Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 15
Weak Formulation

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Hello, welcome to basics of finite element analysis, today is the third lecture for the current week, and yesterday we had discussed weak formulation and we will continue that discussion today as well.

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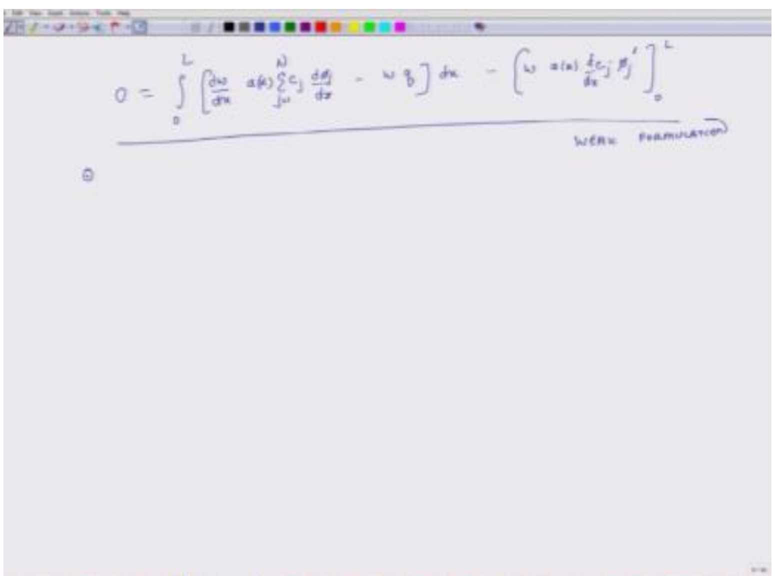

So in the last class we had developed a weak formulation for a particular differential equation, and that weak formulation was 0 equals 0 to L dw over dx $a(x)$ c_j, so this is the weak formulation okay. Now I wanted to make a couple of very important points in context of this weak formulation. First is that as we have several times earlier discussed, that the requirement because of this weak formulation for differentiability on the, on the dependent variable u, it gets reduced, if it is a fourth order differential equation we do it two times I need only a function which is differentiable two times. If it is second order then I need only at, a linear function which will meet the requirements. So that is one thing, the second thing is that,

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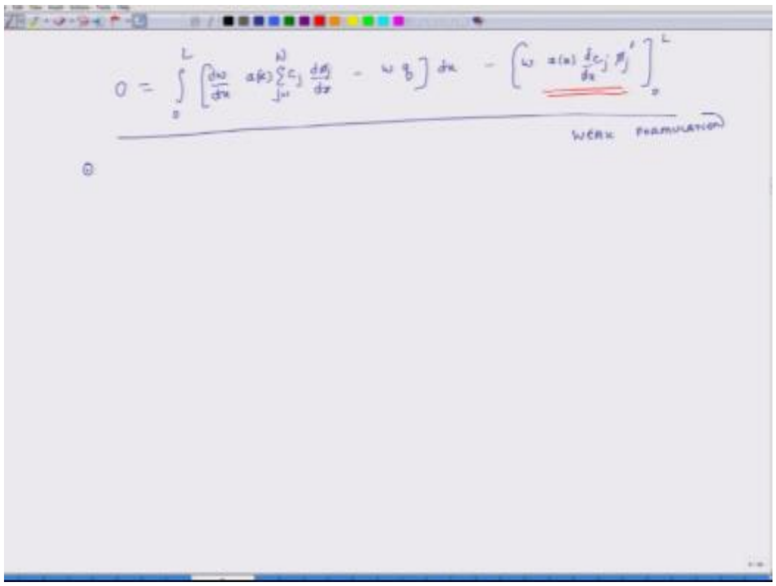

This and this is also we had discussed in the last class was that, this boundary term which comes out through the process of integration by parts it actually helps us, it actually helps us satisfy the natural boundary conditions identically. So in their strong form we would have to choose these shape functions in such a way that they meet natural as well as essential boundary conditions. If I use the weak far, then I have more flexibility in choosing the shape function and the only thing I have to worry about is that the shapes and functions should satisfy the essential boundary condition.

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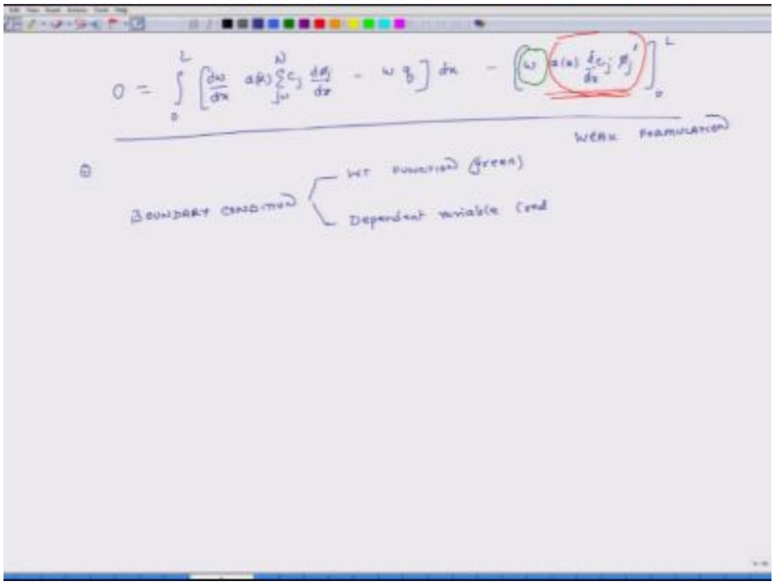

The third thing is, it also helps us classify what kind of boundary conditions are involved. So we will discuss this in detail, so let us look at this, so you have in the boundary condition there is this thing which is the shape function which is in green, and then there is something in red okay. So there are two parts, so the boundary condition it involves weight function and that is in green, and then it involves dependent variable in red.

