**Indian Institute of Technology Kanpur**
**National Programme on Technology Enhanced Learning (NPTEL)**

**Course Title**
**Basics of Finite Element Analysis**

**Lecture – 16**
**Weak Formulation & Weighted Integral:**
**Principal of minimum potential energy**

**by**
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Hello, today is the fourth day for the current week and what we are going to discuss in today's lecture is the relationship between the weak form and the minima of some quantity in context of solid mechanics problems, this quantity corresponds to potential energy so that is what we will try to understand, and that will give you some understanding as to what is actually happening to the physics of the problem as we are developing a weak form and using a weak form to solve a problem.

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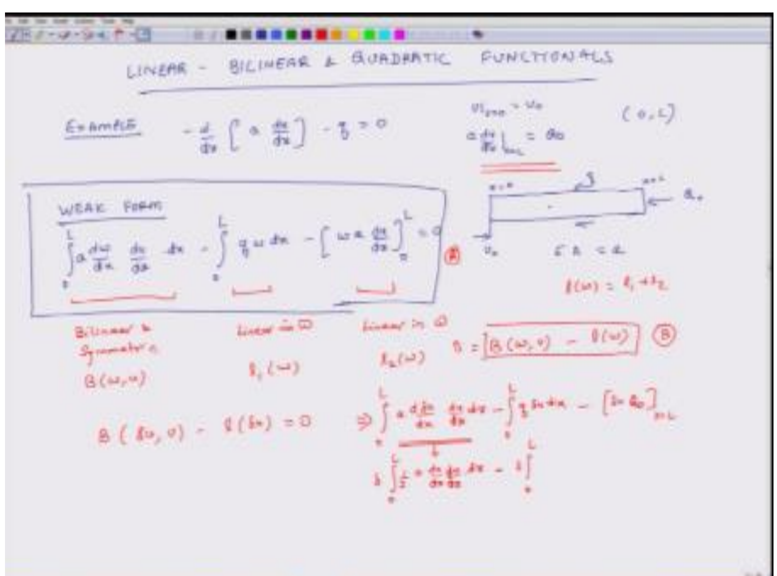

So we will discuss linear by linear and quadratic functionals okay, so the aim is to explore the relationship between the weak form and the minimum of a quadratic functional so we will again go back to our regular example. So this is minus d over $d_x$ a times d over dx minus q = 0 and the boundary conditions are u at x = 0 is $u_0$ and a times $d_u$ over $d_x$ at x = L =$q_0$ where the domain of the system is 0 to L. Now we will look at this problem in context of the problem of a bar under compression or tension, so suppose I have a bar okay.

And I am applying some traction here, I am applying some traction here and then I am specifying, I am pushing this bar such that the displacement at x = 0 is $u_0$ okay and also at x = L this is my x sequence L I am putting some force $q_0$ then u is the displacement in the bar at a position x some position x use the displacement, so I am interested in finding u how it changes from 0 to L, so this is the thing.

And also the young modulus of the bar is e and let us assume in context of this particular example that a is the cross sectional area which does not change with the length, so EA is the thing and this equals A, okay. So we are going to write its weak form straight away so the weak form is 0 to L $d_w$ over $d_x$, $d_u$ over $d_x$ A – integral of 0 to L $q_w$ times dx – $w_a$ 0 so for this particular problem this is the weak form.

And now please pay attention to these things, these things, the first term this is symmetric in w and u, if I replace w/ u and u/ w I get the same term also the co-efficients of w and $ur_1$ so it is symmetric and w and u and also linear in w and u so it is bi linear and symmetric, and let us say I call it B(w and u ). The second term is dependent only on w, q is known q is the data of the problem right, and it is linear in w so it is, it is linear in w okay, so I call it $L_1$ of w and in the case of the third term this is also linear in w.

So I called it $L_2$ of w okay, so I can write this entire equation as B( w, u) – L(w) where L is nothing but $L_1 + L_2$ , so equation a and b are same question a, b , B represents this bi linear term and L represents all other things so of course I have to put a 0 here, and then in the last class we had discussed that w has to be chosen such that it represents the variation in u so I will rewrite this equation like this okay. Or if I want to expand this I can write it as 0 to L a, d – and just think about it x is known at x = 0.

