Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 18
Rayleigh Ritz Method

by
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Hello, welcome to basics of finite element analysis mook course, now today is the last day of the current week and what we plan to do today is continue the discussions which he had in the last couple of classes by actual examples, that is in context of some specific differential equation which we are going to solve using this weak formulation approach.

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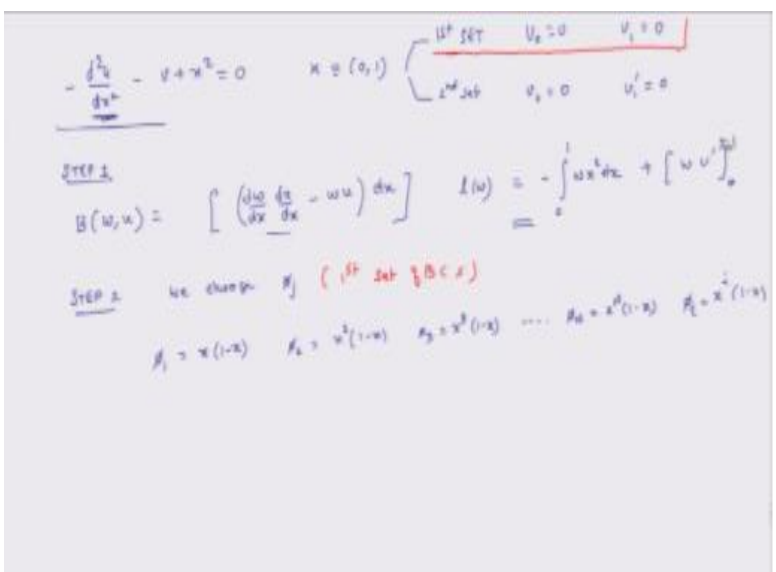

So that is what we are going to do, so the equation we are going to solve is this one. And the equation is such that the domain is from 0 to 1. And then we will solve this equation for two sets of boundary conditions, in the first set of boundary conditions, $u_0 = 0$, $u'_0 = 0$, and $u'_1 = 0$. In the second set of BC's, $u_0 = 0$. But at x is equal to 1, the derivative of u is zero. So the first step is that I express this equation.

I, so the first thing we will do is that we see what kind of differential equation is this. So we see that the equation is linear in u because all the powers of u are 1, there is no u^2 or u^3 victim, right? So this is first thing is it is a linear equation, the second thing is that the order of this differential equation is two, okay. So we had seen earlier that if the equation is of an even order and if it is linear.

Then we can create a bilinear symmetry functional for it. So what is that functional? And this is equal to, so this is bw, how do we get this? Basically we are integrating, we are multiplying the error with the weight function W and integrating this thing by parts. So we get this thing, right? And then this u corresponds to w times u.

So this is the bilinear functional associated with this differential equation. And the linear functional associated with this differential equation is l(w) and that equals minus integral from 0 to 1, wx^2 dx, plus some boundary terms which are, if you are not unsure, sure of how we get this is basically.

You again follow the same process which we have explained several times earlier also that you multiply this whole equation by W, integrate it with respect to x, integrate it by parts and you will get this form, okay. So this L functional will give us the F vector or the forcing vector and the b functional will give us the n/n matrix, okay. So the second step is that we choose $ø_j$ and initially we are going to choose $ø_j$'s

That are consistent with these boundary conditions, first set, first set of BC's. So we have said that the shape functions are the approximation, approximation functions should be such that the values of those functions should vanish at points where the primary variable is specified. Now in this case the primary variable is specified at u = 0 and no at x = 0 and at x = 1, okay. So we will choose functions which meets both these requirements.

So this is what we are going to choose ø 1 is equal to x times 1 – x, this is 0 at both ends, ø 2 is equal to x^2 (1 – x), ø 3 x^3 (1 – x) and $ø_n$ is equal to x_n (1 – x). So in general I can write it as $ø_i$ is equal to x^i (1-x), okay. One condition we had also mentioned when we were picking up these functions was that none of these functions should be a linear combination of others.