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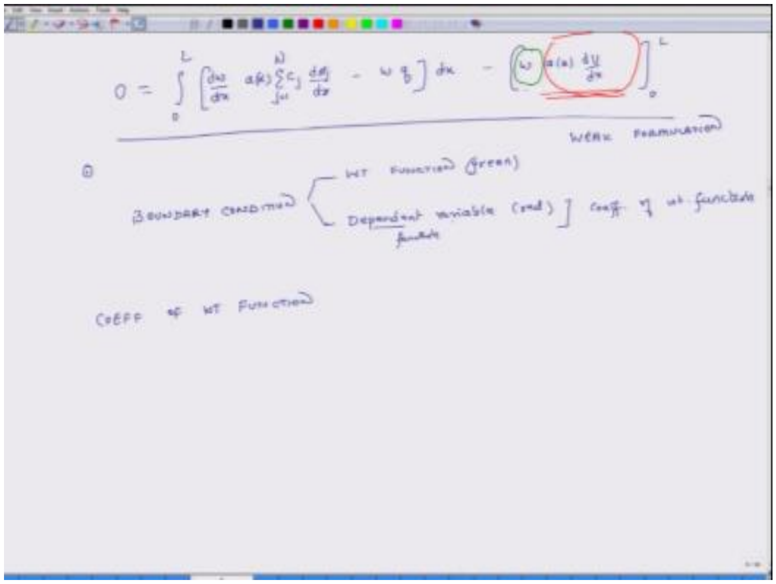

Actually at this stage I will replace this by u, du over dx okay which is mathematically the same okay. So it is dependent on yeah it is, it is a function of a dependent variable, dependent variable function. It is not dependent variable itself, but it is a function of dependent variable. So this is like the co-efficient of weight function, there is w and we can treat a(x) du over dx as the co-efficient of weight function.

Co-efficient of weight function, now in second order equation there is weight function, in a fourth order equation we will have, we will have to integrate two times so first we will get a weight function and second time when we integrate we will get a derivative of the weight function okay. So if there are,

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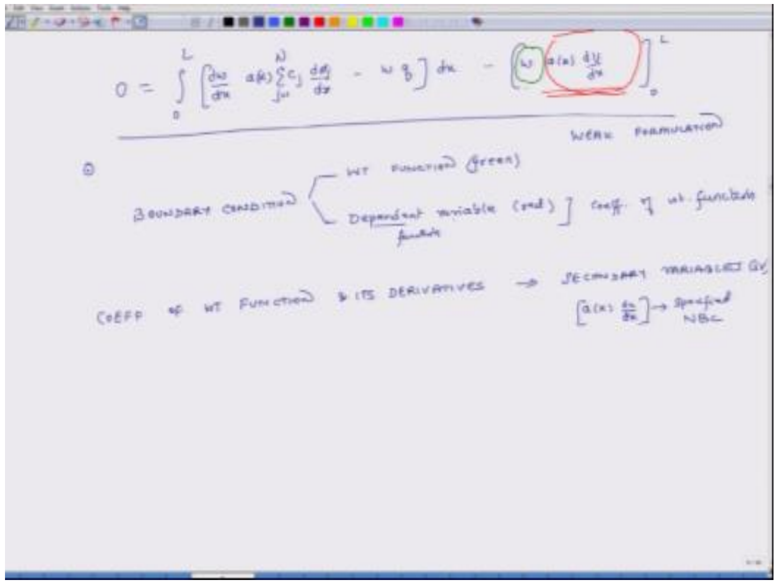

Weight functions in the boundary terms or there are derivatives of the weight function then they are coefficients, co-efficient of weight function and its derivatives okay, they are known as secondary variables (SV) secondary variables. And if I, so in this case it is a(x) du over dx okay that is my secondary variable. And if I specify this, then I say that I have specified the natural boundary condition okay. So this is the logical way to say which one is natural boundary condition and which one is the essential boundary condition.

