Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 17
Variational Methods : Rayleigh Ritz Method

by
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Hello, today is the fifth day of this course, welcome to the course and what we are going to discuss today is how different types of variational formulations and one of these particular formulations known as the Rayleigh Ritz method is very frequently used in finite element analysis, but we will not only discuss the Rayleigh Ritz approach but maybe in the next class or one or two other classes we will also discuss other approaches which are used to formulate our problems and the fundamental difference in all these formulations are is of two types.

First is that either we use the strong form or the weak form that is one difference, and the other difference is that what type of weight functions do we choose, so what we had discussed earlier was that the choice of weight functions is to be such that it has to meet, it has in context of weak formulation it should represent the variation in the dependent variable, in some formulations it is a little more restricted so we will discuss all these details things in detail.

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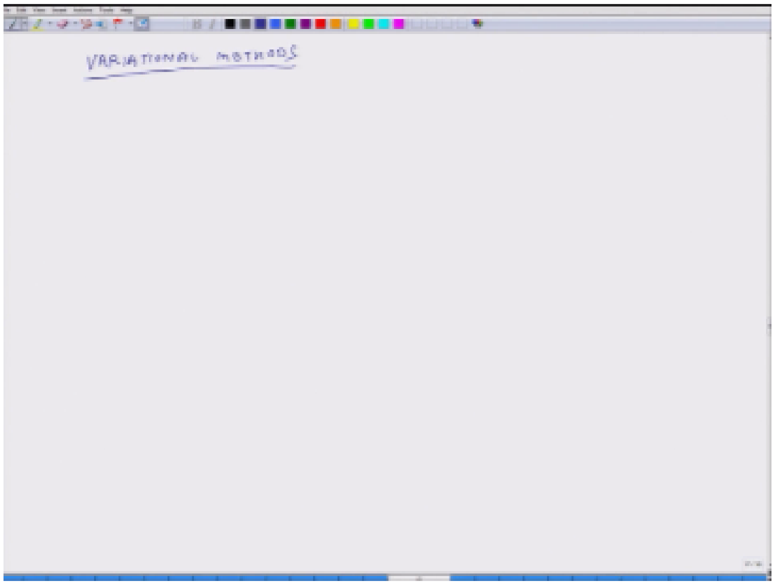

So we are going to discuss variational methods and till so far we have not come to the element level but maybe in one or two lectures once we have covered we have done an overview of different methods we will see that these methods which we are discussing can be applied either over the entire domain of the system or we can apply them at element level also, so we understand these methods then graduating from here to the element level and developing element level equations.

Is a fairly straightforward exercise so it is important that we understand the fundamental principles underlying these methods and then we will start working on the finite element formulation.

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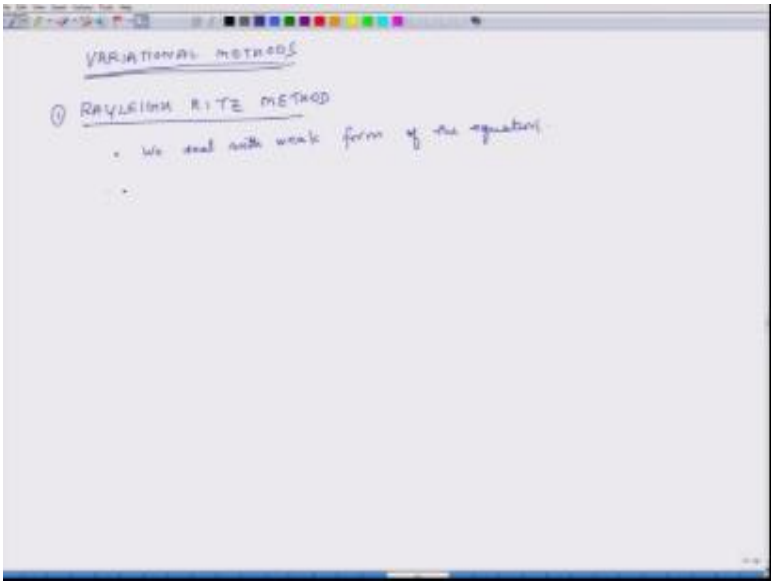

So, so the first method is known as Rayleigh Ritz method okay, and in this couple of important points that we only work on weak form, second is that when we pick Ø.

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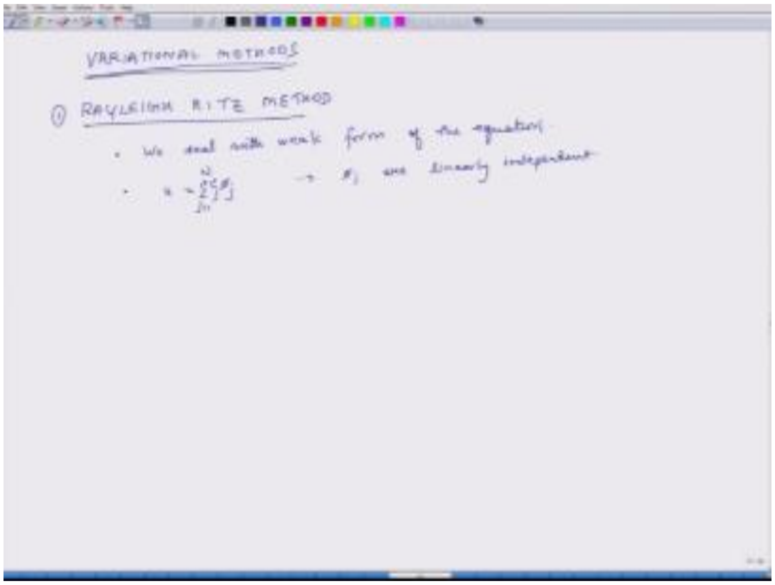

So we pick them in the way that u equals Øj, c_j j = 1 to n and these Ø are linearly independent, we cannot make up functions which are linear functions of others okay, otherwise you will get simultaneously, simultaneous equations and equations in n and unknowns, but when we will start solving it we will not be able to solve the solution unknowns.

