Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 14
Weighted Integral & Weak Formulation

by
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Hello, welcome to basics of finite element analysis, in the last class we were discussing the concept of variation and we were, discussed at length the variational operator so what we will do in the reaming part of this week is we will use that particular concept in context of different types of formulations and starting today we will be discussing weighted integral and weak formulations.

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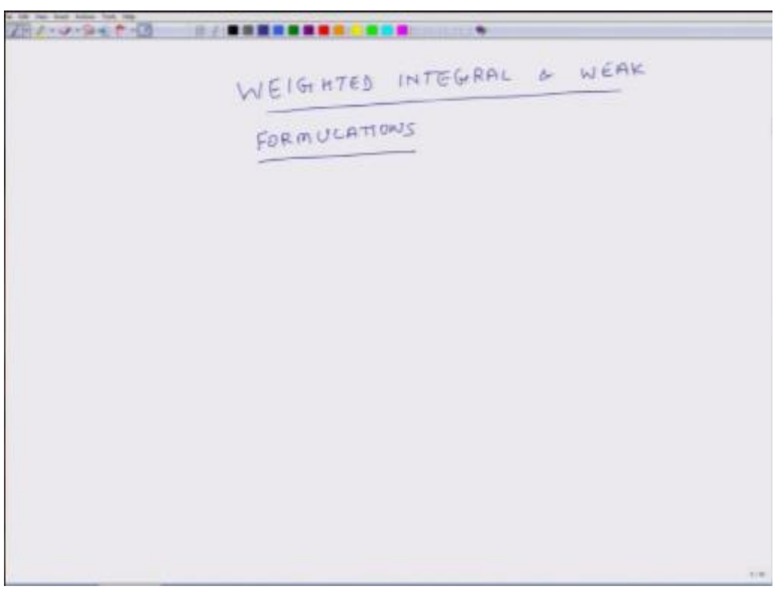

So we will illustrate what I intend to talk today and then later generalize through an example for starters so what we will do is let us look.

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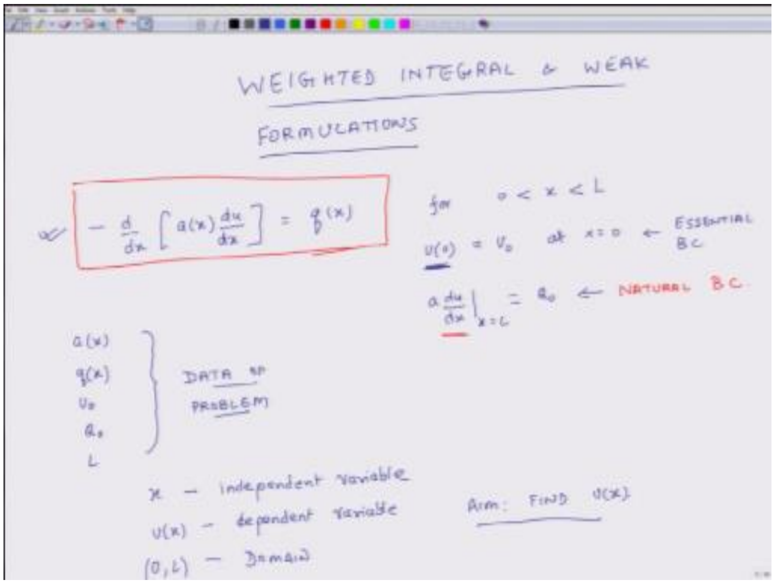

At this differential equation and this equation is let us say it is valid for the range 0 to L, so the domain is 0 to L and the boundary conditions which are specified are that u value of u which is the unknown in this equation is equal to some constant u_0 at so okay at x is equal to 0 that I have already mentioned in the parenthesizes and a times the slope are the derivative of u at x=L is equal to another constant let us say q_0 so at x is so we have two ends x=0 we have one boundary condition, another boundary condition exists say at x=L, this first boundary condition where I am specifying u is known as an essential boundary condition.