So we have to verify whether this is true in this case or not and what we see is that I cannot express any of these functions in terms of other functions, right. So that condition is satisfied. A third condition which I had not mentioned earlier was that this should be a complete set. What does that mean? That if I am choosing ø1 as this, ø2 as this, and I am going up to ø n.

I cannot say that I will not choose ø 3. So whatever comes in the natural sequence of things I have to choose that complete set of functions because if I do not choose complete set of functions I may inject some inaccuracies in my solution, okay. So I have to choose a complete set starting from the lowest possible order.

And going up to whatever degree I want to, okay. So, so three things, they should meet, they should be, these functions should be homogeneous at known points at points where u is known, second is it should be a complete set, and the third one is that these functions should be not linear combinations of each other.

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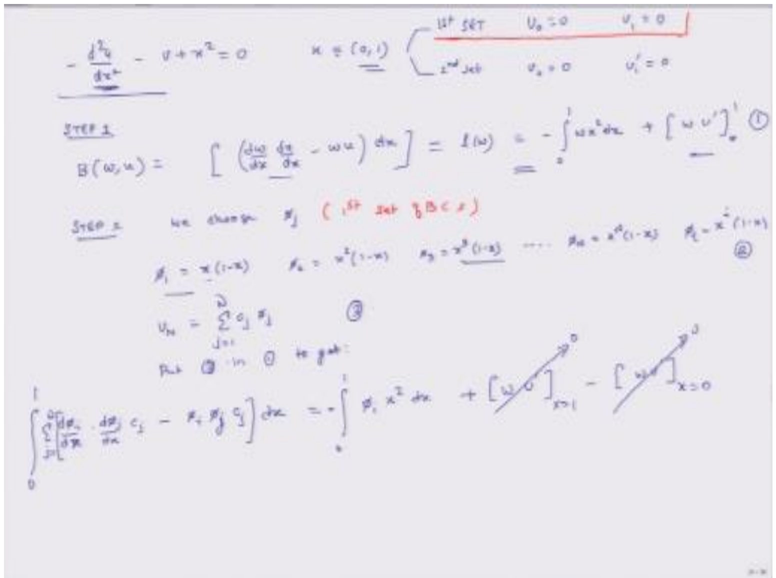

So now using this I calculate my B metrics and L vector, okay. So my u_n if it is a n term solution is equal to C_j $ø_j$, J is equal to 1 to n. And if I put this, so the overall equation is this thing. So put it in this equation, equation 1, this is equation 2, so we put this is three so put three in one to get 0 to 1. So why I am integrating from 0 to 1 because the domain is 0 to 1, if it was 0 to l then accordingly I have to integrate.

So 0 to 1 integral of $dø_i/dx$ times $dø_j/dx$ times C_j, j is equal to 1 to n, minus $ø_I$ $ø_j$ times C_j and I am going to put a bracket around it that is why erased it not that there were something wrong, dx okay. So the entire thing is being summed over in these j and this is equal to 0 to l minus $ø_I x^2$ dx plus sorry 0 to 1 and plus I get this boundary term and let us look at this boundary term.

So u times u' at x = 0 minus no excuse me at x = 1 minus wu' at x = 0, okay. So let us see these boundary terms at x = 1, all these functions are zero. So this and this is how we had chosen the nature of functions. So this is 0 and at x = 0 also this is zero. So both these boundary terms in brackets they vanish.

Because the boundary conditions related to the primary variable is known at both the endpoints, which may not be the case in the second case.