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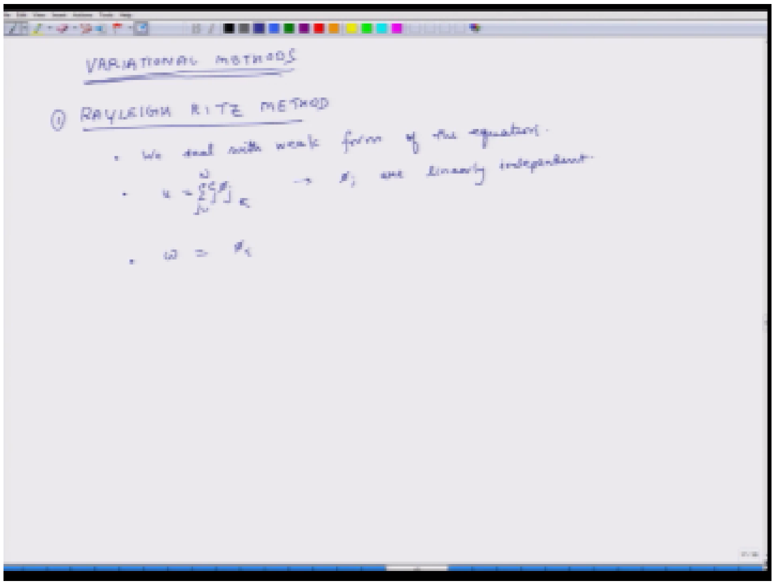

The 3rd thing is that in Rayleigh Ritz method w is nothing but fi so once we have made a choice on these Øj i is essentially use the same functions in place of w okay, these Cj's

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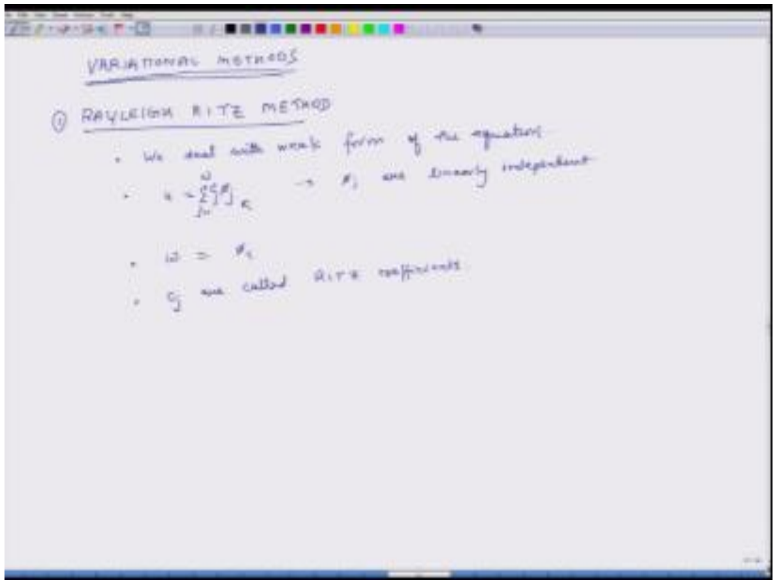

Are called Ritz coefficients, all Ritz coefficients so if there are n unknowns I know that there are n weight functions also, we got the number of weight functions associated with n unknowns is also n, so I get n equations, n unknowns and I can solve them.

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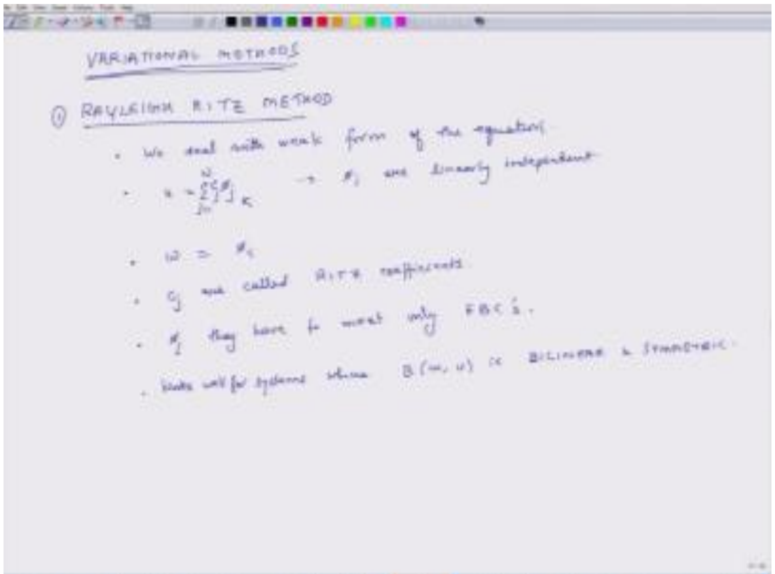

Further what we had discussed earlier in the in the last class and the class before that is that these functions Øj they have to meet only essential boundary conditions, we do not have to worry too much about the natural boundary conditions because when we developed the weak form the natural boundary condition automatically comes into the equation and if its values is known we just plug in those values in the equation, and the last and very important point is that all this thing works well only for, for systems where this B (w, u) is bilinear and symmetric.

I will not say it works only for but it works really well, works well for systems if it is not bilinear in symmetric then we may end up getting non linear equations in C, C's so then we have to figure out how to solve them but for bilinear symmetric functionals represented by B_w it works really well and it is pretty straightforward, so we will discuss this further, so example.

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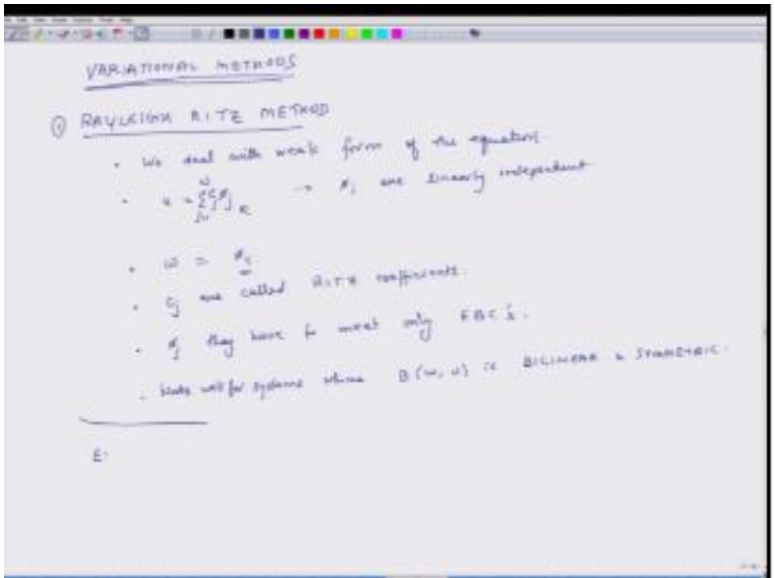

So we had seen that in context of that bar that is one differential equation it can solve there may be others also it could be in general represented, represented as B(w and u) is equal to L of u, excuse me W and we say that d is bilinear and symmetric, so if that is the case then I can also represent the it has i equals 1/2 of B(u, u) - L of u and so this is the equation which I have to solve, this is the functional and if I minimize this functional I get this equation, we have seen this in the last class.