And how do we figure out whether a boundary condition is essential or not we will discuss this later but I wanted to introduce this term right now. So the first boundary condition is essential boundary condition and the second boundary conduction which relates to a times du over dx is known as natural boundary condition okay. Couple of other things a(x) is known it is a function of x, similarly q(x) is known, u_0 is known which is relates to this boundary condition, q_0 is also know and L which is the value of x all these are known things, so all these are referred as data of the problem.

Whatever we know about the problem that is referred as data of the problem and then of course x is the independent variable and u(x) is the dependent variable, and our aim is find u(x) okay, and finally 0 to L is domain of the problem so that is the domain of the problem. Now before we discuss how to solve this problem I wanted to explain the physical meaning of some of these parameters, meaning of a, meaning of q, u_0, q_0 and L and it just stands out that this particular differential equation is valid for a very large number of problems so.

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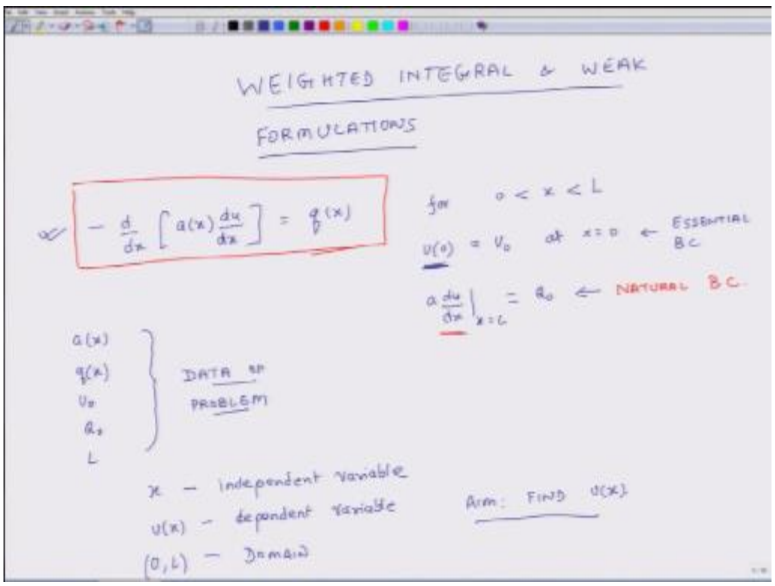

We will look at what types of problems this particular differential equation is valid for, so I will just rewrite the differential equation.

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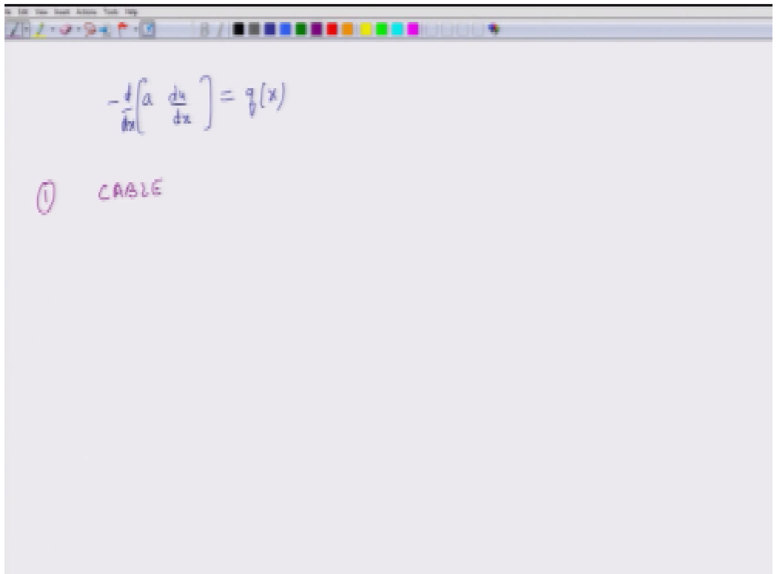

Okay so the first area where this is valid is a cable okay cable, so if I have a robe or a cable this.