At x is = 0 x is known right oh sorry, x =0 u is known yes, which means the variation of u at x = 0 will be 0 so that term will not exist at least in context of this problem so the only term which will exist is the term associated with x = L position, so this is equal to w and a, du, dx so a du, dx is L, oh sorry at x = l $ad_u$, $d_x$ is defined as $q_0$ from this boundary condition okay, so this I have to evaluate at x = L and since we said that w is nothing but variation in u so I will replace w/variation in u.

Now this term I can write it as variation of the bi linear term I can express it as this agreed? Also here I can move this variation operator outside the integral based on the discussion which we had 2 weeks back that I can move, I can switch the differential operator and the integral operator.

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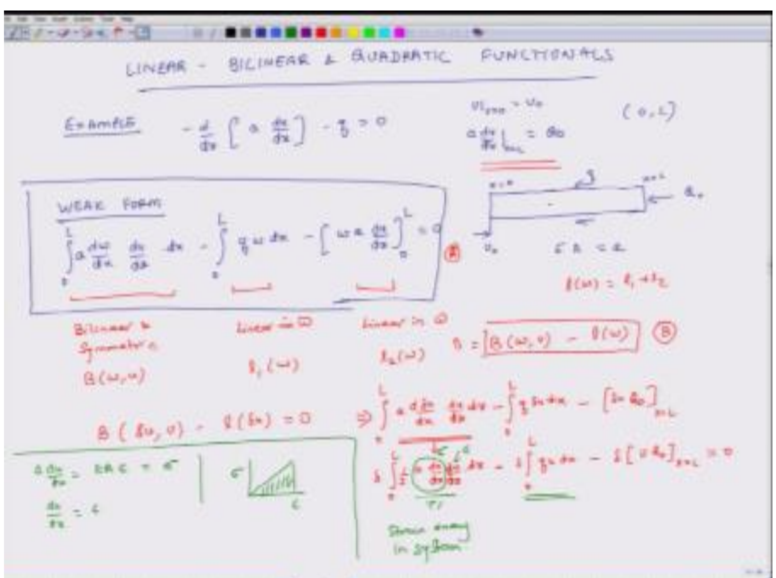


And excuse me, I can switch the differential and the variation operator back and forth and for the same reason I can also switch integral operator and the variation operator back and forth. So I can take variation out so this becomes $qd_x$ and then this is variation of u times $q_0$ at $x = L$ and this is equal to 0, understood. Now a is, now we will look at term 1 first term, a is e times a, a is e times a and du were $d_x$ is the strain in the system, if I am pulling a rod it will stretch by little and the differential of that stretching is the strain.

So this is nothing but stress $xC_L$ is stress and the other $d_u$, $d_x$ is the strain, so when I multiply so this is by stress, this is by strain and when I multiply stress by strain so this is stress this is strain and that is my stress when I multiply, if the linear system so the slope is $E_A$ when I multiply this thing E/A then I get basically the work done in the system and that is the it is stored in the system as potential energy okay, it is stored in the system as potential energy so this is strain energy actually.

An actually this and that is why we have this term half also because when I compute the area of this triangle I have to have half there okay, so this is the strain energy in the system and not in the system this, this is for per unit volume and when I am integrating it to other whole domain this is the strain energy in the overall bar, if I do not integrate then I am getting the strain energy per unit volume at that point of interest, when I am integrating it which is over the domain 0 to L then it is in the entire bar, in the entire bar, this is q times u times dx so q is traction, suppose the bar is go is.