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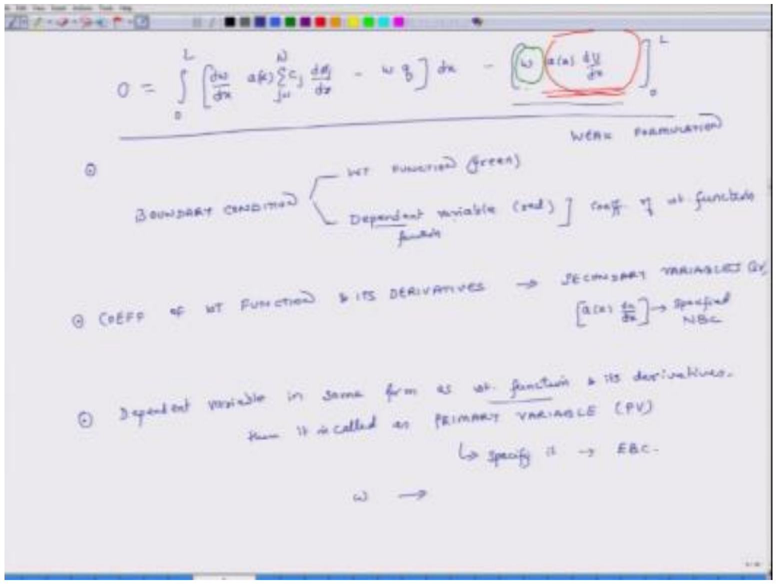

So this is one, the second thing is if we express the dependent variable in same form as weight function and its derivatives then it is called as primary variable. And if I specify it I say that I have specified the essential boundary condition okay. So to make things clear in this boundary term the weight function appears as w, in the boundary term weight function appears as w, if I express the dependent variable in this equation the dependent variable is u. We had discussed mentioned this right in the beginning. So if I,

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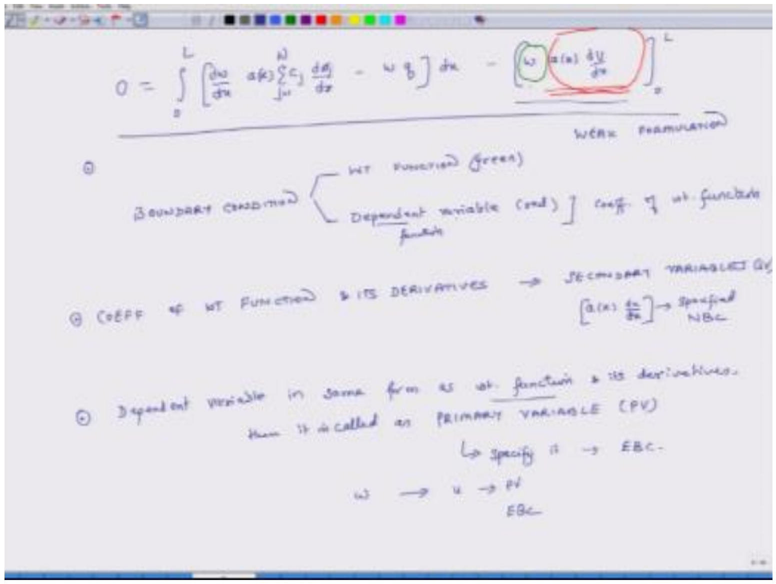

Replace w by u in this boundary term, then u is my primary variable, and if I specify it I say that I have specified the essential boundary condition. In case of a beam, so this is the equation for a bar right, this is, this equation represents a bar or rope intension or electro static problem and all that, but suppose there was a beam.

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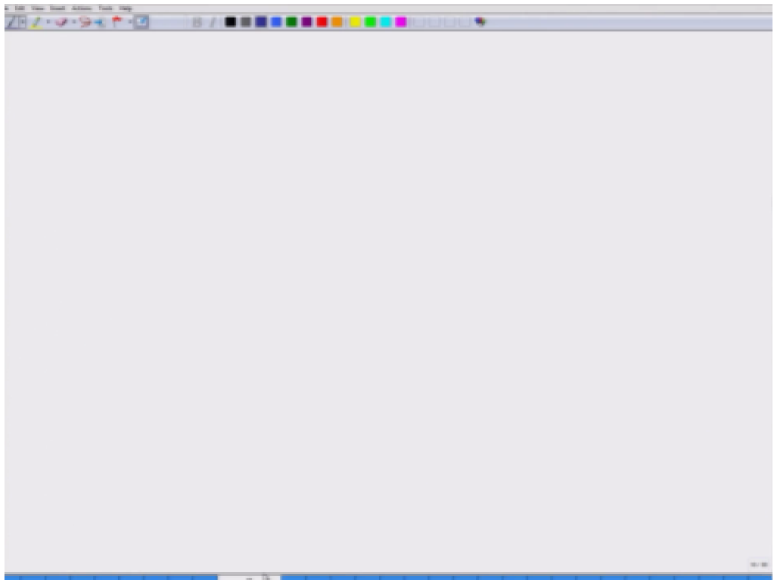

It will be a fourth order differential equation. And there when we integrate it two times.

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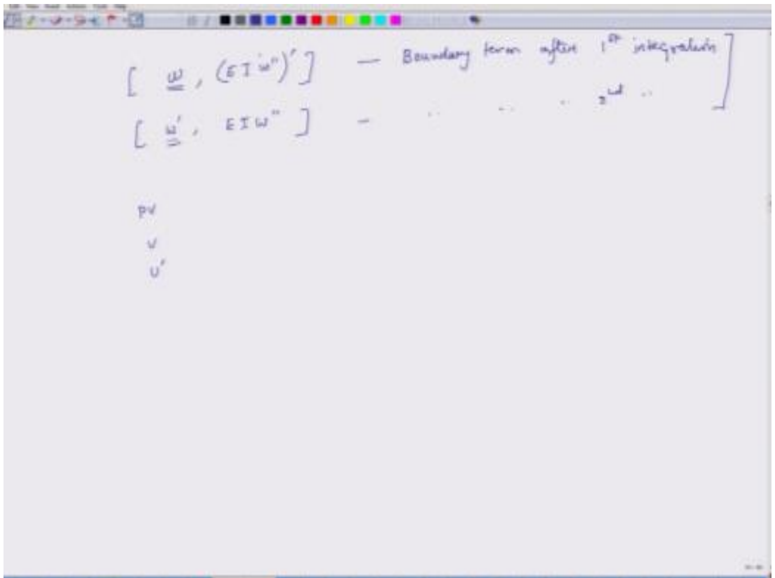