Because the variation in u is w so this is w okay, so now we assume that you and let us say that it this u is having n terms in its solution so that is why I call Un so it is a n term solution Cj, Øj is some constant okay, so I can, so let us call this equation, 1 call this 2, so if we put 1 in 2 what do I get I get, so u_n is this entire thing u is this thing and also w is no, no that is fine but what I am saying is I know that u is this equation represented by equation 2 and w is the weight function and I said that in Rayleigh Ritz the weight function is same so it could .

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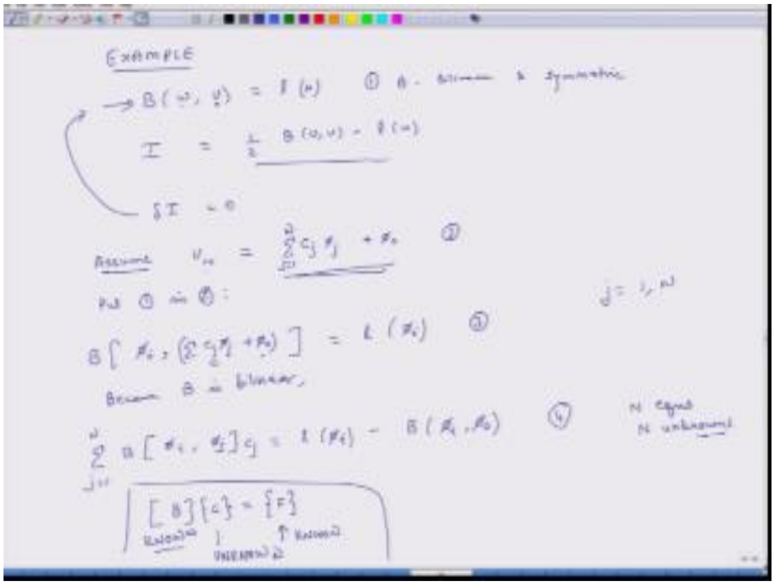

It is F_i so instead of w I am putting $f_{I,}$ c_j, Øj +$f_{0,}$ so this entire thing in parenthesis it represents u is equal to L of $f_{i,}$ this is equation 3 and we will actually do one example right now, this is very general but we will actually do an example so you will see how this works out so because, because B is bilinear what I can do is that I can pull out C_j out of this bracket and we will actually do an example and we will see what that means so I can write it as summation of j= 1 to n to e of fi, okay, so because it is bilinear.

I can separate this F_0 separately and also because it is bilinear I can oh where did C_j go, C_j I can take C_j also out of the bracket, now when you look at this equation 4 you see that for different values of I so j= 12_n okay, for different values of i, this is one equation and on the in this J I am adding up everything right. So for different values of i, I will get different equations, if I put F_i = Fi1 I will get one equation, fi = fi2 I will get second equation and all these C_{js} are unknowns so I will get n equations involving Cj's which are n unknowns so I will get n equations .

And n unknowns okay, and essentially what I will get is a b matrix multiplied by a Vector C and this is equal to a vector f so this is known, this is unknown, and this is known okay, so now I can solve these equations and get the values of C_s.

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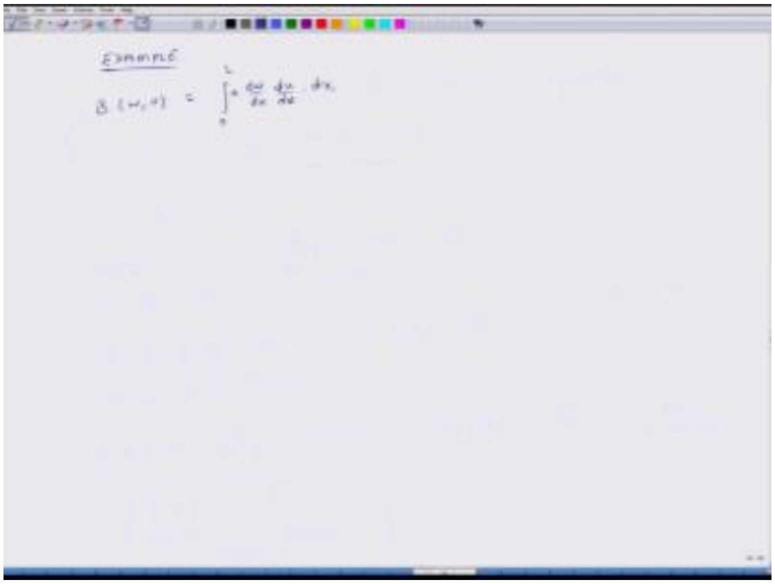

So we will do an example, so this is which formulation the Rayleigh Ritz formulation okay, the main thing about Rayleigh Ritz formulation is its weak form, it uses weak form and here the choice of weight function the same as the choice of polynomial for you know approximation functions, so in context of the bar problem B we had seen it was defined as 0 to l dw over dx , du over dx times dx and there was this constantly or some parameter way okay, so what we will do is.

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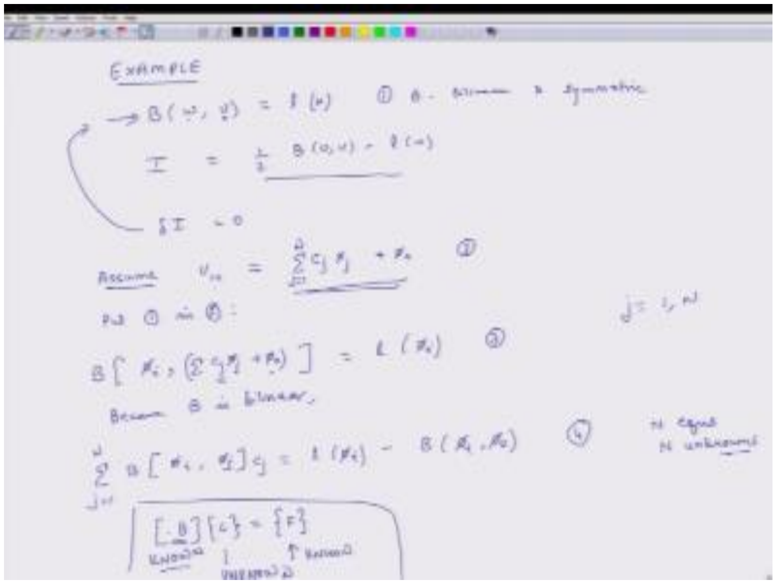

We will compute this b metrics for this type of a function.