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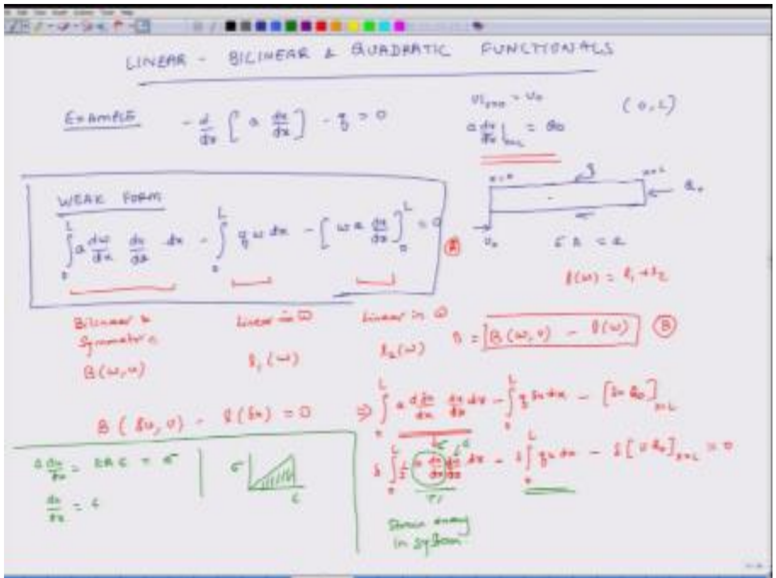

In a call it and I am pulling it if there is some friction or some traction.

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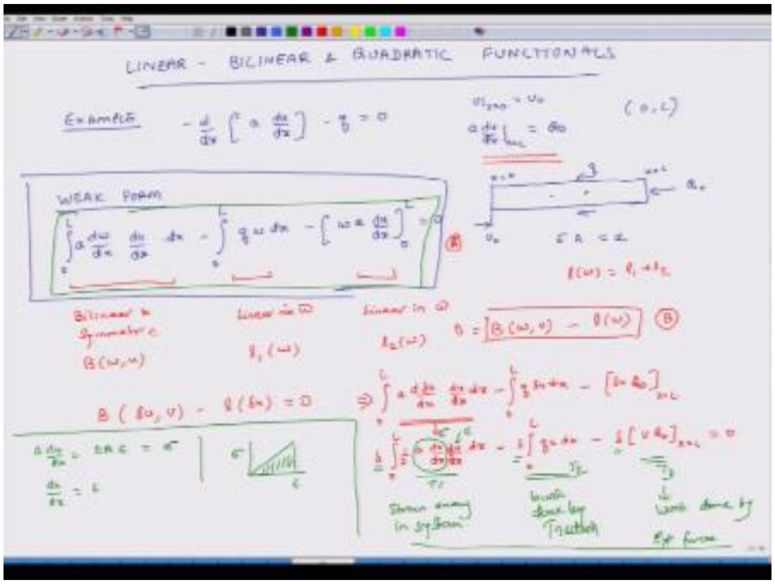


This thing will also do some work and the work which will be done by this is suppose q is that tractional force, and the displacement at this location is let us say u then this is the work done by traction right. And this the third term $t_3$ this is $t_2$ what is this, q is the external force and u is the displacement at x = L, so this also work represents work done, work done by external force agreed, so what this means is that at least in context of this bar problem this entire thing, see I have put this variational symbol outside so what this means is.

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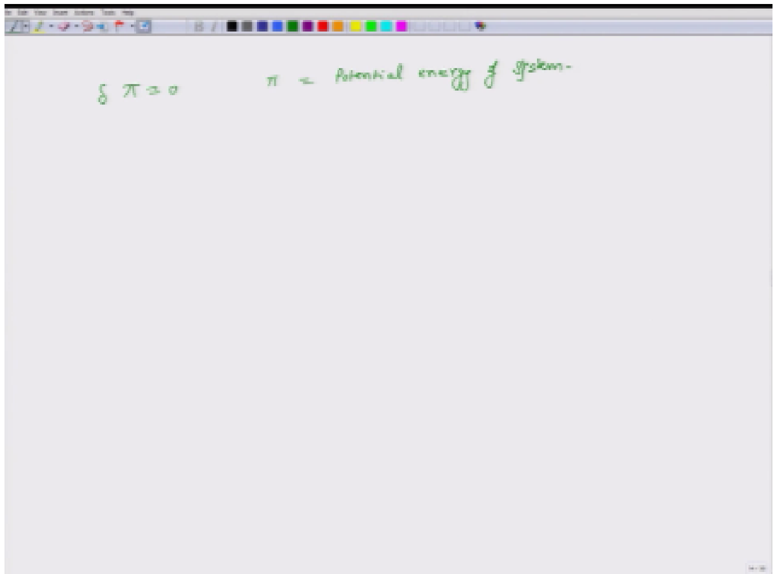


That I can write the entire equation as where л is the potential energy, i is the potential energy in this system.

