Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 19
Method of Weighted Residuals

by
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Hello welcome to basics of finite element analysis, today is the beginning of the fourth week of the series of lectures, and in the last lecture we had discussed the weak formulation, weak formulation type of approximation method and specifically we had discussed the Rayleigh Ritz method where the choice of our weight function was same as the approximation functions which we used while representing the dependent variable.

So today we will be discussing a different category of approximation solution approach, and this is known as method of weighted residuals.

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So in this particular method what we do is we first calculate the residue multiplied by a weight function, integrate this weighted residual over the domain of the problem and equate it to zero. What we do not do in this case is, as we saw in a case of Rayleigh Ritz method is that we do not weaken the differentiability. So this is one very fundamental difference.

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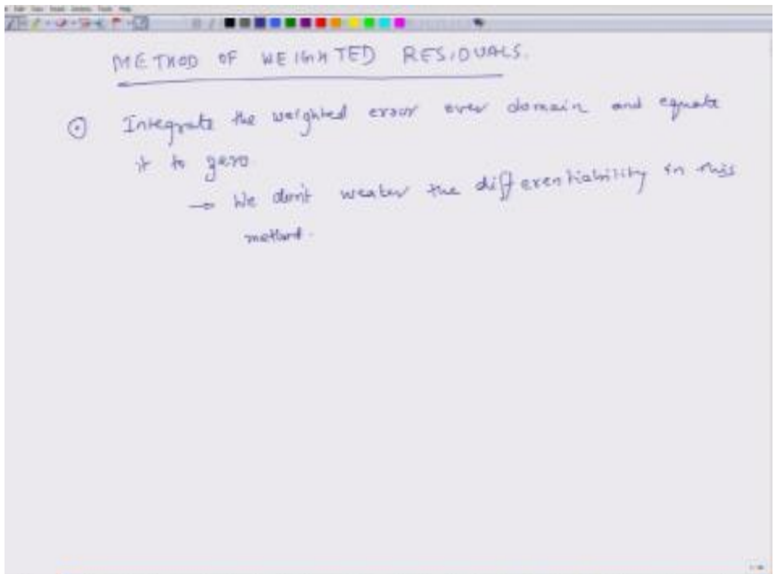

So, so the first thing is that we integrate the weighted error over domain and equate it to zero. We do not weaken the differentiability in this method so this is one very important difference, what this directly means is that in Rayleigh Ritz approach when because we have a weakened differentiability, when we do this weakening of the differentiability and shift the differential operator from u which is the dependent variable to w which is the weight function, then some boundary terms come as in this process right.

The secondary variables or the national boundary terms come in this process, so when we are trying to enforce these boundary conditions related to natural boundary conditions or secondary variables we do not have to worry too much about that because they automatically come through the mathematics. In this case because we are not weakening the differentiability

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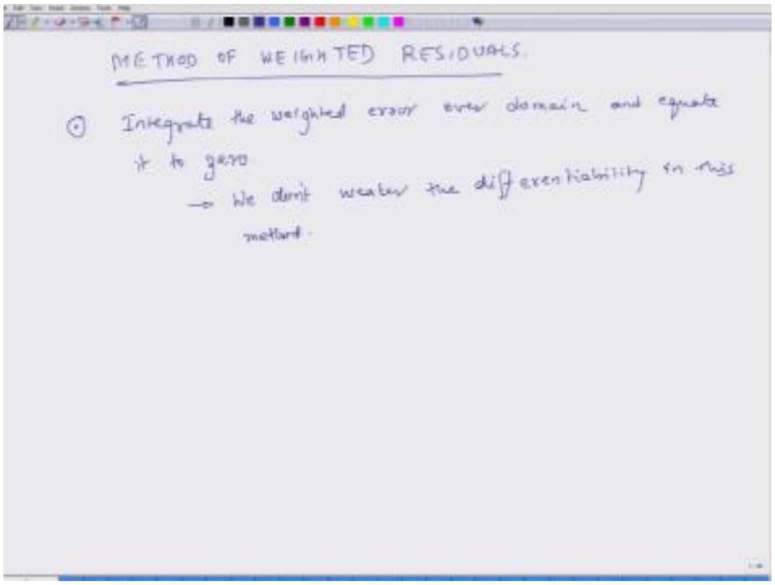

So, these secondary variables do not emerge in the integral explicitly. What that means directly is that when we choose the approximation functions in this particular weighted residual approach, these approximation functions not only have to satisfy all the essential boundary conditions but they should also satisfy all the natural boundary conditions, this very important.