We will get these things, there will be a weight function and there will be its co-efficient. This is what we will get in the first integration, and when we integrate it second time we will get this and, so this will be the first boundary term and this will be the boundary term after second integration okay. So in this case and this happens because the governing equation for a beam Euler Bernoulli beam is a fourth order differential equation right.

So in this case my second primary variables will be, there is w so corresponding to that there will be u, and there is w′ so corresponding to that will be u′, where u is the deflection of the beam okay, u is the deflection of the beam.

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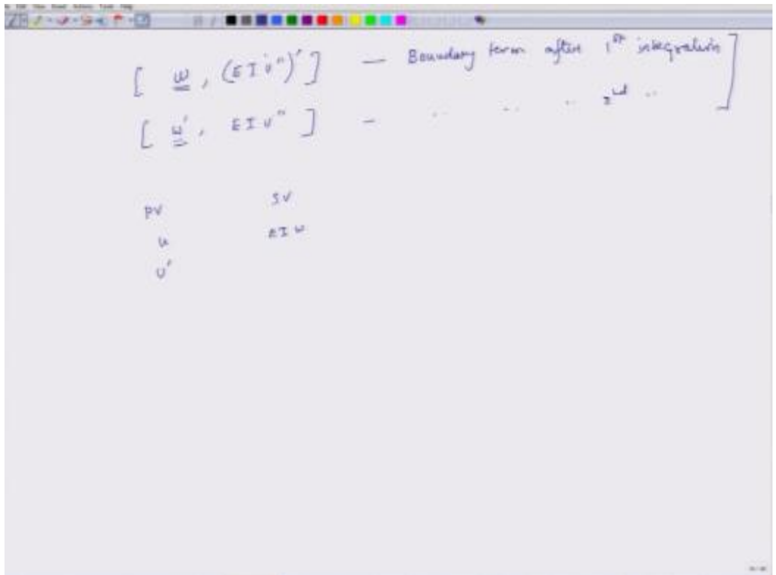

And the secondary variables will be ei, so excuse me this should be, this should have been not w is should have been u, because u is the unknown we are trying to solve.

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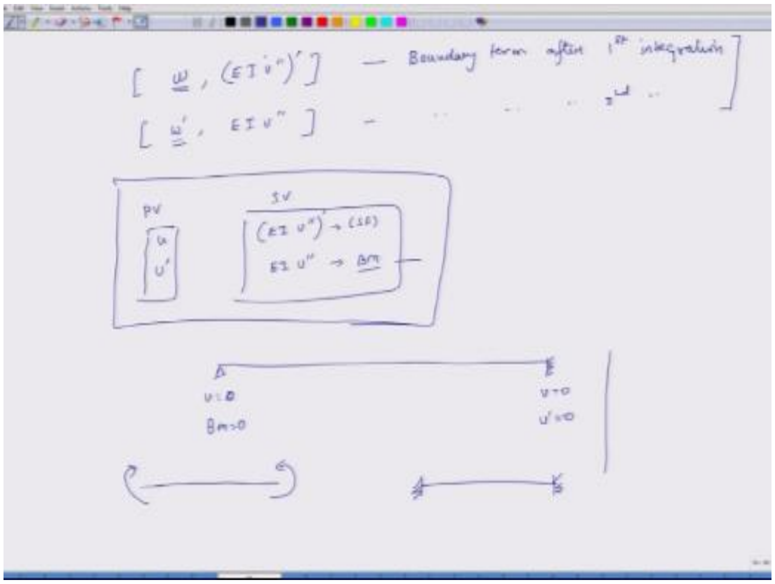

So it is (EI u″)′ and this is what it is the sheer force, it is the sheer force in a beam this is the formula for the sheer force right. And EI u″ this is my sheer force and this is EI u″ is moment bending moment okay. So my essential boundary conditions are u and u′, my natural boundary conditions are sheer force and bending moment. So if I have a beam, I have to specify two conditions at both the boundaries, at either I specify so total number of boundary conditions I have to specify are four and two on each end.

So either I specify u and boundary can a bending moment or u′ also have combination of these two. So at a pin end my displacement is zero right. And corresponding to the second thing my bending moment is zero, and at the fixed end at the rigidly fixed end can till leave you know. My u′ which is deflection is zero and the slope is zero okay. So all these boundary condition terms they naturally come out when we do the weak formulation, and you can classify them and separate them, and you then by looking at the physics of the problem you can figure out which conditions are known, and which are unknown okay. So you can have situations where all the essential boundary conditions are known, for instance you can have a beam like this, in this case u′ and u are known at both the ends, so all EBCs are known or you can have another situation so it is a beam but I am bending it by applying a moment at both the ends.