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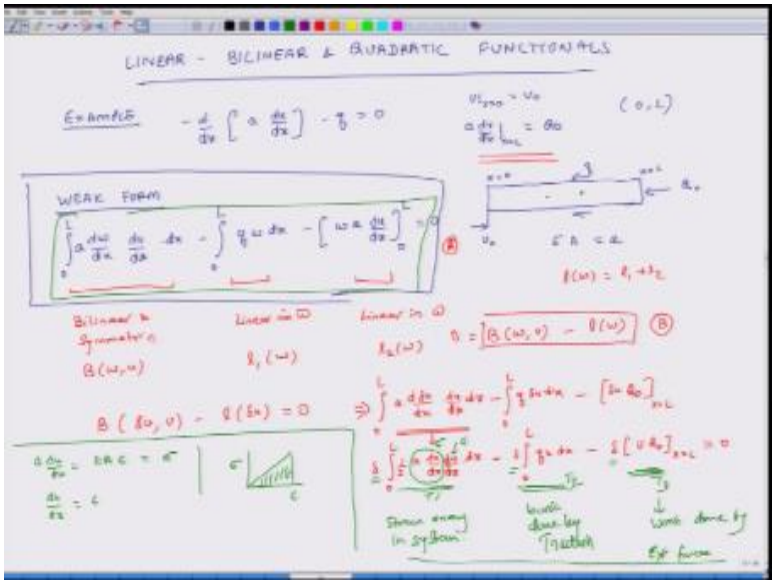


So the potential energy of the system is basically the sum of $t_1$ + sum of $t_2$ + sum of, I am sorry the sum of $t_1$, $t_2$ and $t_3$ okay.

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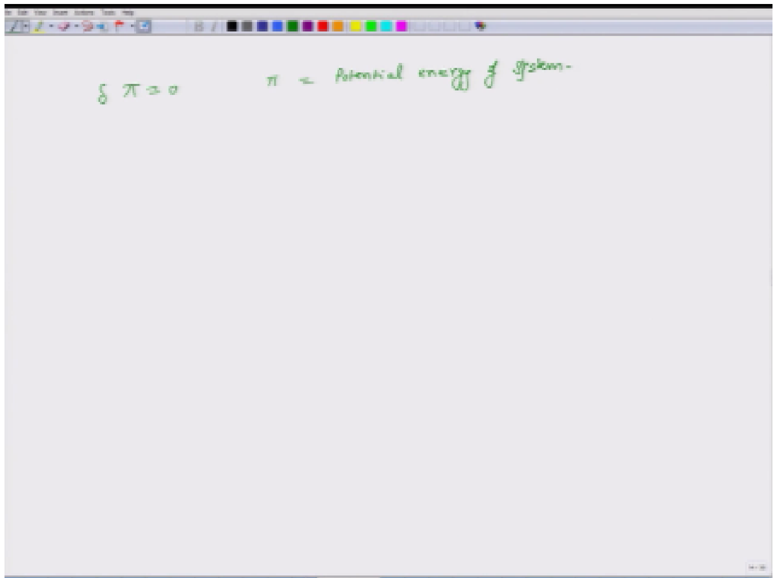


The  potential energy л it represents.