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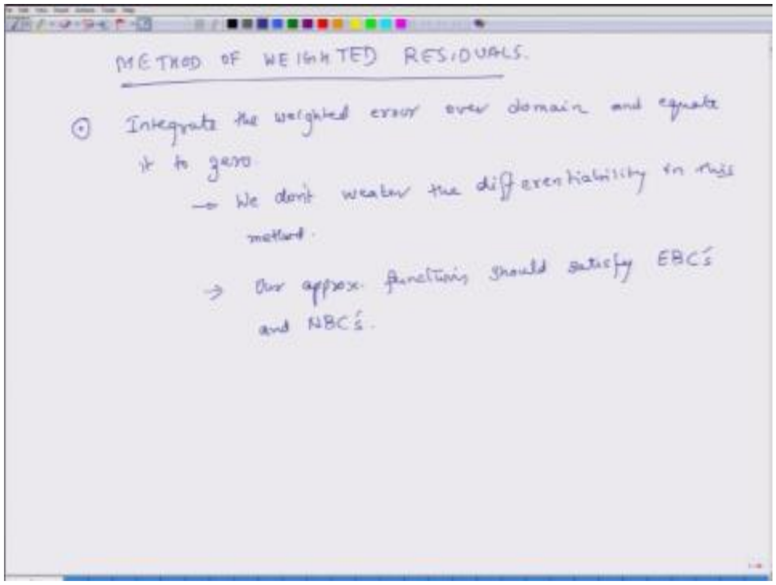

So our approximation functions should satisfy both essential boundary conditions and natural boundary conditions. In the Rayleigh Ritz approach we did not have to worry about the natural boundary conditions because they were getting automatically satisfied, here that is not the case. So what this means is that if you want to use the weighted residual approach we have a stricter choice in you know stricter rules in terms of what kind of functions should we be picking. In Rayleigh Ritz approach the choice was pretty flexible, so this is a very important condition.

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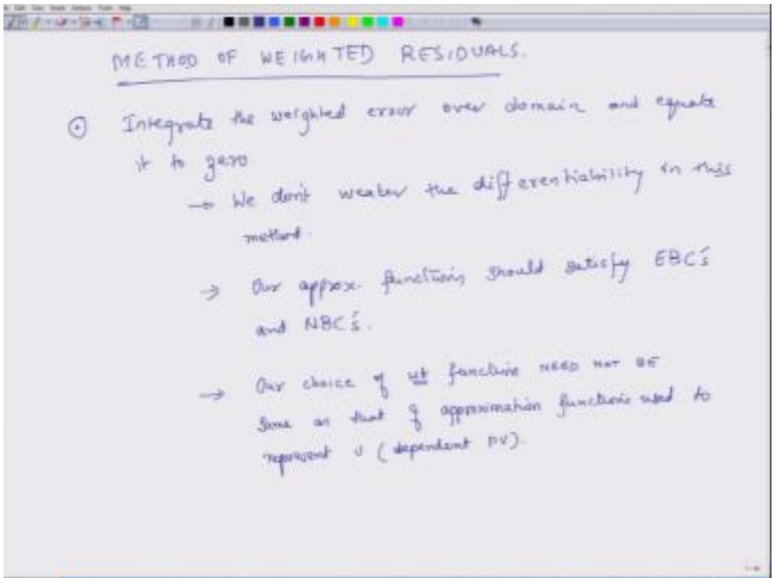

The third thing is that our choice of weight function, weight function need not be same as that of approximation functions used to represent u which in this case is the dependent primary variable, this need not be and depending on what kind of function we choose that particular weighted residual approach has a different name. So there are different weighted residual approaches, there different weighted residual approaches and for instance there is a Petrov Galerkin method and that if we make a particular choice for the weight function then it is called a Petrov Galerkin method.