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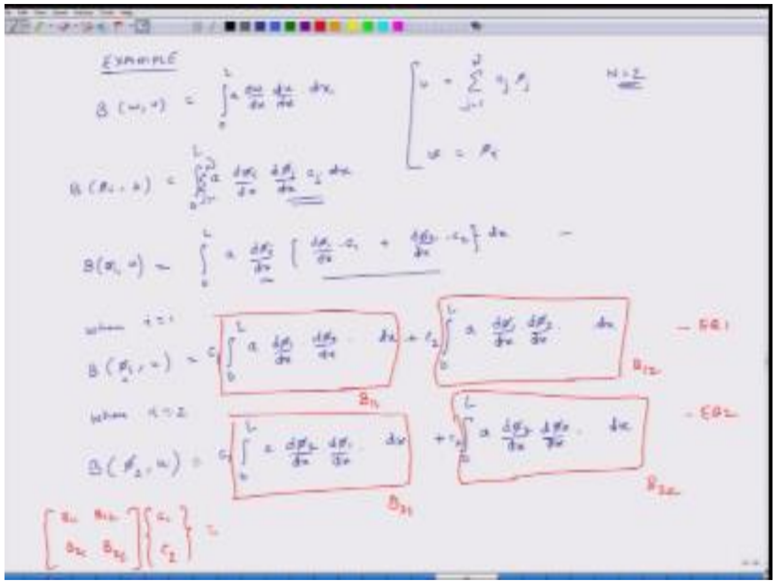

So first step is that we say that u = c_j, Ø, j = 1 to n and in this case for purposes of simplicity we limit n = 2. So the size of our B metrics will be 2/2, let us calculate how that, how do we figure out what are the terms in that matrix and then w is Øi, then b of Øj = 0 to L a times right, so I am right now just working on the B matrix and I have to sum it up over j = 12_n so this equal to integral of 0 to L a times d Øi over d_x right, and I have to sum these things on the end is G on the end is j right. So what do I get d Ø1, see Ø1 could be 1 and 2 it can know only 2 values right.

Because n = 2 so d Ø1 over dx times c_1 + d Ø2 over dx times c_2 d_x agreed, okay so now in this equation I can vary, in one case I will be 1 and in another case I will be 2, I can have 2 values because n = 2 right, okay. So when i = 1 what do I get, B Ø1, Øj = 0 to L so i equals one times d Ø1 over dx times c_1 + integral 0 to L, a d Ø1 over dx times d Ø2 over d_x times d_x and when i = then my B functional is. So here i = instead of 1 now I will put 2 right and this equals 0 to L.

c_2, c_2 where should I put c_2 and there has to be a dx also here and actually this should not be it should be u, because I have this, this entire thing is u like this entire thing is u, so this is equal to dØ and instead of if i = 2 then I get here this thing okay, so I am getting two equations, one is for B of $Ø1_u$ and the other is B of $Ø2_u$ and now I identify these things so what I do before I move this c_1, c_1 is a constant but it is unknown so I move it here outside the integral sign I moved it outside the integral sign.

And then this term is B 11, this integral term is B 12 ,this integral term is B21, and this integral term is B 22, understood so this is how I calculate the values of B's. I know what kind of Ø I have so I can integrate these and get the entire B matrix which is in this case 2/2, if n was 3 then this B would have been 3/3 and so on and so forth. So my equation is B11, B12, B21, B22, C1, C2 see c1 and c2 are unknowns, so these equation 1 and this is equation 2 can be represented by this 2/2 metrics multiplied by this column vector, this vector on C and on the right hand side what do I get.

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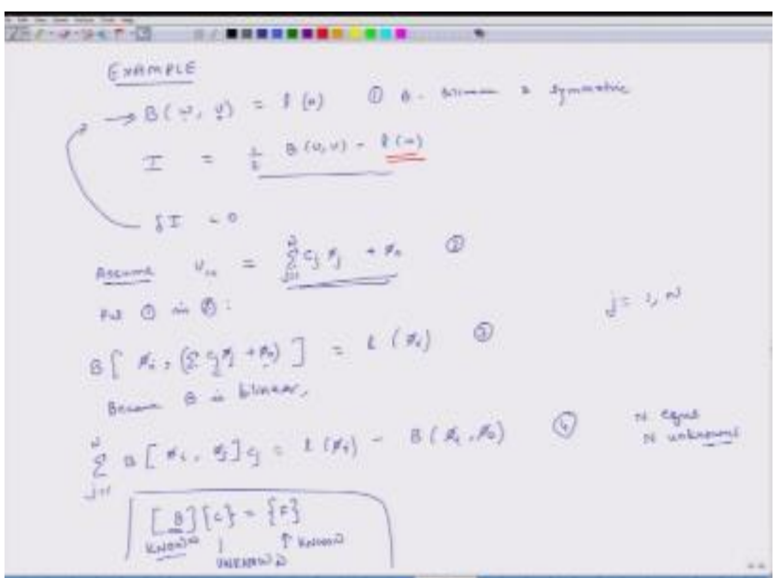

I get the f equation you know the terms involving the f vector I basically do the same thing on l_u the linear term, I do the same thing on l_u and I will get.

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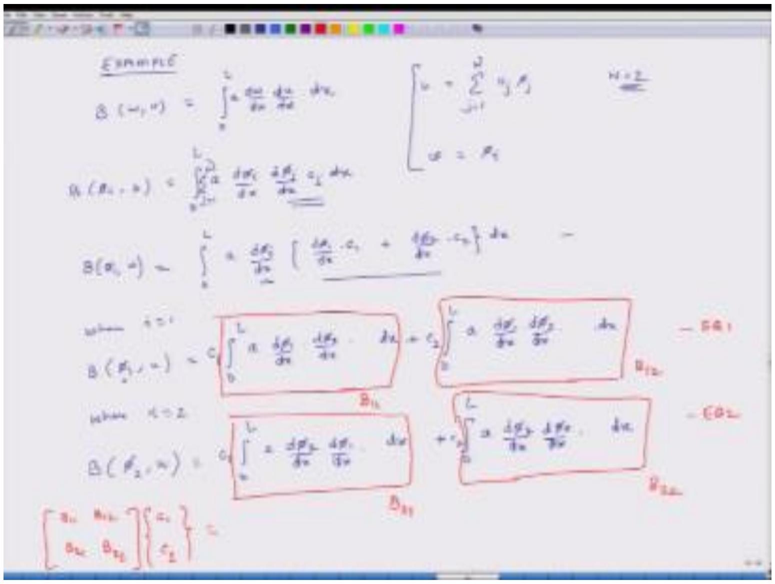

My f1 and f2, any questions confusion how I get f okay, so this is how we get this B metrics and then we multiply it by this vector of unknowns which is the c column and that equals the f vector. Now the way you calculate this f vector is that you have to go back to your definition of this L_u term right, okay so that is what we will do in next 5-7 minutes okay. So and for that again we have to go back and refer to the original equation so let us go back and see the definition of L okay.