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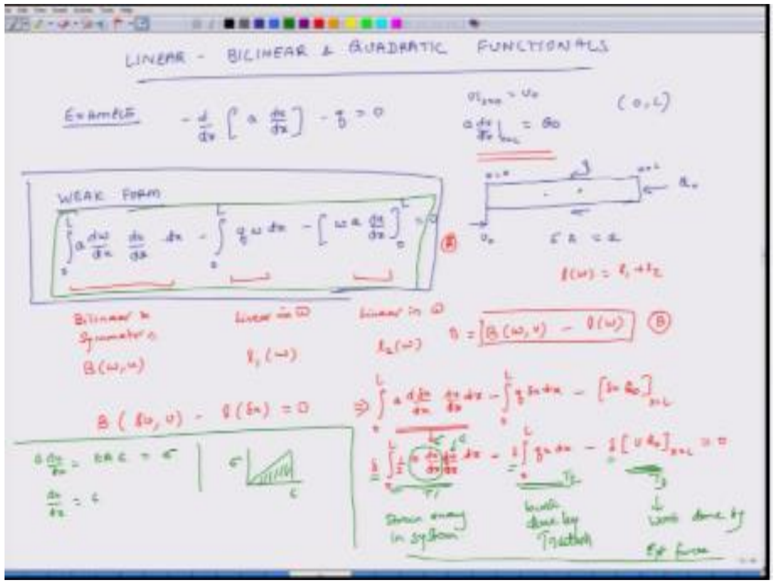


Sums of t1 and t1 includes the integral it is not just at a point it is the integral of all the small, small amounts of potential energy due to strain.

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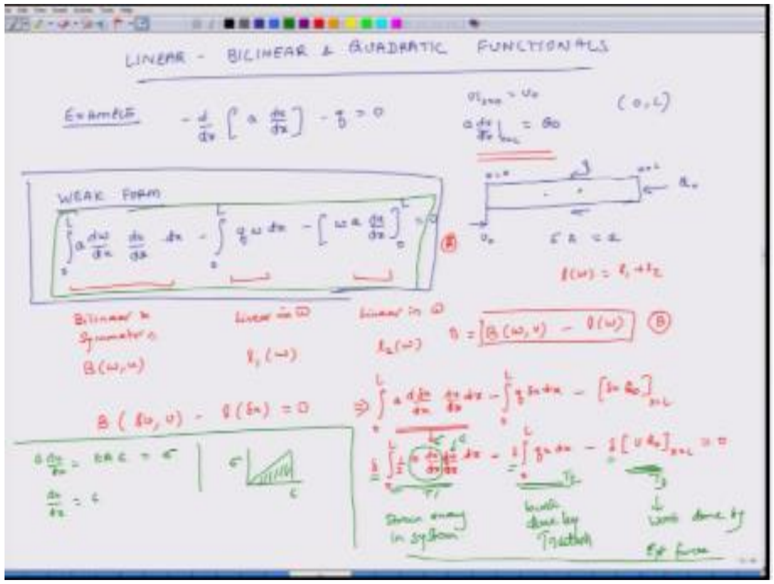


Potential energy due to traction and due to external force right, so it is a total integral quantity so it represents.

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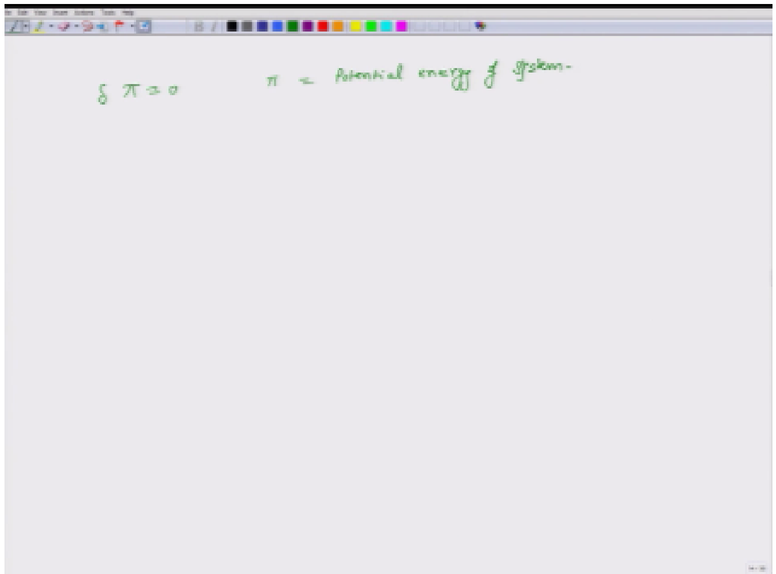


The total potential energy in the system л is the potential energy in the system and I am taking its variation to be 0. Now in differential calculus we know that a function is at a minima or a maxima only when its first derivative is 0 right. The function is at its maxima or its minima only when its first derivative 0 right.