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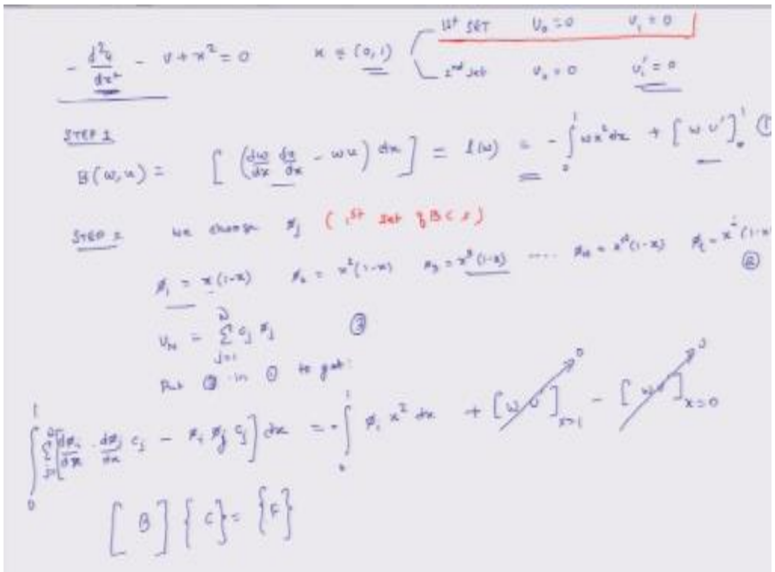

Where u is prescribed at x = 0 but it is not prescribed at x = 1, okay. So that is there. So, so from this we can calculate this B matrix, okay. And, and this c vector and this is equal to F.

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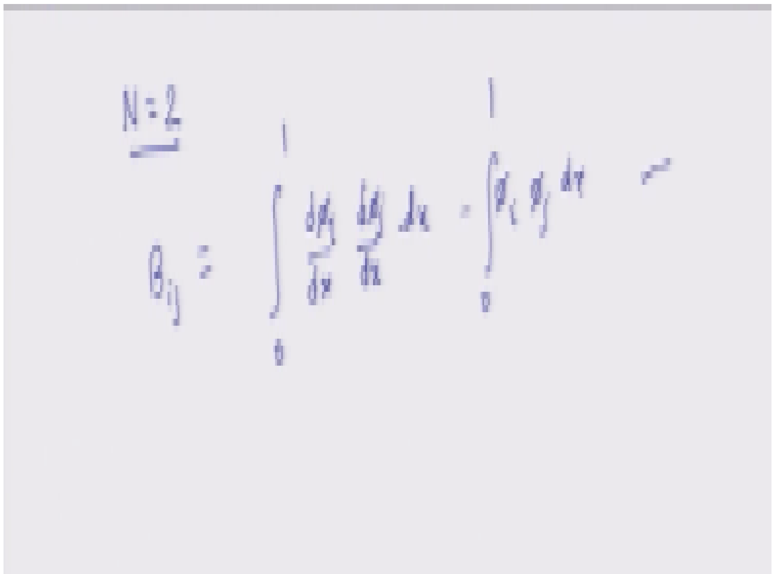

So for simplicity purposes let us say n is equal to 2, suppose you want to develop a 2/2 equation. Bij is equal to integral of, so everyone understands how we get Bij, right?

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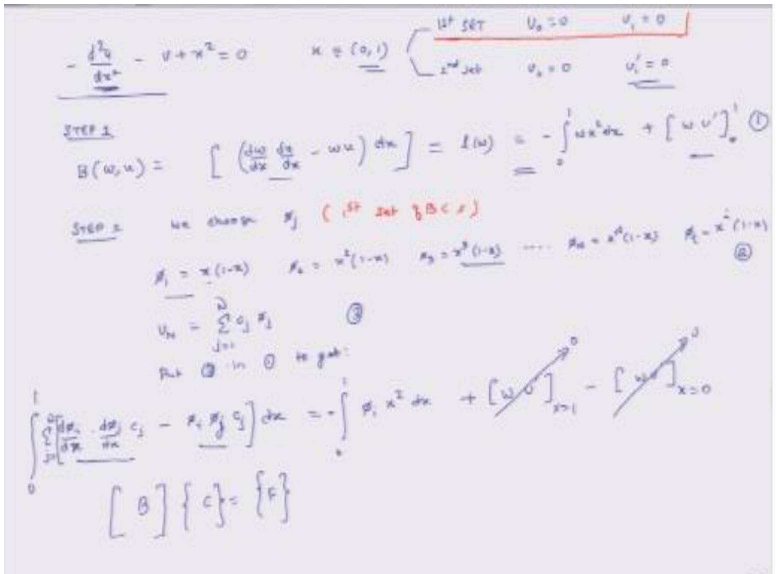

It is basically is the same thing, okay.