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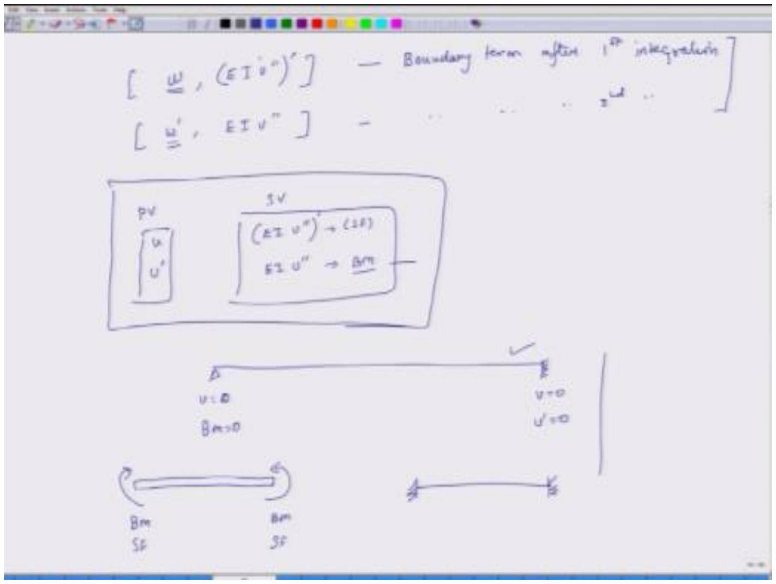

So in this case, bending moment is known at both ends and because I am not applying any vertical sheer force, so sheer force is also known. So in this case, both at both ends I know the natural boundary conditions, and in the third case which may be something like this, I may have a mixture of some essential boundary conditions are known and some natural boundary conditions are known okay.

So this is the way we can specify and classify different boundary conditions, and the next point in context of weak formulation we are going to discuss is, that how do we make the choice of weight functions, because still so far what we have been saying is that we, if there are n unknowns corresponding to n c_j's right c_1, c_2, c_3 then we will choose n different weight functions and because of that we will get n different equations and we will solve those to find the values of c's. So now we will discuss that atleast in context of weak formulation how do we go around choosing appropriate weight functions.

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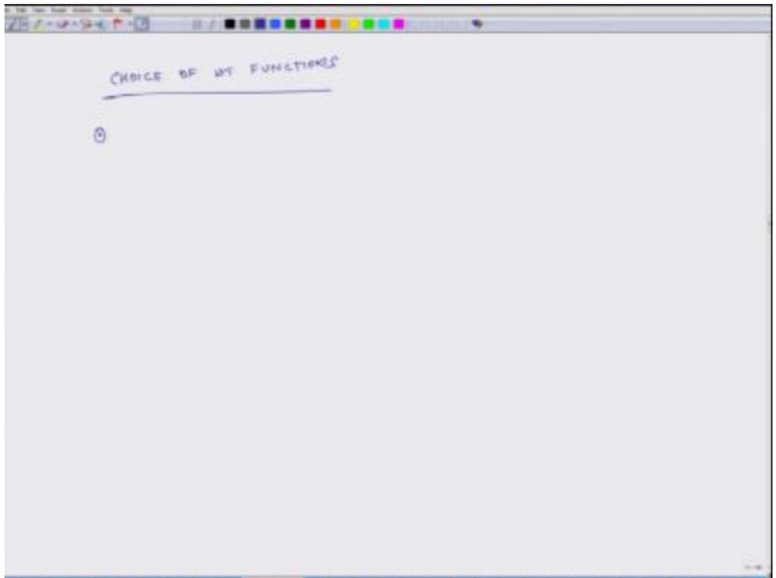

So choice of weight functions okay, so in general we choose w such that it vanishes at the boundary if the primary variable is known at the boundary okay.

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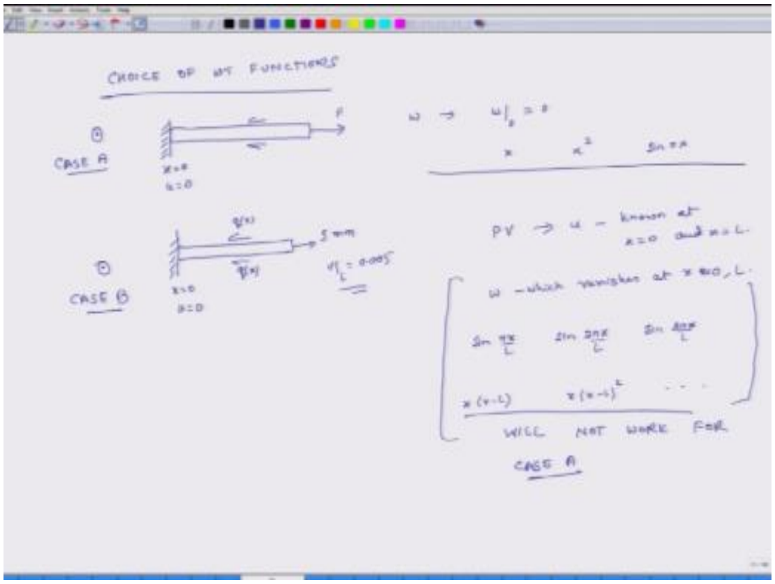

What does that mean, suppose I have a bar and I am pulling it and I am also applying some traction on this, x at x=0, u=0 right. Then when I am making a choice for weight function, I have to ensure that w is zero at x=0 okay. What could be an example of such a weight function, one example could be x, another example could be x^2, another example could be sinΠx and so on and so forth okay.