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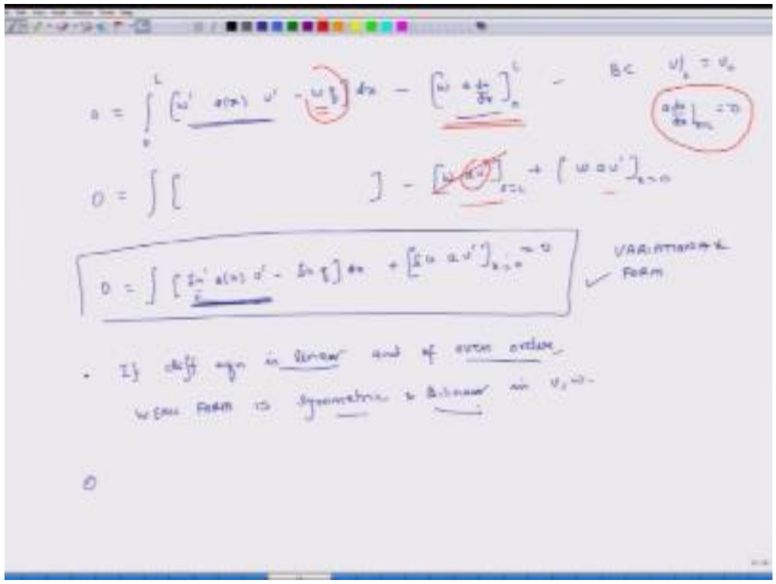

So in my definition of L I have this wa d_u over dx and also w times q okay, now in this case a x = 0 u_0 is I mean excuse me, a x = L at x = L I have specified that this is a du dx is 0 so this term is 0 right, and at x = 0 u_0 is not specified so w will be 0 because it represents the variation and because the, that thing is specified then it is 0 so, so in my L functional which is a linear functional only this term will appear, only this term will appear in context of these boundary conditions, if the boundary conditions differ then it will it can change okay it can change.

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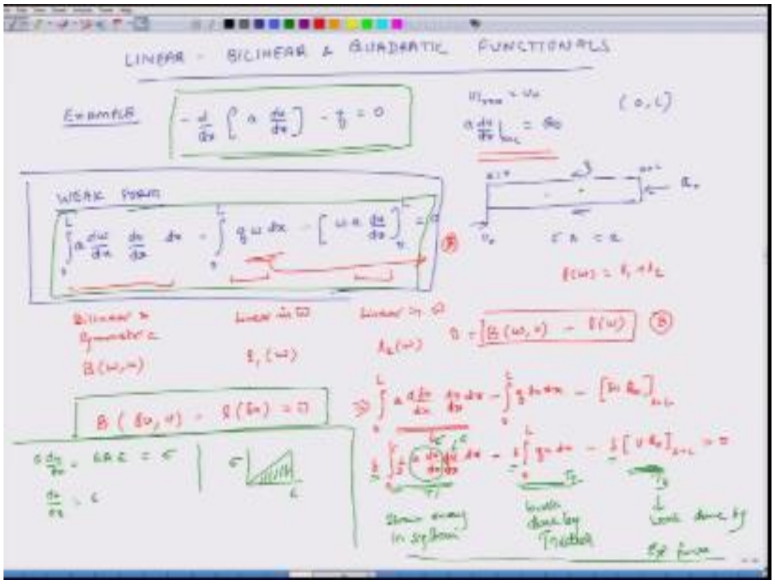

So, so what we will do is so if I move this to the right side this will become a positive thing so that is what we are going to calculate.

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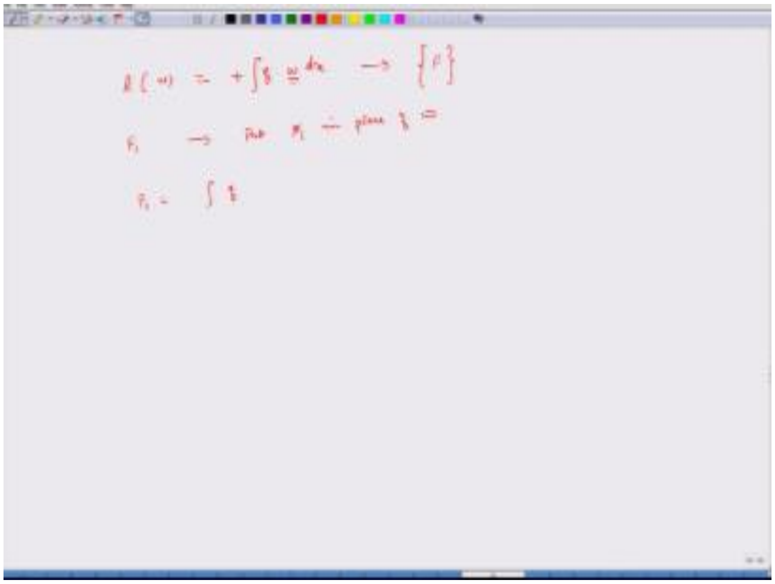

So my linear functional was what, actually it was q times w d_r right, and it has a positive sign because I moved it to the right side, now this will give me the f vector. How do I get a vector for f_1, I put, I put Ø1 in place of w so I get f1 = integral of q and I know q, q can be a constant or it is a varying function I do not, it depends on the nature of the problem but that is known .

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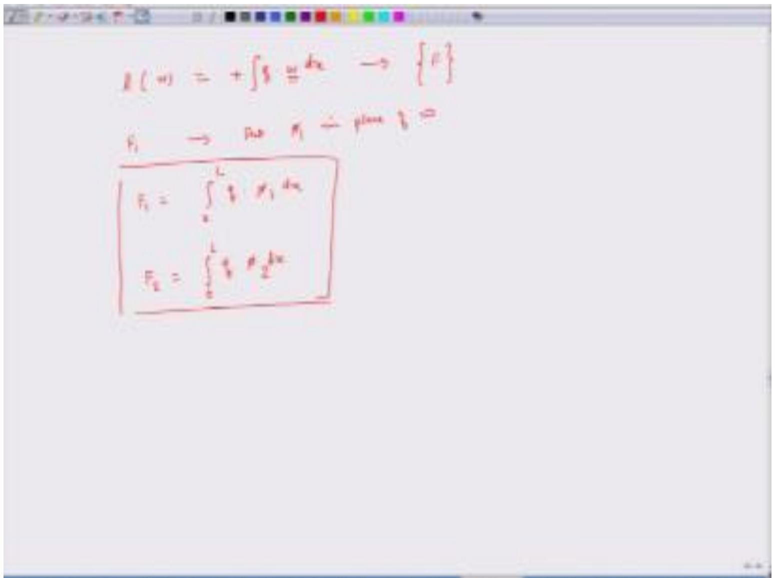

So this time Øi times d_x 0 to L okay, and similarly f_2 in this case the choice of w is Ø2 so it is 0 to l q times, Ø2 times d_x so here i is 1 and here i is 2, so these are the definitions of f1 and f2 and what kind of Ø1 and Ø2 I have to choose which these Ø have to meet which conditions, the essential boundary conditions they need not need meet the natural boundary conditions.