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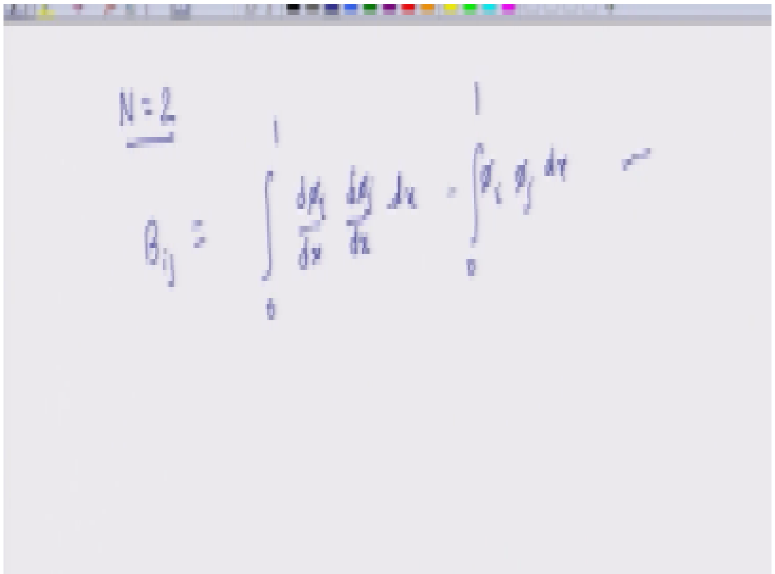

So this is how I get Bij.

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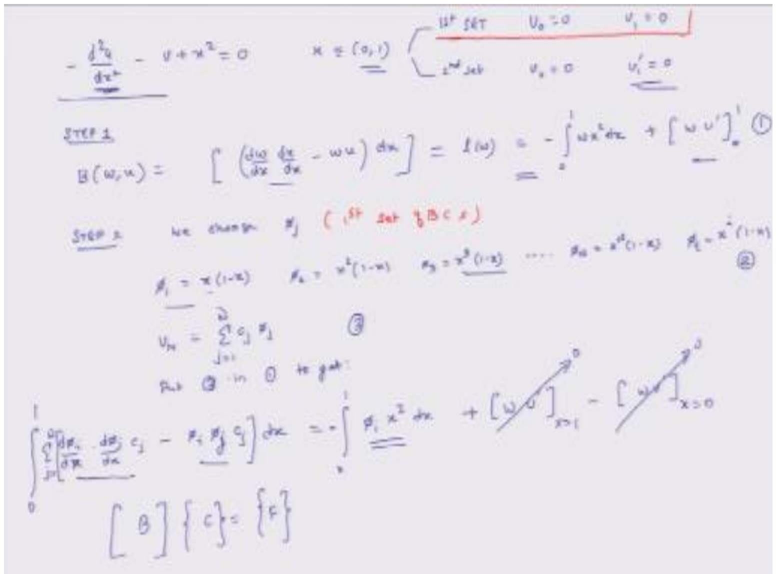

And the force vector is related to integral of ϕ_i times x^{2}.