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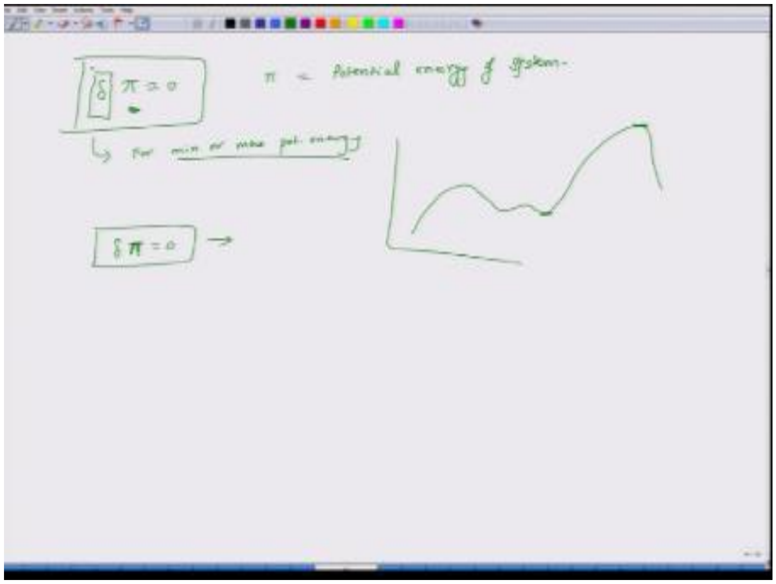


I mean you have suppose so at this place your derivative will be 0 also at this place derivative will be 0, this variational operator we have seen that it behaves very similar to a differential operator so this also tells us that л this is the condition, this is the condition for minimum or maximum potential energy okay, it is an extremer it is a condition for an extremer and at least in context of this course we will not worry whether it is going to be a minimum or a maximum because the problem which we will discuss will be only dealing with.

Minimum situations so if we use this criteria when that tells us, so what does this mean that the potential energy of the system will be at its minimum when whatever variations we take and how do we get these variation, by changing these $w_{'s}$ by small amounts right, so whatever variations we take in w in a use they will cause correspond to small changes in л, and if those small changes are 0 then we will see that the system is in equilibrium, the system is in equilibrium.

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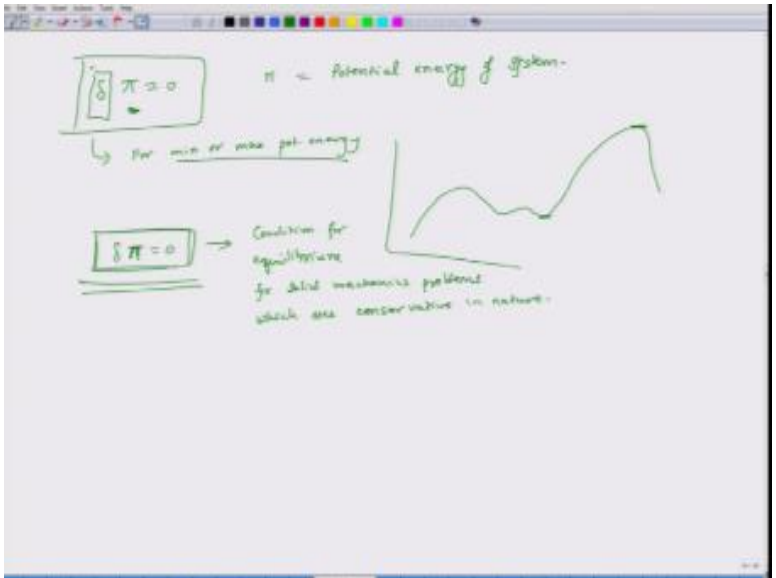


So this is the condition okay, so this is the condition for condition for equilibrium for solid mechanics problems which are conservative in nature, so if some heat is getting dissipated and all that then maybe this is cannot be exactly true, but we have also seen.