If we make another choice of weight function then it is known as a Galerkin method. Then there is a method known as method of least squares, there we choose a different weight function. So there are different types of weight functions.

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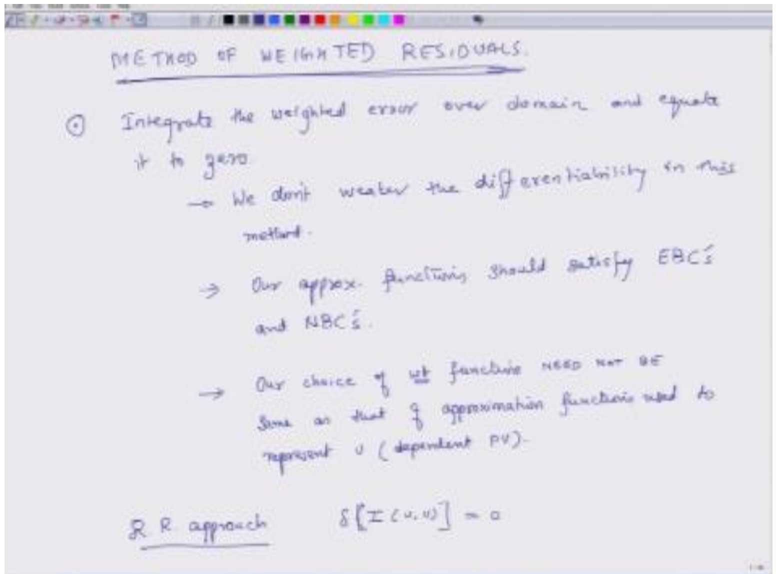

The advantage of this weighted residual says in our Rayleigh Ritz approach what we had shown was that effectively in Rayleigh Ritz approach, in Rayleigh Ritz approach effectively we had shown that we were taking the variation of some functional right i which was bilinear in nature and symmetry and that variation we were equating to zero. And when we take this variation the interpretation that this vary, the first variation of this bilinear symmetric operator i is equal to zero is it meant that there is some function which is getting minimized, there is some function which is getting minimized.

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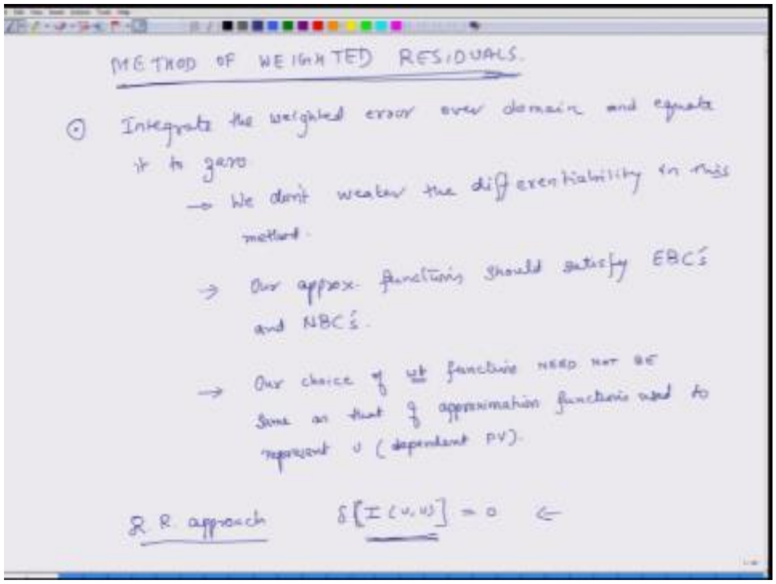

There is no reason to think that all functionals will be necessarily bilinear and symmetric right. There may be a lot of problems, things may not be symmetric in nature and they may not be it may not be possible to represent them in form of I u of u, so they may not necessarily be bilinear or symmetric or both okay. So in such cases, this rally rates approach need may not necessarily work, because this functional may not exist. So if the functional does not exist you cannot minimize that functional.

