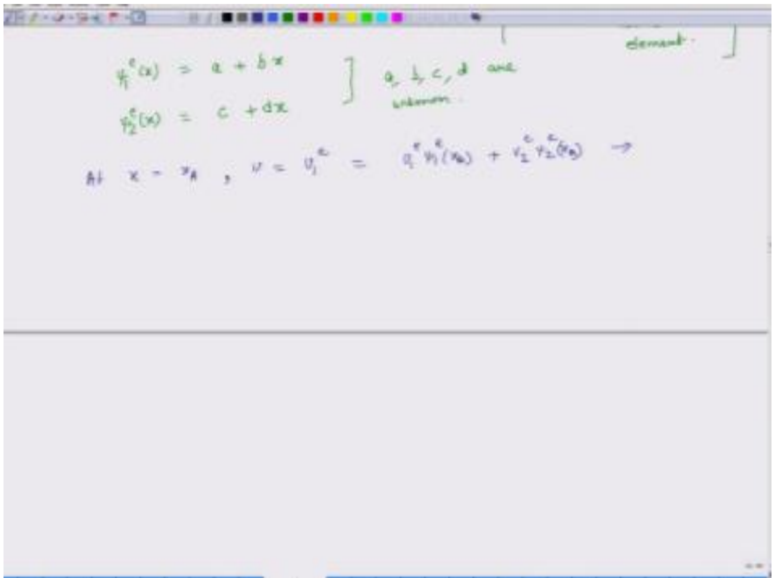

Relation and I am plugging in x is equal to X_A agreed, so this is there, now this relation is true for any value of X_A you know, the node could be located here at this location or it could be, because this is e^{th} element so it is valid for all values of X_A and it has to be valid for, it should not depend on values of u_1 and u_2 also, u1 could be 0, then also this equation should hold true if u1is something else, then also this equation has to hold true in all situations.

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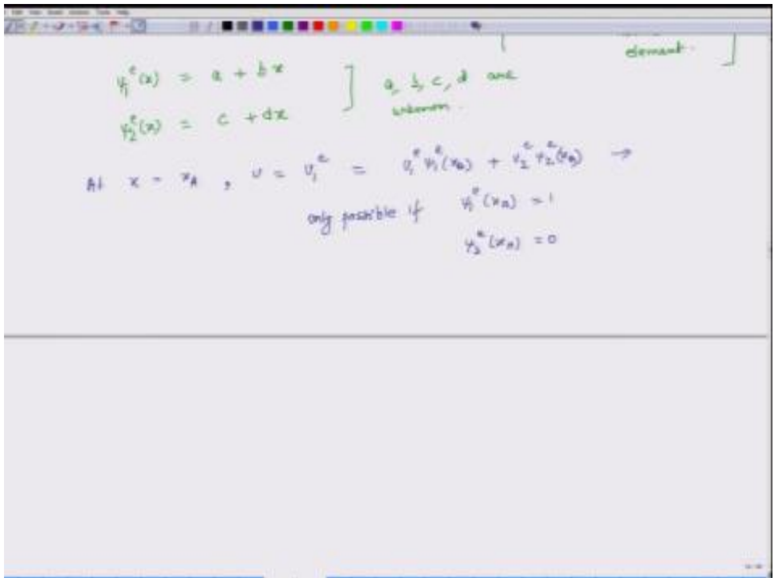

So this is going to be possible, only possible if the value of ψ_1 function, at X_A a is equal to one and the value of ψ_2 function at X_A is equal to 0, only then this will be possible for all values of u's and all values of right, because if, if these two conditions are met.

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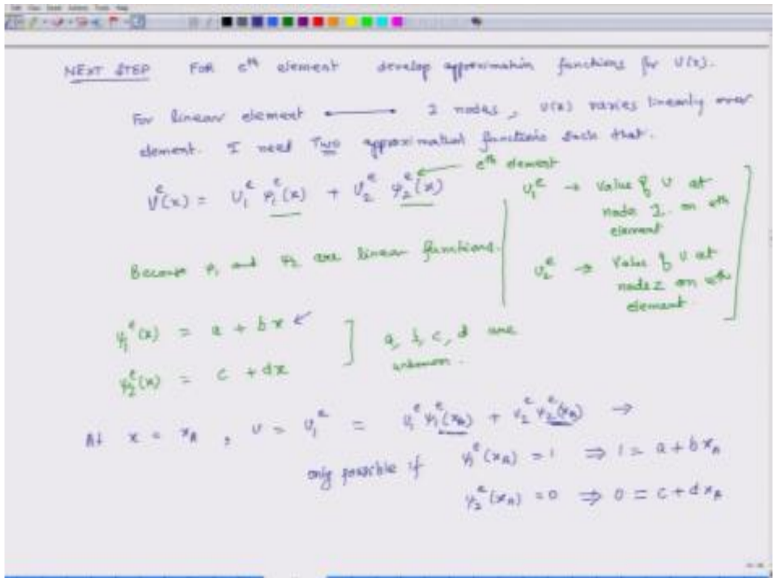

Then this will be one, this will be one, and this will be zero, so it u_1 will be equal to u_1 so it will be identically satisfied right. So from this what do we get, so now go back, ψ_1 is equal to a+bx, ψ_1 is equal to a+bx, and what this means is one equals, I am going to plug in x is equal to X_A in this relation, is equal to a+b times X_A agreed, and here I have do the same thing for the relation ψ_2 c + dx, so this is equal to zero, is equal to c + d X_A right.