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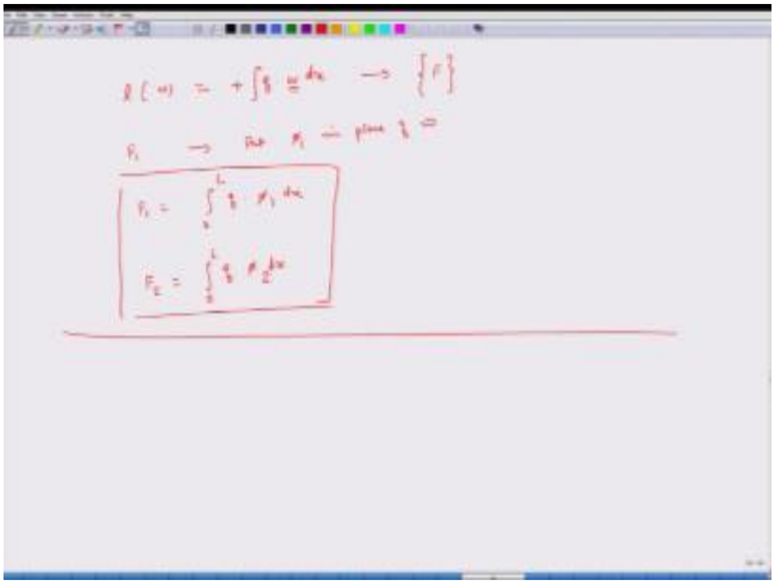

Now they need not meet the natural boundary conditions so that is there, so this is what I wanted to discuss in this particular example and what we will do in the next class is we will actually do one more example so here so initially we have discussed.

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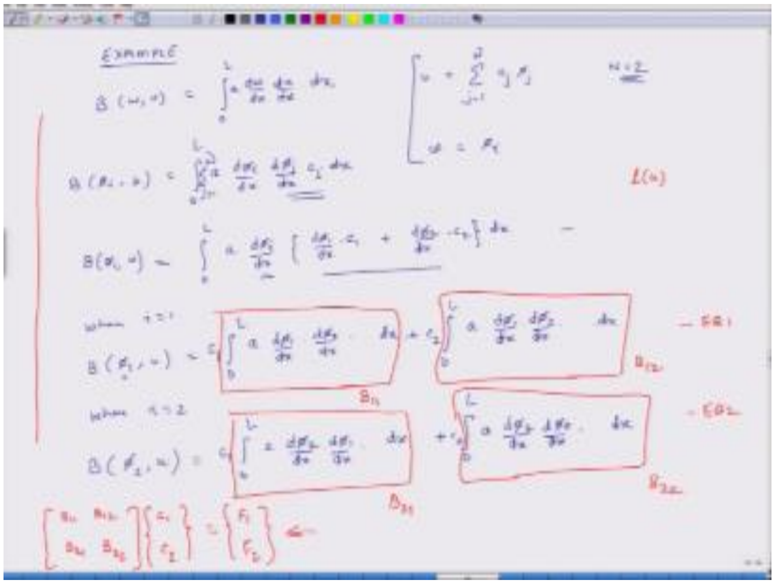

Everything in a very abstract sense.

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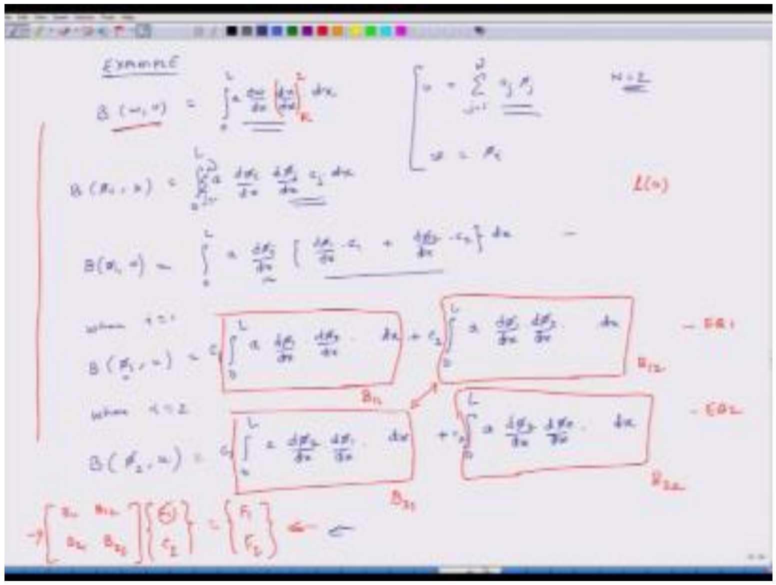

Then we went ahead.

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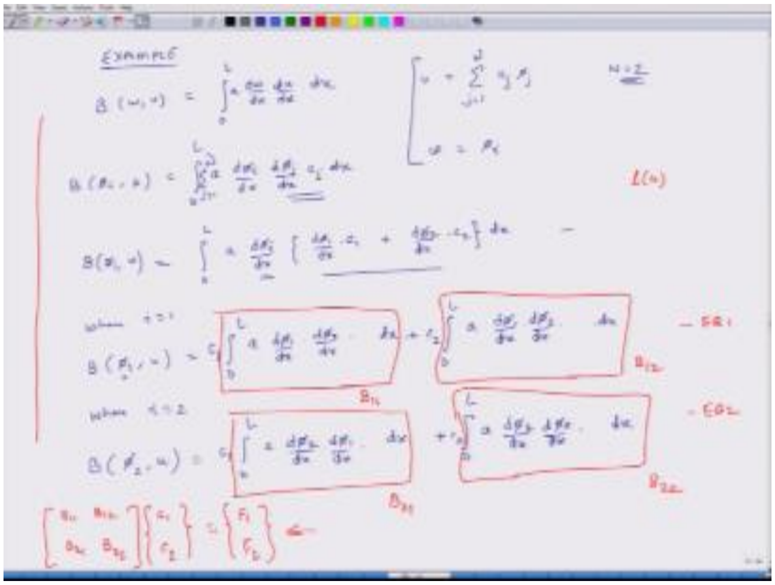

And actually this was.