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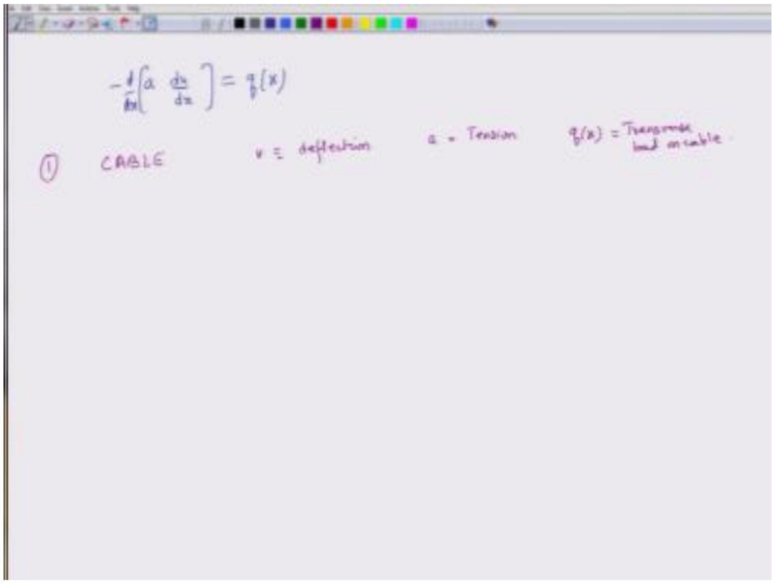

Tells us that how the what the shape of the cable is going to be which is hanging on its own so here u corresponds to a deflection, a corresponds to tension in the cable, q(x)which is the term on the right side of the equation it is the transfers load on the cable so an example of transfers slide could be the weight of the cable itself or if there is something sticking to so that is the transfer or I can apply some external force also okay so that is the transfers load on the cable.

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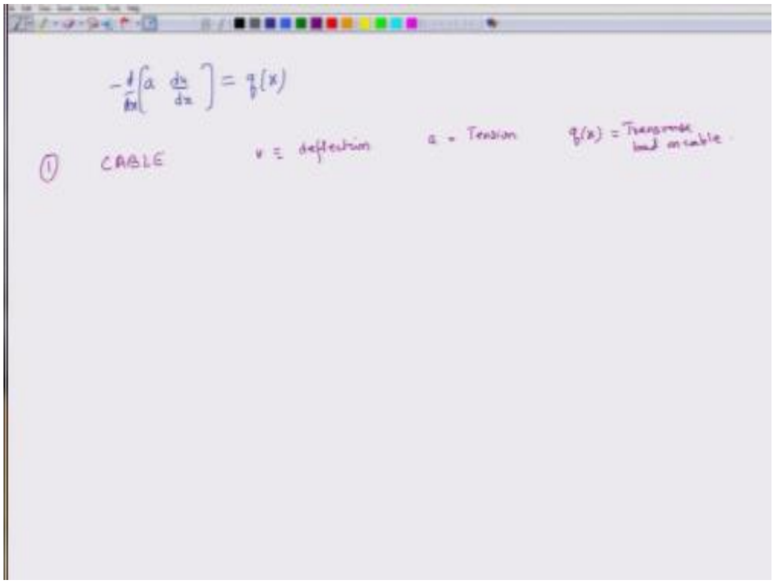

And

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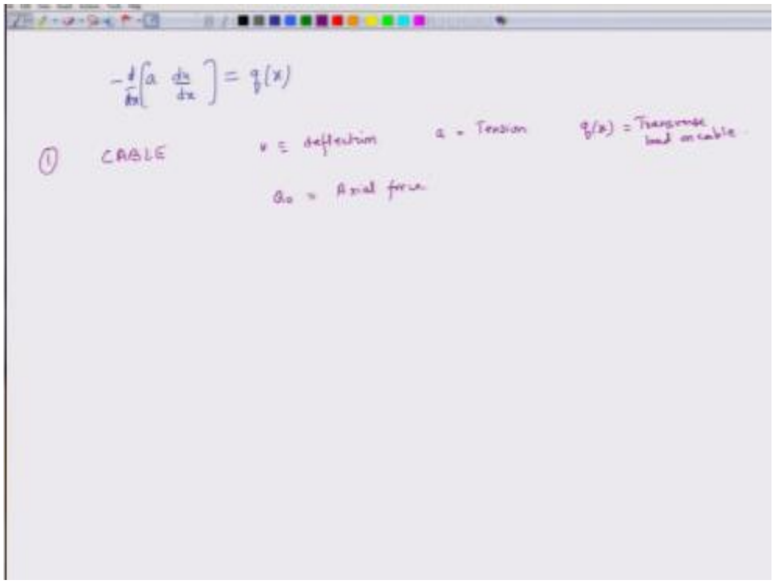