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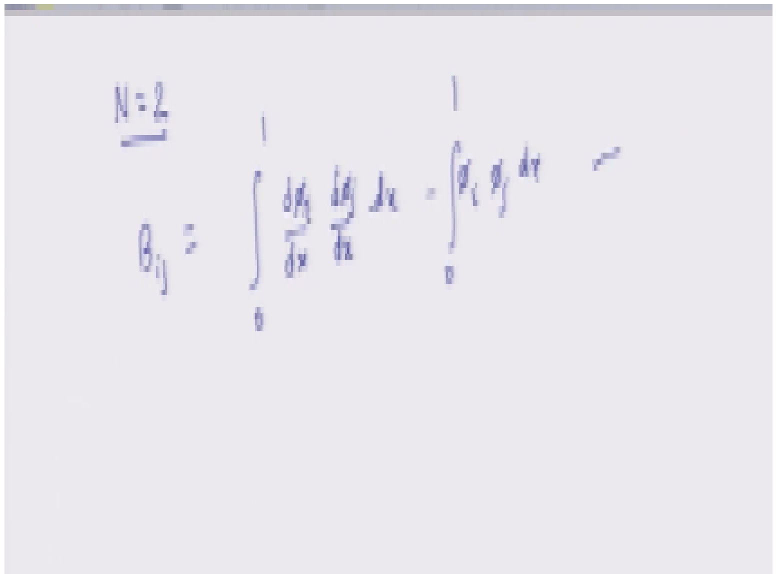

And while calculating Bij, I substitute these Ø's with the assumed functions.

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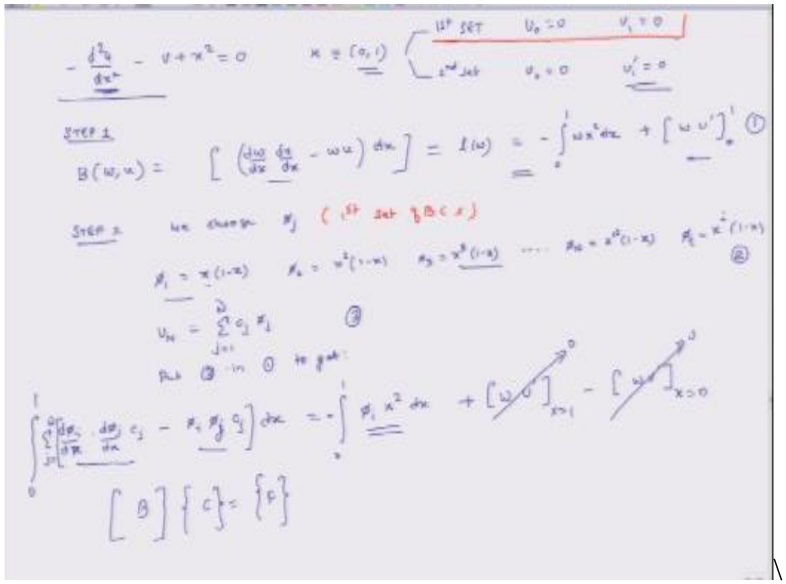

Which are ø1, ø2, ø3, and so on and so forth, so I can calculate fairly easily.

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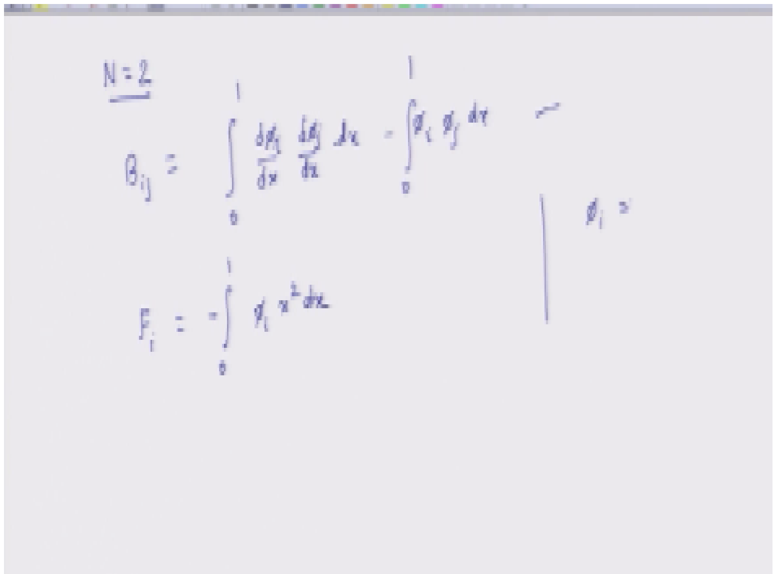

The terms which are in B matrix and for calculating force, it is integral of minus ø$_{I}$ x^2 dx, okay. And we know that, what is øi ? øi equals, what was our resumed function?