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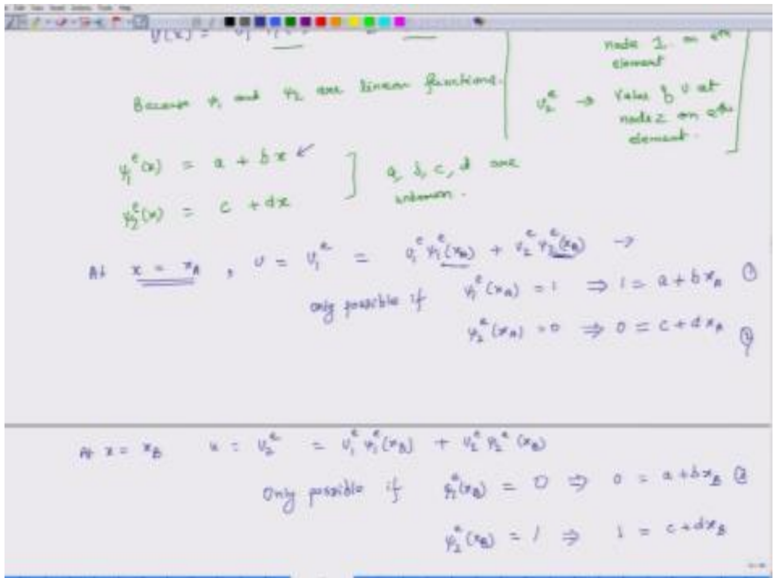

Then so this is the condition at x is equal to $X_{A,}$ at x is equal to $X_{B,}$ at x is equal to X_B what is the displacement, u_2^e, u is equal to u_2^e agreed, and this is equal to u_1 ψ_1 and I am, I am computing this function at $X_B + u_2^e\psi_2^e$ and I am computing function at X_B, again this is only possible if ψ_1 when I compute ψ_1 at X_B this is equal to what 0, and ψ_2 when I computed at location X_B this is one. So again this gives me another relation is equal to 0, is equal to a + b X_B and this gives me one equals c+ d X_B okay, this is equation 1, this is equation two, this is equation three, this is equation four.

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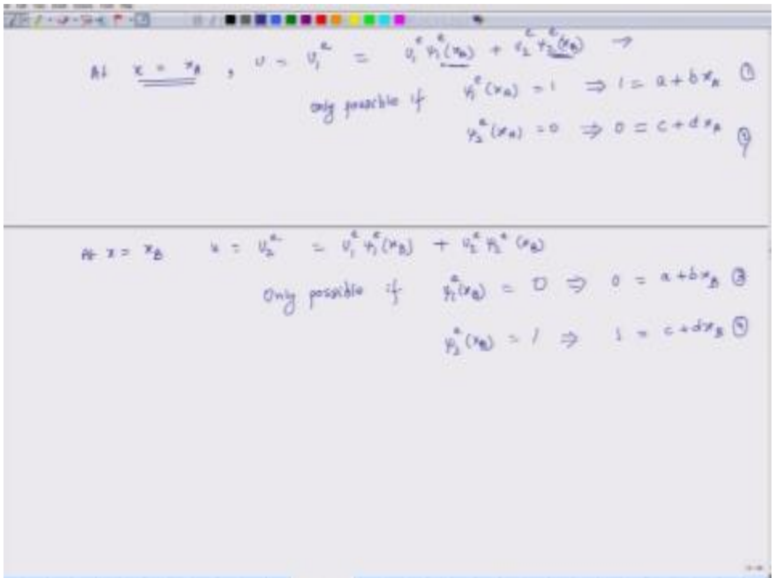

I have four equations, and four unknowns a, b, c, d. So I can solve for a, b, c, d, right, I can solve for a, b, c, d, and I can plug them back in.