So in such cases, we rely on these weighted residual approaches, even though we see,

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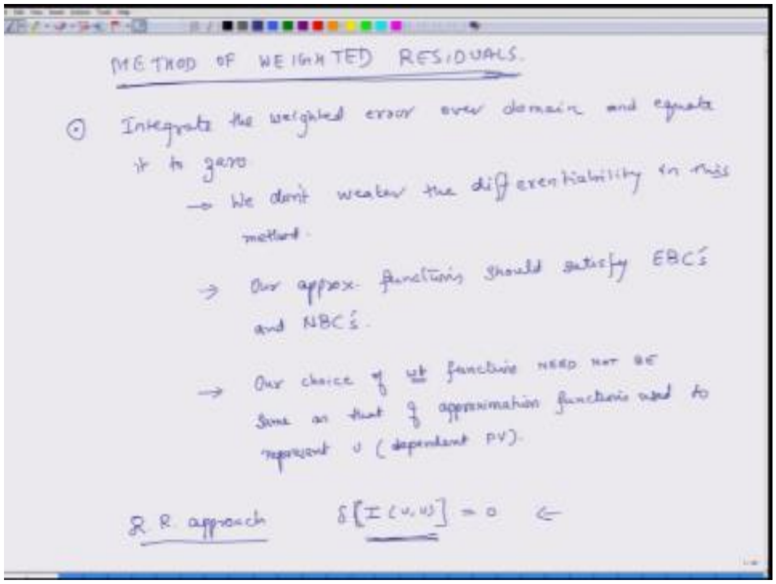

That they are more restrictive in terms of the choice of displacement or approximation functions for to represent displacements. But they still enable us to provide us approximate solutions especially, and they are especially helpful when things become nonlinear or when this i functional it does not exist.

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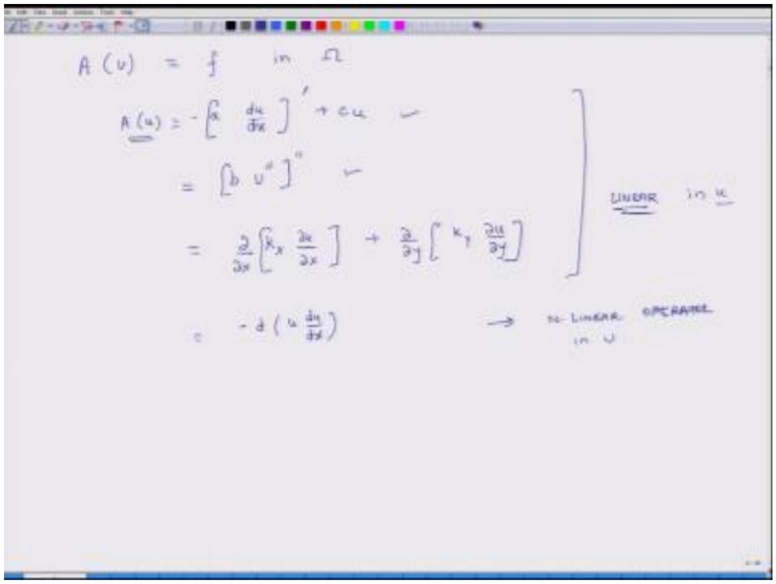

So we will consider different operators, just consider different operators, let us say that there is an operator u, A(u) and that equals f in some domain okay which is Ω, and we will list down different types of such operators, so one example of this operator could be A(u) is equal to a times du over dx and then I take the differential of it plus cu okay. So this particular operator we have seen several times in our previous lectures. Another operator which we can represent using A(u) would be some parameter of some function b(x) times some dependent variables, second differential and this entire thing is being differentiated twice.