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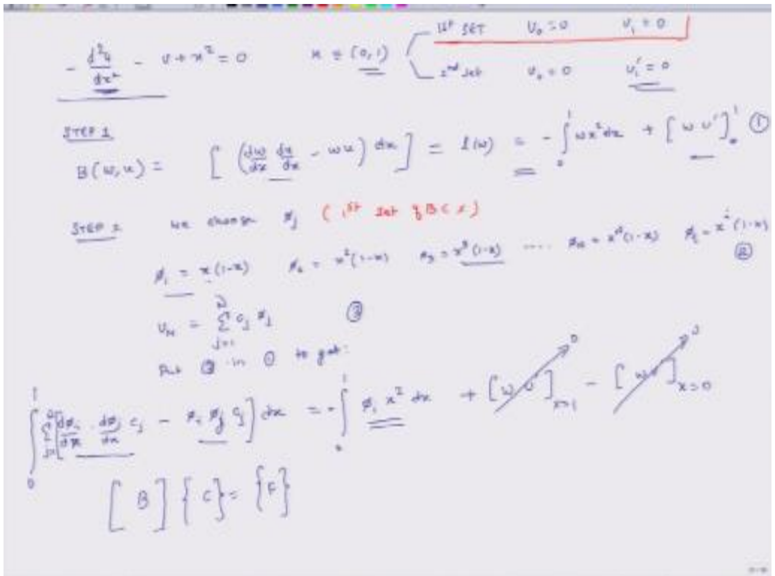

X ^{I} times (1 – x).

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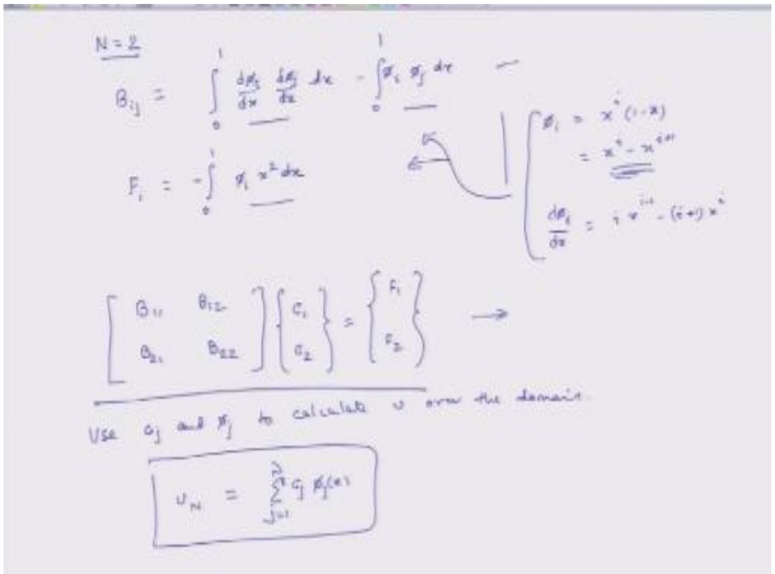

So that is equal to $x^i - x^i + 1$. So I can substitute these things, this relation in these equations and I can very easily calculate the B matrix and the F matrix, okay. So because this I can also write dø or dx is equal to i times x $^{i-1}$- $(i+1)x^{i}$. So I may call these substitutions in this and I can calculate my B matrix, so I get for a 2/2 system, I get this 2/2 system and I can calculate from this so here F is known.

Because I know øI and I can integrate it so F is known, also these B's known, the only unknowns are c1 and c2. So I can calculate c1 and c2 in terms of F and B matrix, okay. Once I know C is then, what is my, what is the original unknown? U, right? So then I use C's, Cj and øj to calculate u over the domain.