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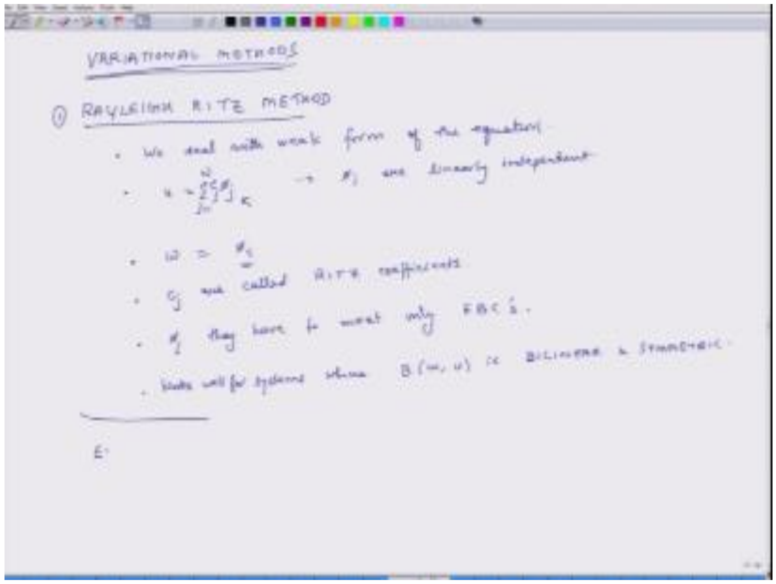

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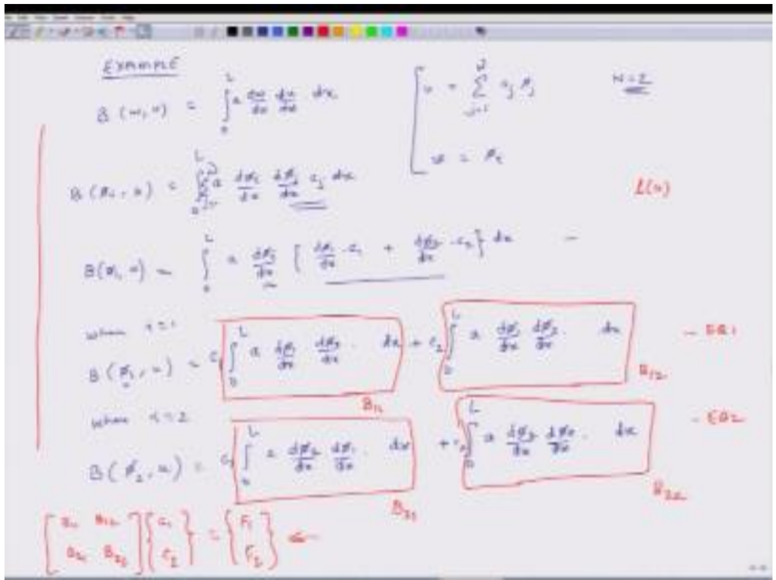

Is the abstract sense where we said that we will get an set of equations b times c = f and then we actually made it more tangible and for n = 2 we have computed the values of B11, B12, B2 to B1, and also f1 and f2 and then the next example we will actually do a real equation and we will solve for some of these terms. Now 1, 2 important things I wanted to make.

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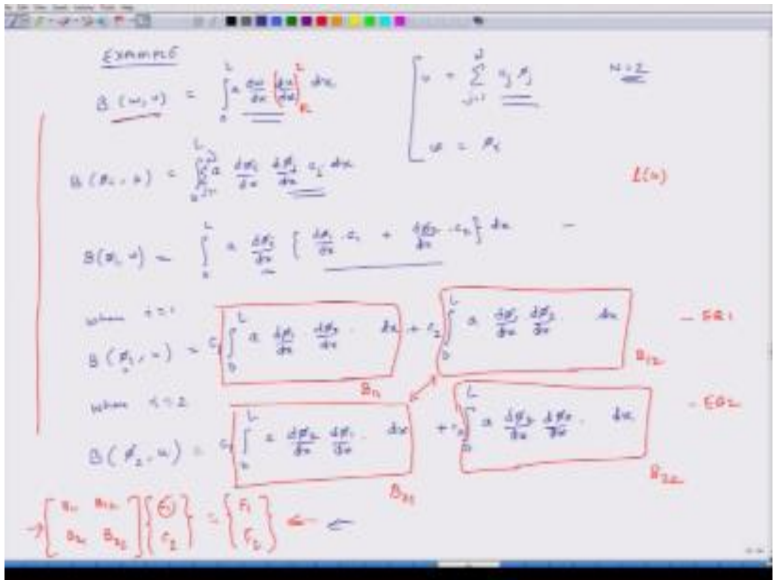

These were. first thing is that if you look at this B metrics and if you look at these terms B12 and B21 they are same right because if u d Ø1 over dx times d Ø2 over dx and here differential of Ø2 with respective x times differential of Ø1 so it is asymmetric matrix and the symmetric comes because our original function B was what, it was symmetric, it is directly because of that reason, second thing suppose we had instead of our weak form was something like this, suppose our weak form was something like this.

Suppose it had a excurse so this is not a linear system right It is not a linear system if that would have been the case then when we would be when we would compute the values of B's and C's we would not get linear equations in C okay. Then we would not get linear equations in c because when we replace u with cj, Øj we will get squares and cross multiples of Cjs, you may get $c1^2$ + $c2^2$ + c1 , c^2 + $c3^2$ + c1, c3 + c2, c3 and so and so forth okay, so these equations will not be linear.

These equations will not be linear and neither they will be symmetric, they are symmetric because B is symmetric and these equations are linear because B is bilinear okay, if they are linear equations they are very easy to solve, if they are non linear then it is a much tougher problem so in at least in context of the discussion we will be having in this course we will be only solving linear problems okay.