This particular operator is applicable for beams, EI and then the deflection for derivative that equals moment or equals the q right, so it is q. Another example of another variation of this A could be some constant k_x times δu over δx and then you differentiate it with respect to x plus partial derivative of some function k_y times δu over δy. So in this case, this is an operator which is working in two dimensions okay, another operator example could be du over dx, so these first four operators are linear in u, in u okay, but this is a nonlinear operator, so nonlinear operator in u, so the point what I am trying to make is,

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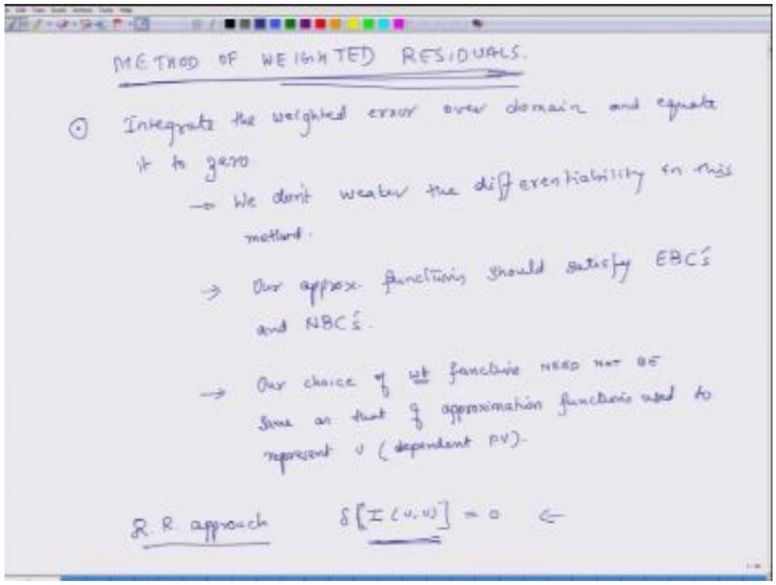

That this particular approach method of weighted residual approach.

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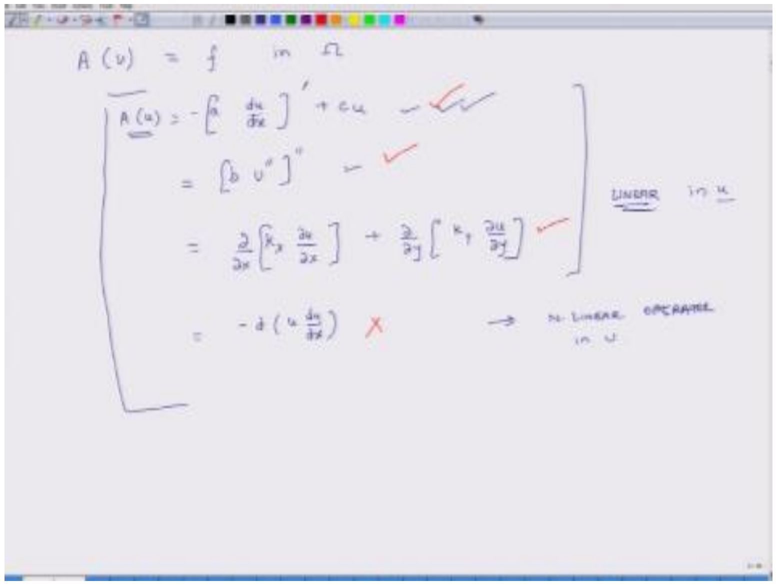

Will be fine regardless of the nature of these operators, our Rayleigh Ritz approach will work for this thing, you know this, it will work for this, it may work for this, but it will not work for this, but these weighted residual approaches may work for all sorts of situations.