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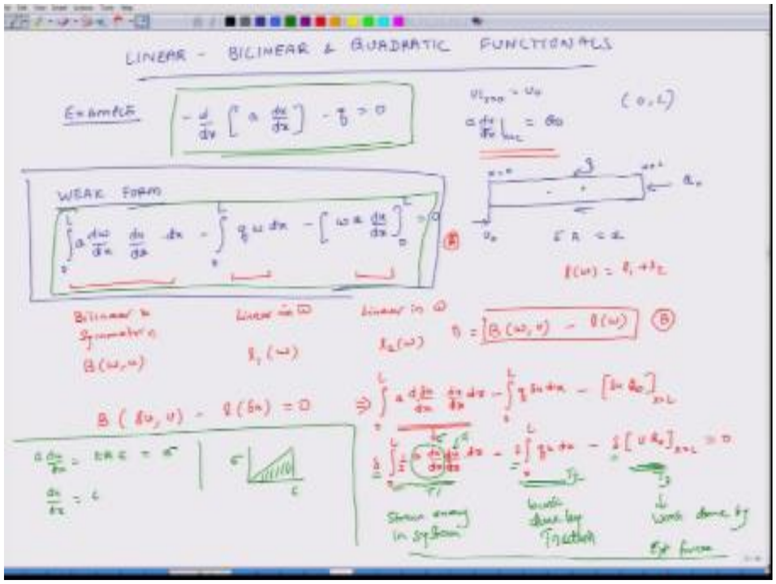

That this equation is not only valid for solid mechanics problem right but it is valid for a big range of problems which we had seen, it is valid for cables and bars so these belong to solid mechanic situations, but it is valid for heat transfer laminar and all that so in context of solid mechanics.

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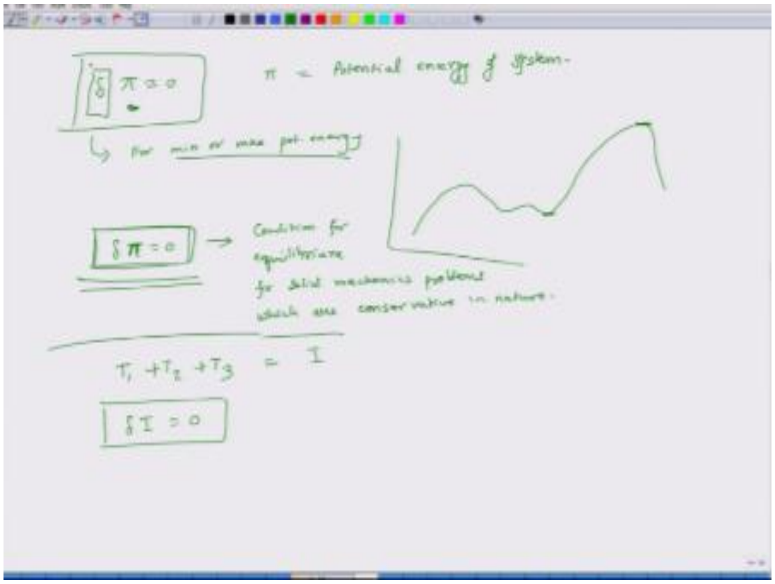


I can say that $t_1 + t_2 + t_3$ represents the potential energy but in general I cannot say that this is potential energy so I make a general statement that $t_1 + t_2 + t_3$ represents some functional i right. Some functional i and the condition for that functional to be at a minimize, that its variation should be 0 understood. I can also write this as so if I go back and look at.