So then I can calculate the value of u at any point in the system. So u for an end term solution is Cj, øj , J is equal to 1 to n. We have done some calculations and it will be worthwhile for you to go back to your homes and see how you get these numbers.

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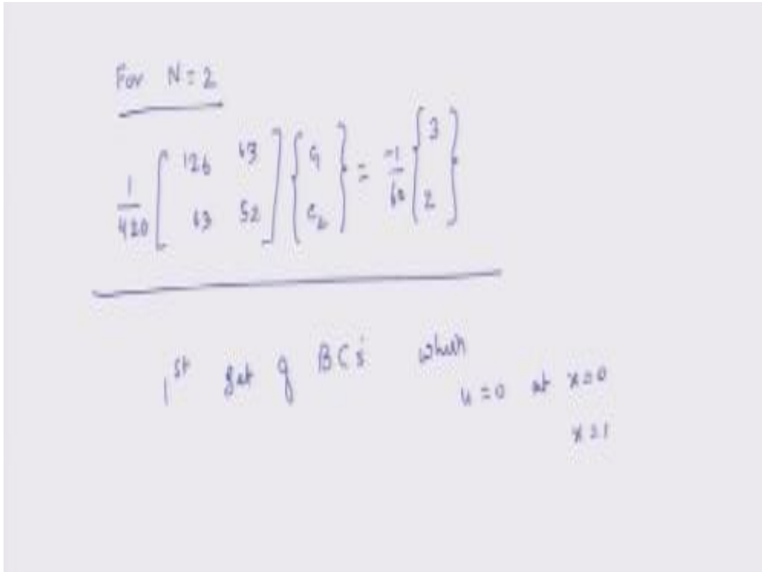

That for n is equal to two the overall equation is this. So this was for first set of boundary conditions when u was zero at x is equal to 0 and at x is equal to one but we said that, we will also solve it for.

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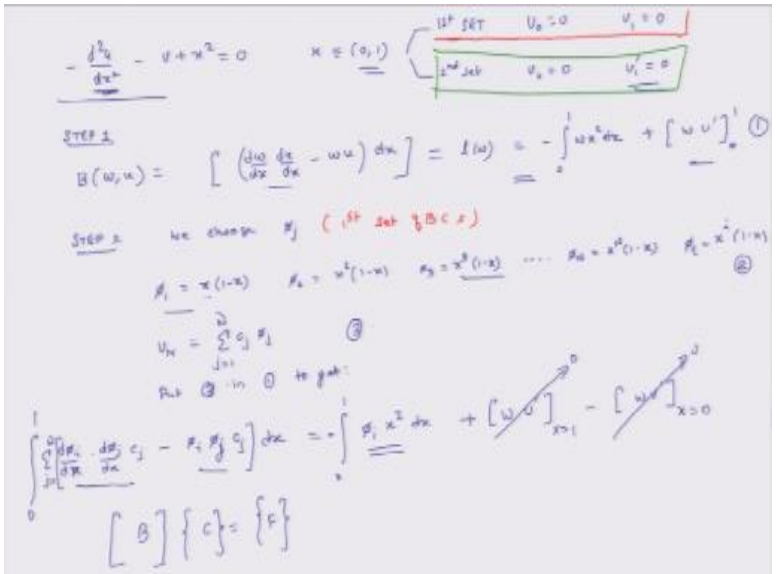

The second set of boundary conditions so this is a different problem. Here u is as specified at x is equal to 0, but it is not specified at x = 1, right? Rather it is differential, u and prime is specified at x is equal to one. So we will solve this problem also.