So this is one, another case so, so here I am applying a force in this case, this is same case but here instead of, instead of applying a force I am displacing it let us say by five millimeters. So u at L is equal to 0.005, in this case what is our primary variable, primary variable is u right, we are discussed what is the primary variable and secondary variable. So primary variable is u and u is known at, at x=0 and x=L.

So if u is known at both x=0 and x=L, then I have to choose w which vanishes at x is equal to at both ends okay. It has to be zero at both ends, w should be such that it is zero at both ends. It could be any function but it has to meet because not because u is known even though u is non zero at x=L, my choice of w has to be such that it is should be zero. In the first case u was not known, I was applying a force but I do not know how much the bar is going to stretch.

Here I am actually pulling it by five millimeters so x is known this u is known at x=L, and also u is known at x=0, so, so in this case we will look at few boundary conditions so it could be one sinΠx over L, another could be sin2Πx over L, sin3Πx over L, this could be one set of functions right, another function could be x times x-L, another function could be x times $(x-L)^2$ and so on and so forth.

So this is how we choose the functions, now these functions will not work so let us say this is case A and this is case B. So these functions will not work for case A why, because if we choose these functions then we have overly excessively restricted our choice of functions okay. Here the only restriction is at x=0, u_0. So I cannot have the weight function behave in such a way that the weight function becomes zero at the other end of the rod okay.

So the next thing is that why do we make such choices, why do we make such choices okay? So the first reason is, that we have to choose these weight functions in such a way that they represent variations in the primary variable. They represent variations in the primary variations, so.

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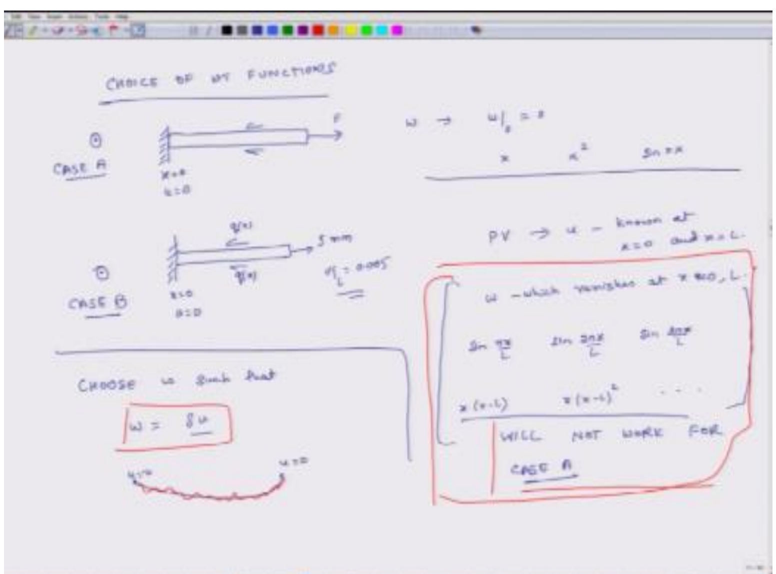

Choose so in a general sense choose w such that w is a variation of u okay. So variation of u and if you remember our discussion of variations the variation of u suppose you have a string and it is, and it is suppose u is zero at this end and u is zero at this end, then the variation of u could be any function as long as, as long no not small because the smallness is dictated by that amplitude alpha if you remember.

As long as it does not violate the end conditions right, wherever the variable is known at that place variation has to be zero. At all other places it can be non zero, so if I choose these functions for case A I will not truly represent the variation in u because I will be specifying that variation at the other end of the rod is also zero. So once again I will like to restate that we choose w in such a way that they represent variations in the dependent variable.

Wherever the dependent variable is known at those places the value of w has to be zero, wherever it is not known the w need not be zero, w need not be zero.

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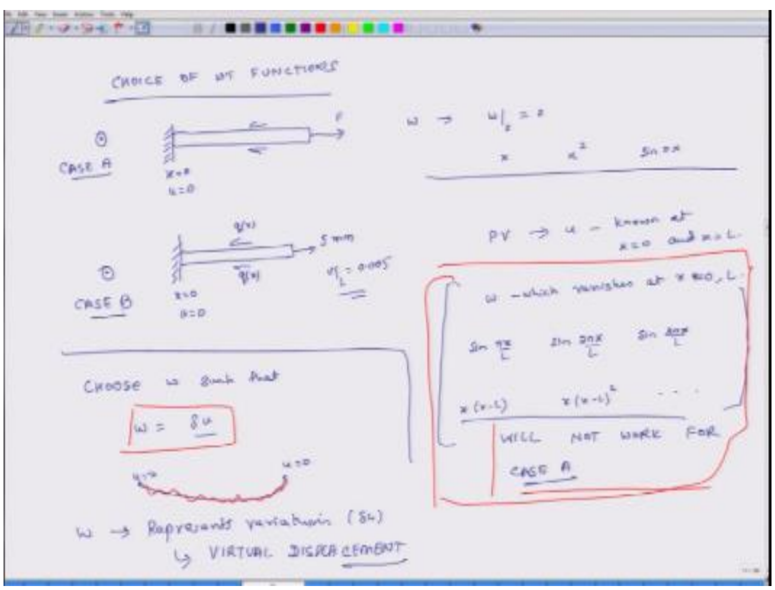

These variational so these, so w represents variations or δu and in mechanics it is also known as virtual displacement, and maybe in the next lecture we will see why we are doing all this. Why we are doing all this? So if I do this then my governing equation.