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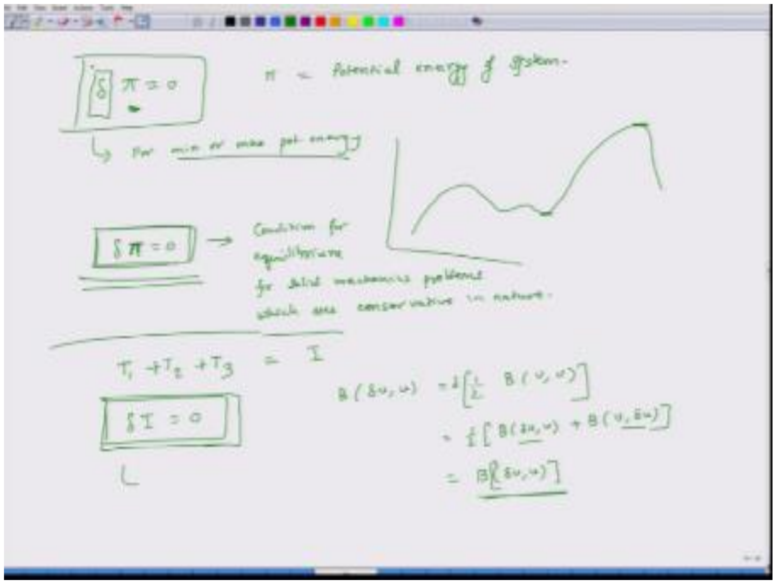


This equation I can rewrite it as, why is that, I first take the variation on first thing so I will get 1/2B this thing how do I say that this is equal 1/2 B right, and because this is linear and symmetric and u so this is nothing but b of, so all this discussion about weak form and about this criteria for having a minima is valid only if we have B which is bi linear, it has to be bi linear, if suppose it was u2 then when we do the math it will not work out like that okay, so it has to be bi linear so this criteria.

Is valid only if B is bi linear and it is symmetric. There maybe situations probably not in context of this course that B maybe symmetric but not necessarily bi linear, then we can still get this.

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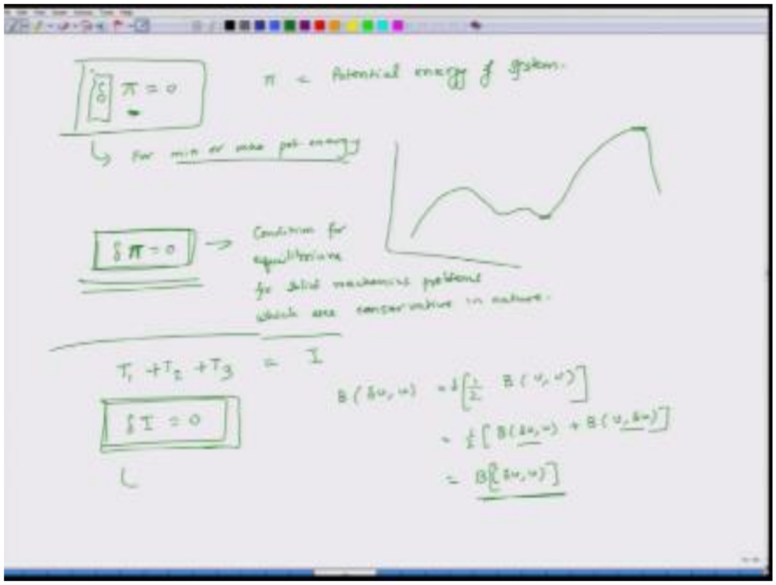


Variation of I to be 0 but not through the process which I have explained but through some other process which we will not discuss in this course, but in this course we are saying that variation of I = 0 and we show that only in context of functionals which are by linear and symmetric functionals there to be both okay. And one second i represents some quantity and if its first variation is 0 then it means that i is either at it is maxima or at its minima, that is attrix extremer and in context of.

Solid mechanics conservative systems i corresponds to the total potential energy of the system okay. So this concludes the discussion for this particular class and in the next class we will start discussing in detail about different variational formulations thank you.

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