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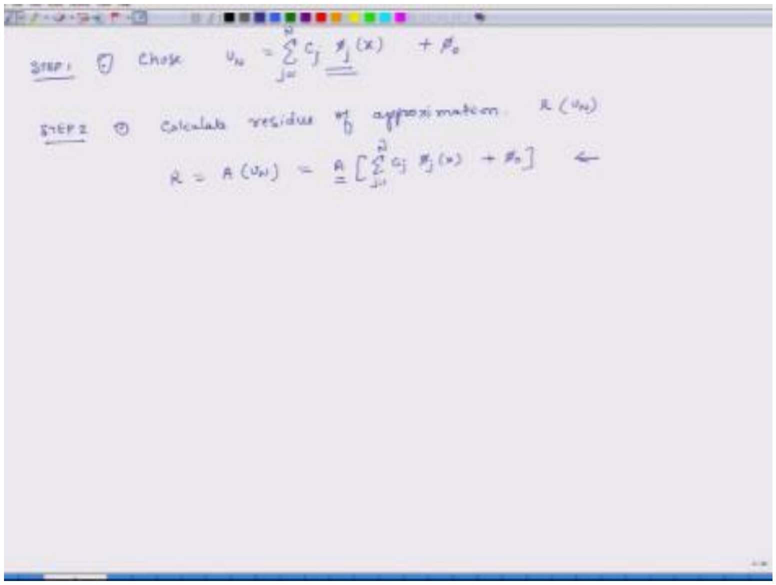

First thing is that we choose an approximation function for u, and suppose u I want to express it in terms of a n term solution expression, then U_N is equal to some constants c_j times ϕ_j which is a function of x and j is equal to 1 to N okay, so this is the first step, step one, second step this involves that we calculate the residue of approximation. So our residue I write it as R and it is a function of this U_N which is a N term solution for dependent variable u.

So my residue is equal to, I apply that operated on U_N if the overall differential equation is this $A(U_N)$ and since U_N is c_j $\phi_j(x)$, so it is equal to I am using this operator on U_N c_j ϕ_j oh I can also add a constant term here ϕ_0 okay, so $\phi_j(x)+ \phi_0$ and the c_j time ϕ_j, I am going to summit over in d's j. So this is my residue, and this residue it depends on the nature of operator.

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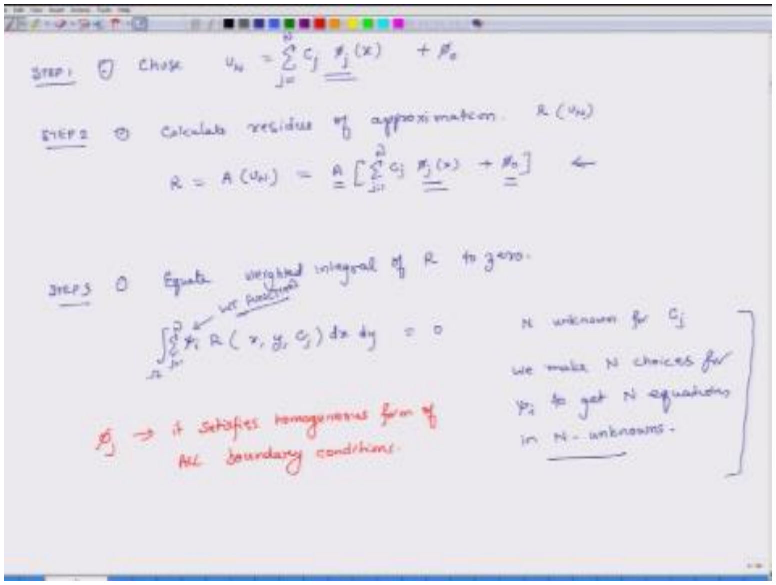

This operator could be one of these or even something different it depends on which governing equation we are trying to solve. So I can, then the next step is step three that equate weighted, weight integral of R to zero okay, so the mathematical statement for this is, I am going to integrate it over domain Ω, my R it depends on U_N, so it is a function of x, y,c_j, dx, dy, and I am going to multiply it by some weight function ψ, so this is my weight function, and this I equate to zero, and of course I forgot I have to summit over j equals 1 to N.