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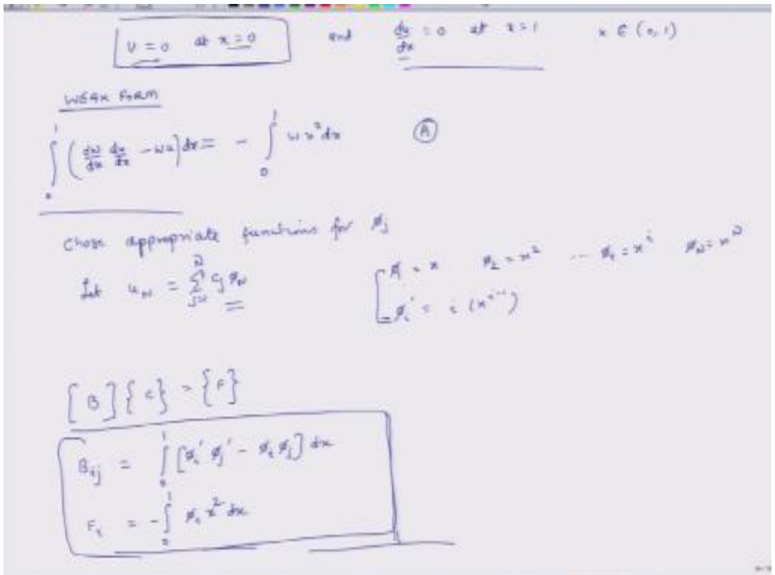

So in the second set of boundary conditions we had said that u was zero at x is equal to zero and the derivative of u du/dx was zero at x = 1. So in this case at x = 0 the primary variable is specified. So we have an essential boundary condition which is specified at x=1, a natural boundary condition is specified and the associated secondary variable is du/dx. So again how do we do this, we go back to the original differential equation?

Develop its weak from. So we will again go back to the weak form. So I will rewrite it 0 to 1 because our domain is, and this was dw/dx, du/dx, dx minus is equal to minus w x^2 dx integral 0 to 1 plus wu' from 0 to 1. I can expand this by breaking it up into two parts one at value at 1, one at value at zero. So that is what I will do. So at one and then minus wu' at zero, and we know that at u is equal to 0 when x is 0 okay. Excuse me.

So at x = 0 u is zero which means the variation of u at x = 0 is zero, right? So this part goes away so this term vanishes. The other thing is that I have specified du/dx is which is slope at x = 0. So this part become 0, if this part was 1 some non–zero number then we would have put that number here, okay. Then we would have put that number here, suppose this number was two slope was specified as two.

Then we would have put the value 2 here, it is not that all the boundary term conditions terms always vanish, it just happens in this case. So but in this case it is zero. So I do not even have to worry about the second boundary condition. Because I specified it and by specifying it I have taken it off from the equation, okay. So that is there. So this is the equation we have to satisfy and the only thing is that so what do we do next?

We choose appropriate, approximation functions, okay. So let u_n is equal to cj øn. Suppose this is a n term solution and what kind of a approximation function do we choose which ensures that its variation is zero at x = 0, but its value should not be 0 at x = 1 okay. So we can choose ø1 = x, ø2 $=x^2$, $ø_i = x^i$ and $ø_n=x^{n.}$ So the derivative of øi prime that is, it will be equal to i times $x^{i-}1$.

So I put this, put this information and this information back into equation A and then I get once again I get a B matrix times a C matrix, C vector is equal to a F vector. And what is my Bij? Bij, oh! There is something I missed in this relation it was W times u dx. So Bij is integral of 0 to 1, $ø_i$ prime, øj prime, minus øi, -øj dx. And the F vector is defined as 0 to 1 minus øi ^{x2} dx. So using this we can calculate different elements of B.

Suppose I take a n term solution I get un equations and once again when I assumed my fees, I again made sure of that the three requirements were met. The first requirement is that the primary variable is vanishing at x = 0. So that is being satisfied, it is not necessarily vanishing at x = 1 same is true for my functions. Second condition was that linearly independent and the third thing was that should be a complete set.

So I have not omitted any intermediate terms, I am taking all the terms starting from the lowest possible term. So with that that, I conclude today's discussion and what we have discussed in discussed today and also in the last class is, how Raleigh Ritz approach works, what we will discuss in next two three classes are some other approaches for solving these equations, but there we will not rely so much on the weak form but strong form of the solution and with this we conclude this week's discussion, have a great weekend and thank you very much, bye.

<u>Acknowledgement</u>
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