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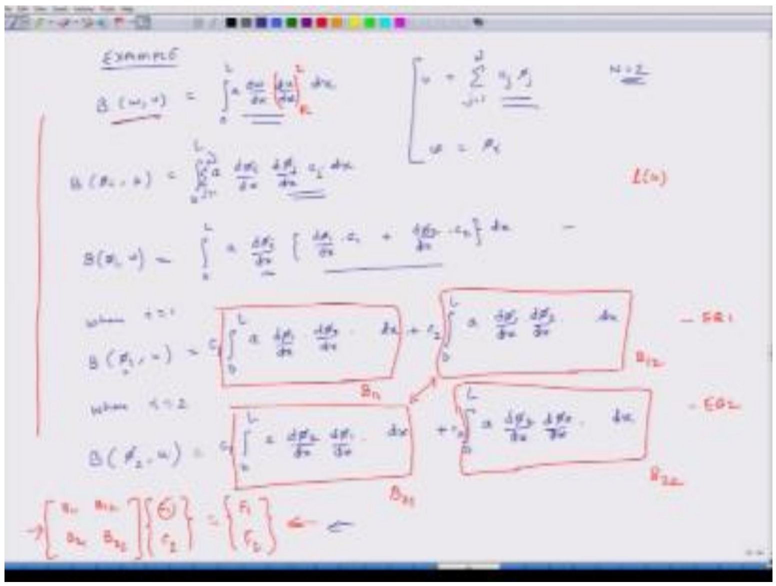

So suppose so that is one thing.

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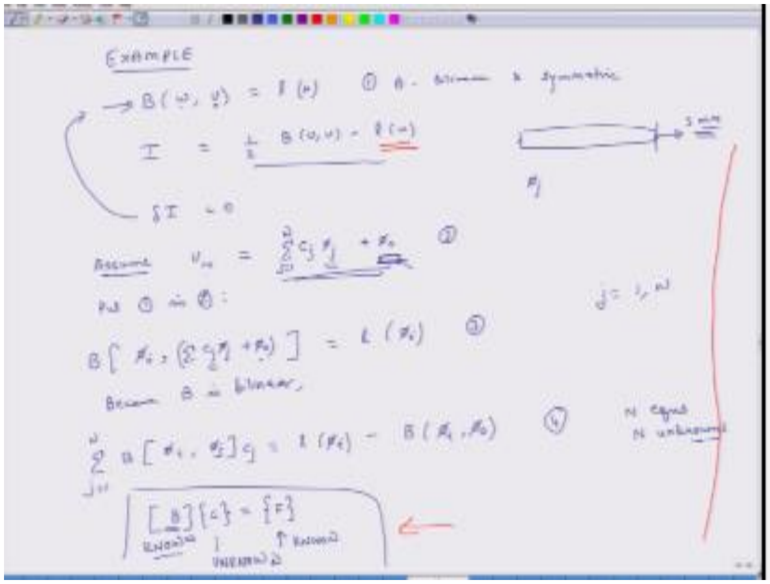

The other thing is that I also had this $Ø_0$ term this $Ø_0$ term, now $Ø_0$ term helps us take care of so suppose you have a bar and there is a displacement 5 millimeters here then all these Ø all the values of $Ø_j$ it will be 0 at this end and this end right that is how we are picking up, but then at the end there is some displacement which is 5 millimeters so that is 5 done through this introduction of $Ø_0$ okay. So that takes care of the non homogeneous boundary conditions related to essential related to primary variables, related to primary variables okay, these Ø take care of the homogeneous conditions related to primary variables.

We had discussed this earlier, homogeneous condition means things are equal to 0 if there is a non-homogeneous condition.

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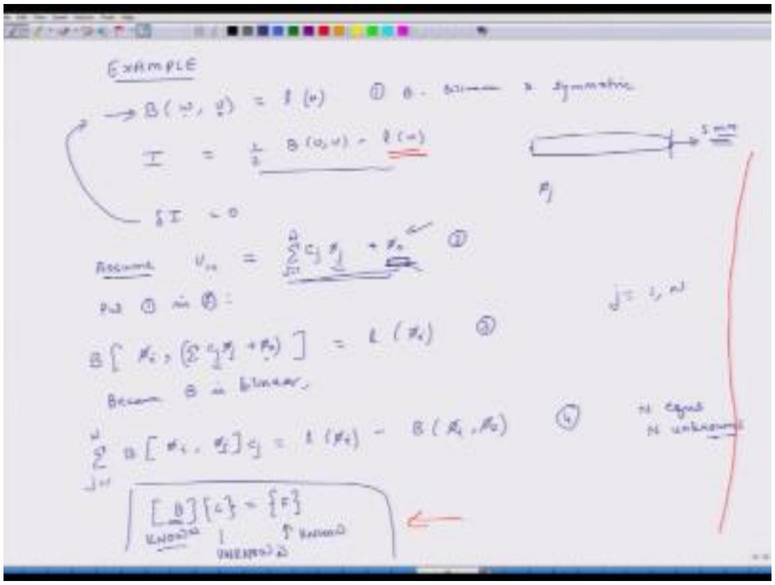

Then this is Ø 0 term takes care of that so that is why we had introduced this term $Ø_0$ okay, so but the mathematics is identical so we do not have to worry about it mathematics, mathematics is identical.

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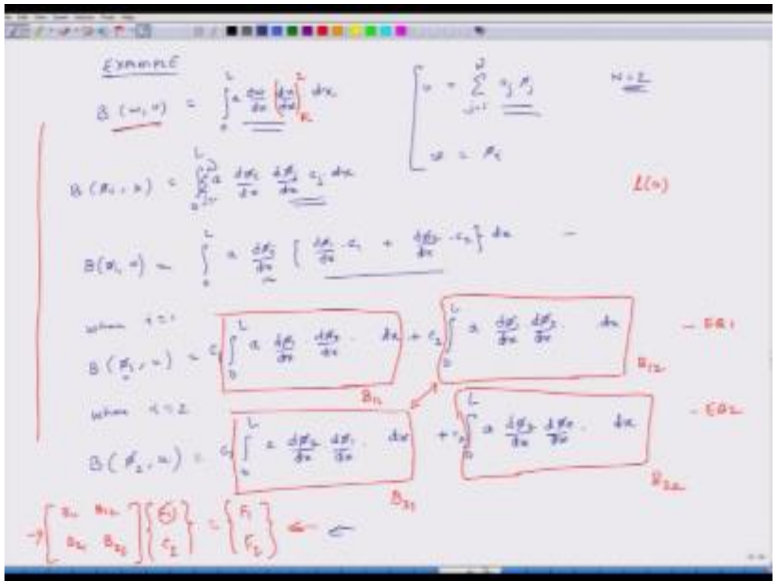

So that concludes our discussion, in the next class we will continue this discussion and do one or two very practical examples of actual differential equations so that you become more familiar with this method, thank you.

<u>Acknowledgement</u>

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