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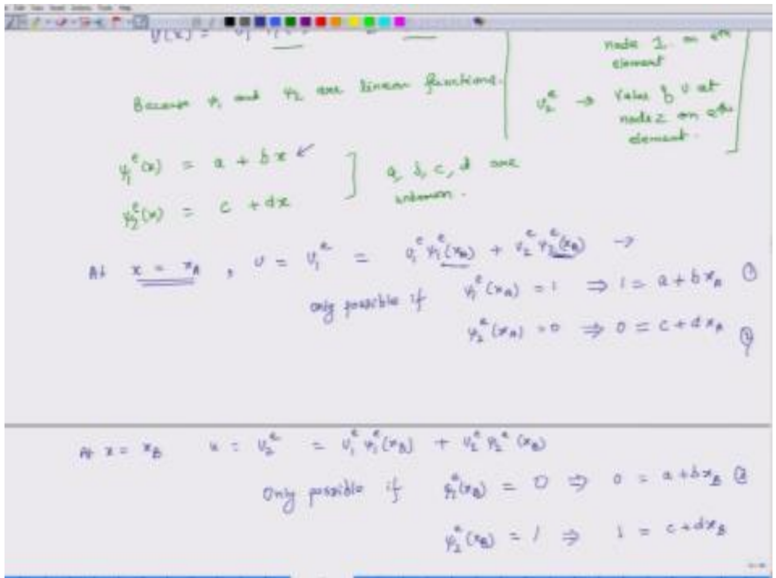

These relations, and once I plug them back in relations I get the mathematical expression for ψ_1 and ψ_2 agreed. So my ψ_1 function,

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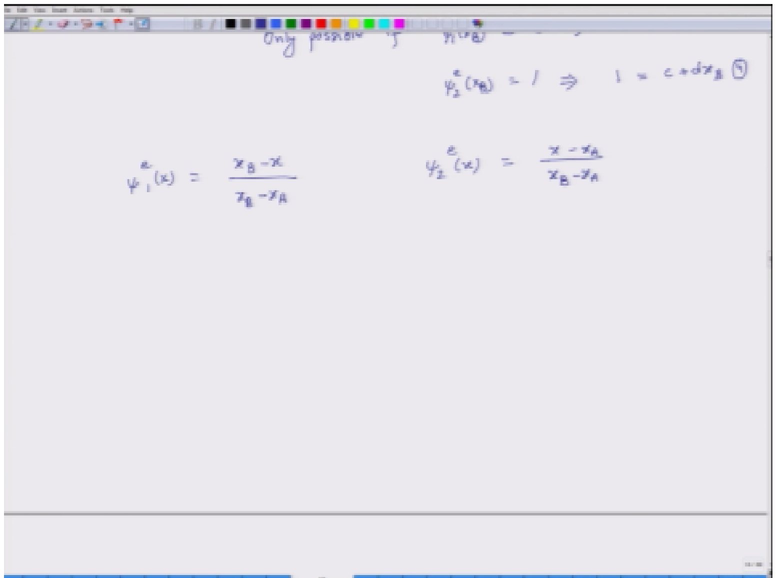

So using this method what I get is ψ_1 which is, which depends on x is equal to x_B-x divided by x_B-x_A, and my ψ_2 function is equal to x- x_A divided by x_B-x_A, so these are my ψ_1 and ψ_2 function for which element, e[th] element so e is general so it is does not matter they are well at this relation as long as I know x_B and x_A in my initial coordinate and my final coordinate I have the ψ functions for all elements okay, I have the ψ function for all elements. So now that I know this,

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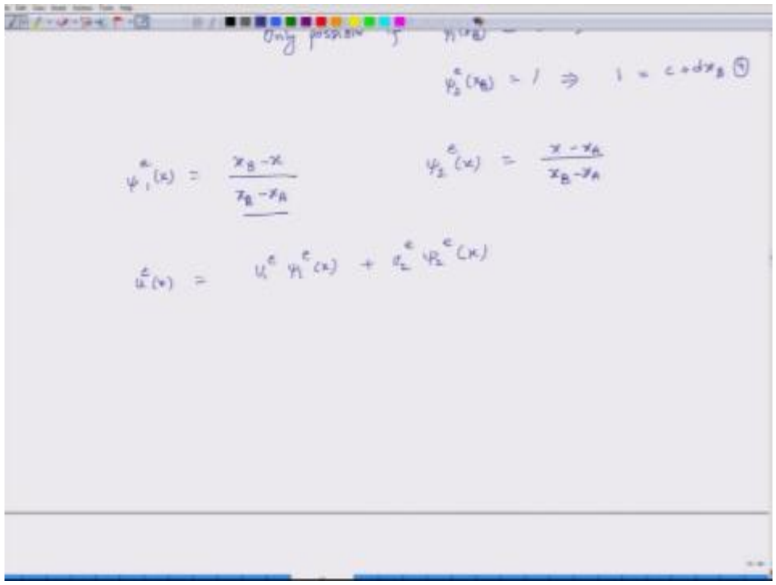

My u for the e^{th} element which is the function of x is equal to $u_1^e \psi_1^e(x) + u_2^e \psi_2^e(x)$ okay. So we will conclude this discussion here, and we will continue this discussion on interpolation functions or approximation functions in the next lecture also. Thanks.

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