So there are total N unknowns, unknown values, unknowns for c_j right, because this is a N term solution, so I have to make N different choices for ψ_i, N different choices for ψ_i we have to ensure that I get any questions for N unknowns, so we make N choices to get N equations in N unknowns, N equations in N unknowns okay. This term okay, so I will like to reiterate some things. Our choice of ϕ_j and ϕ_0 we have explained it earlier. But so here when we are choosing ϕ_j it should be such that it satisfies homogeneous form of all boundary conditions.

And we will actually do an example to show what we mean by this, so all boundary conditions not just natural boundary conditions or essential boundary conditions, ϕ_j should satisfy all boundary conditions, but only their homogeneous form okay, only their homogeneous form. So if u equals a non-zero constant I replace that non zero constant by zero, and this ϕ_j has to satisfy that boundary condition.

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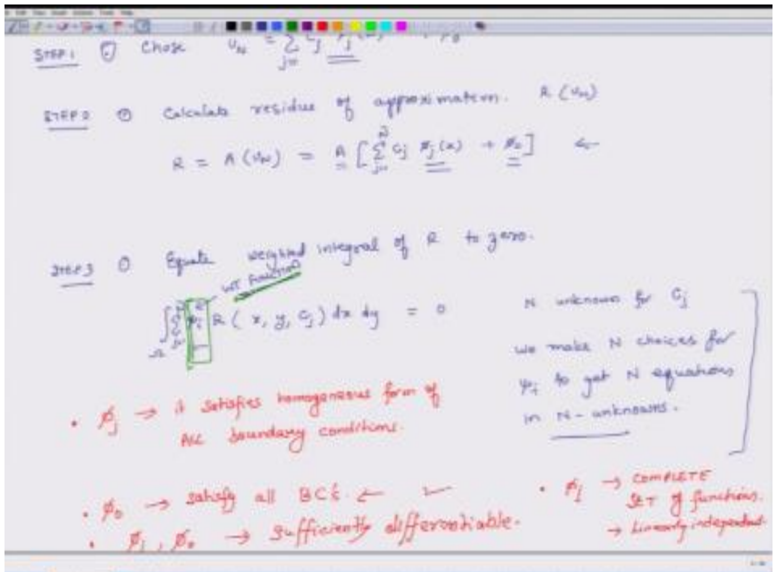

And then ϕ_0 should satisfy all boundary conditions, should satisfy all boundary conditions. So, so if the boundary conditions are non-homogeneous then ϕ_0 has to satisfy those, if they are homogeneous then it will anyway it has to satisfy that. So this is one requirement, second requirement is about ϕ_0, the third requirement is that ϕ_j and ϕ_0 should be sufficiently differentiable, they should be sufficiently differentiable. So if my operator N, I have fourth order derivatives, then ϕ_j and ϕ_0 should have, should be differentiable upto atleast the fourth upto four times atleast.

Otherwise my choice of ϕ_j and ϕ_0 is incorrect, and the last point what I wanted it to make is that ϕ_j should be a complete set right, complete set of functions. So if I am start, if I use polynomial functions then I have to start with x, x^2, x^3 and so on and so forth. As long as these functions satisfy the boundary conditions then that set should be complete in itself, I cannot have missing terms in between okay. So ϕ_j should represent a complete set of functions, so this is there and yeah what he said is true and they should be linearly independent.

So these are the requirements of ϕ_j they should satisfy all the homogeneous form of boundary conditions, sufficiently differentiable ϕ_j should be a complete set, and it should be linearly independent. So we have done step 1, we choose a particular function for U_N, calculate residue, in the third step we multiplied by a weight function, and do the weighted integral of residue to zero, but we do not weaken the differentiability and we have discussed what type of ϕ_j should we be picking up.

Now what we will do is we will look at different types of weight functions, so we have till so far not discussed which kind of weight functions should we be choosing. So we will look at different type of weight functions, and they will be associated with different method solution methodologies, different solution methodologies and we will discuss all these different types of different flavors of weighted residual method in the next lecture okay. So thank you, and we will continue this discussion tomorrow.

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