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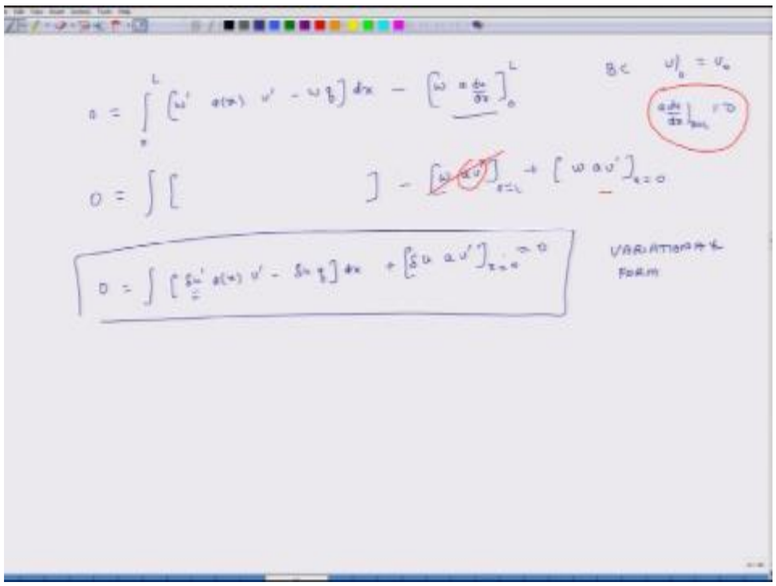

So my original governing equation was this weak form, this is the original weak form of the, for the system right, and my boundary condition was u at zero is equal to some constant u_0 and a du over dx at x is equal to L is zero okay. So what I can do is I get the same term here, [-w au′] where x = L, and +[w au′] x=0 okay, so what I have done is I have just expanded this term in the parentheses.

Now I know that du dx at x equals L is zero I, I am now going to enforce this boundary condition, so this term becomes zero okay, so this, this whole term goes away right or I can yeah, so anyway so this is what I am left with this is the thing. Now if I replace w by variation then I get, this is the equation I get where w is represented by a variation in w, this form where I am replacing w by variations is known as variational form, it is variational form.

And a lot of times this weak form and a lot of times this weak form and variational forms are lot of times they are interchangeably used, they are used interchangeably, other point that if.

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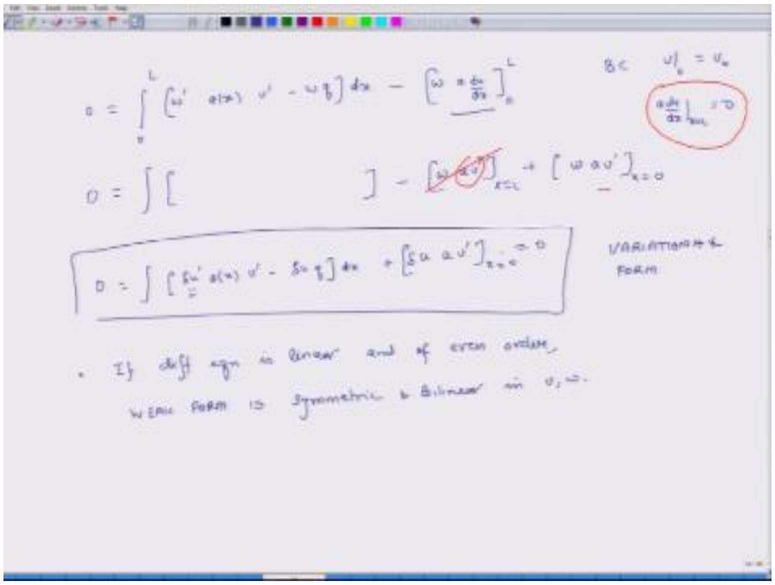

The differential equation is linear and of even order then we will see that weak form is symmetric and bilinear in u and w, so if you look at our original equation.

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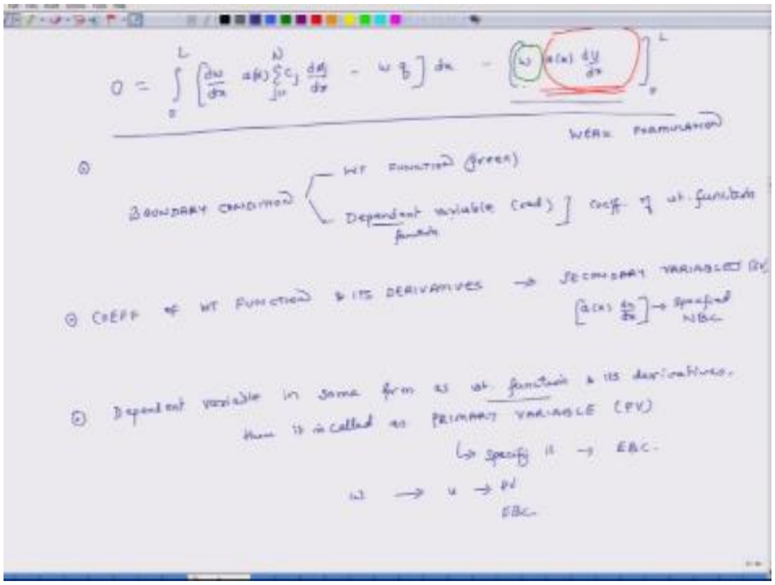

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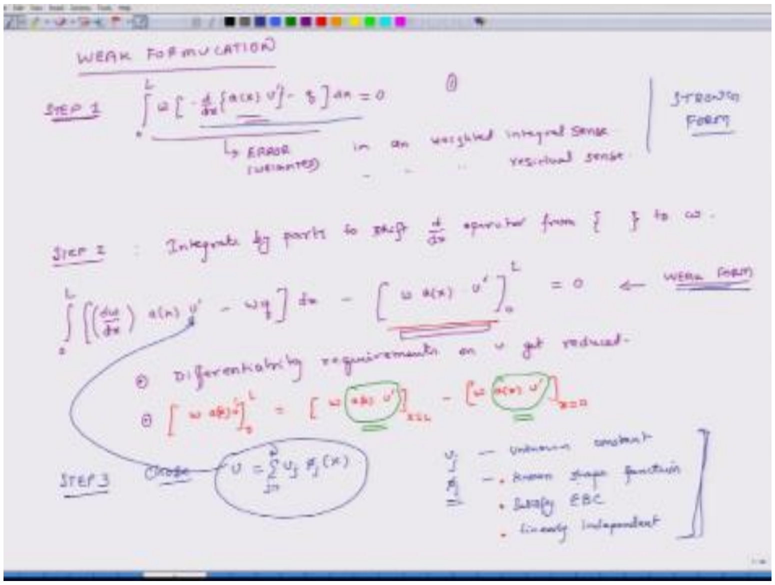

This equation is linear in u, we do not have a u^2 term right, and what is the order of this differential equation, this is a second order differential equation right. Iu′ and then it is going to be a differentiated one more time, so it is a second order differential equation.

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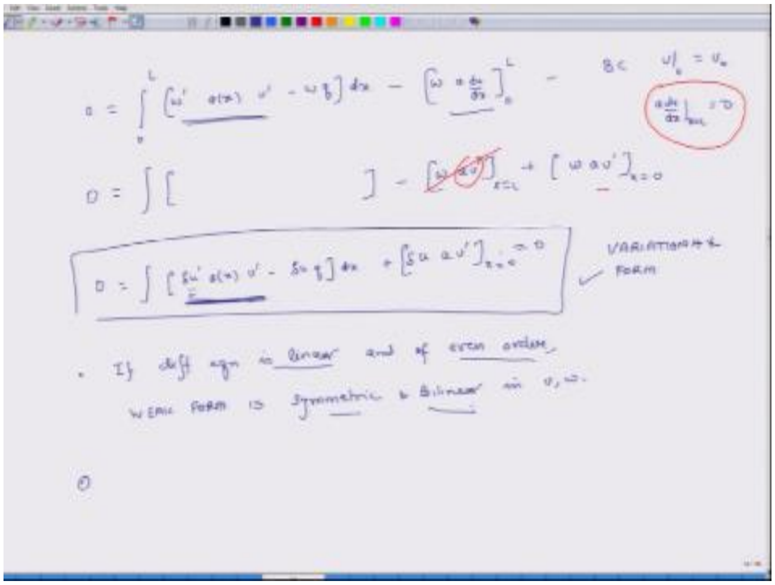

So it is of even order and also it is a linear equation, and for such equations when we construct the weak form, there is another form weak form or this is another, this is weak form then you see that you have this term w′ a(x) u′, or you have this thing. So this term is we have discussed this earlier, it is symmetric; if I replace w by u and u by w I get the same thing right. So this weak form is symmetric and bilinear okay, it is symmetric and bilinear.

The third thing is, so we have said that differential equation is linear and of even order, weak form is symmetric and bilinear. And the next point to make is that because w represents variation in u, it represents variation in u its nature should be similar as that of u, it should belong to the same space as u, it should belong to the same space as u okay. And what that means is that when we are choosing different options for picking up w's, we do not have to see when we were, we had made, we have made some choices for picking a fees, hej's right, which represent u's, we can use same functions to represent w's also.

So that makes our life easier, makes our life easier. So this concludes our discussion on this particular topic atleast in context of weak formulation, now in the next class what we will discuss is three other concepts, linear, bilinear and quadratic functionals, and then we will show that when we are multiplying the error with the, with the weight function and we are integrating it over the domain, and then when we are creating a weak form of this, we are actually minimizing some overall function.

And in context of solid mechanics essentially what we are doing is that we are minimizing the total potential energy of the system, which and whenever in a solid mechanics we minimize some total potential energy that is the state, that is the condition for its equilibrium. So that is what we will discuss in the next class. Thank you.

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