In the boundary conditions we had this parameter q_0 okay, so q_0 that is the axial force on the cable so it could be that I am pulling the cable at one end so that is the axial force it is fixed at the other end so that q_0 is axial force okay.

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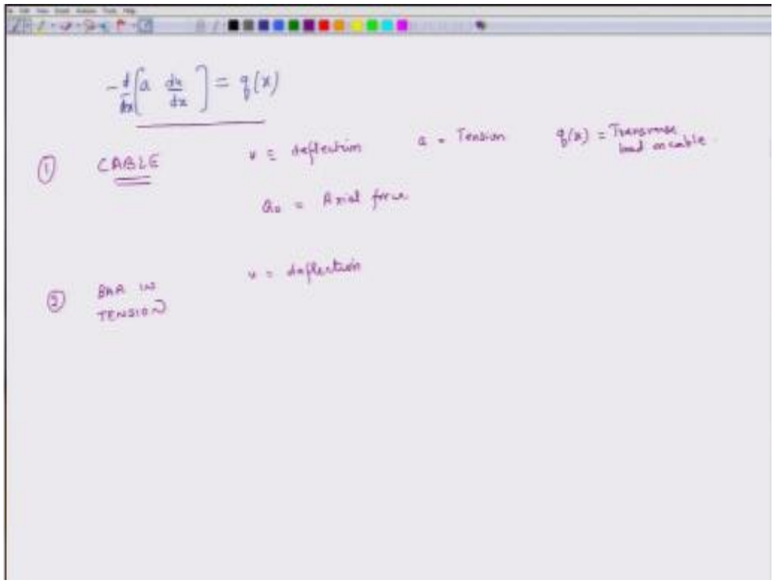

So this is one area in which I can use this differential equation but it is applicable to several areas, the second area in which the same differential equation works is bar in tension okay, so same differential equation if I know how to solve this equation I can solve different types of problems, so this is the bar in tension, here u is my deflection okay it is the axial force, no I am not saying deflection q, q_0

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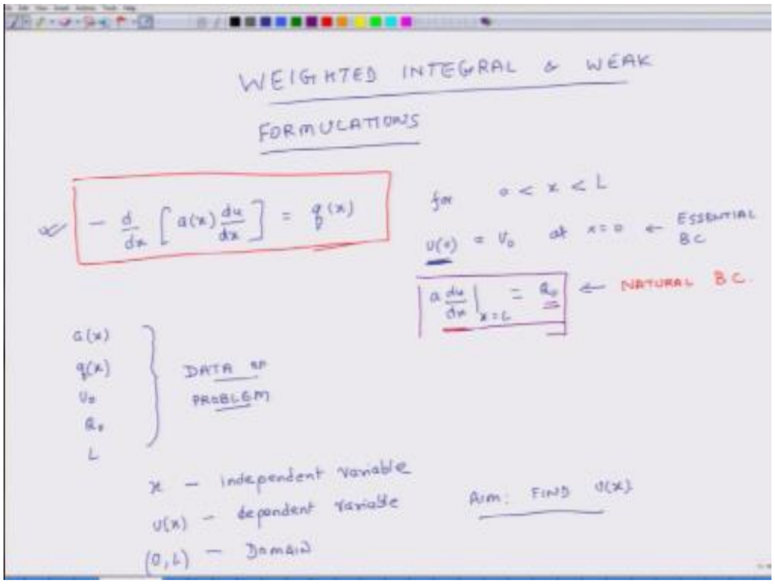

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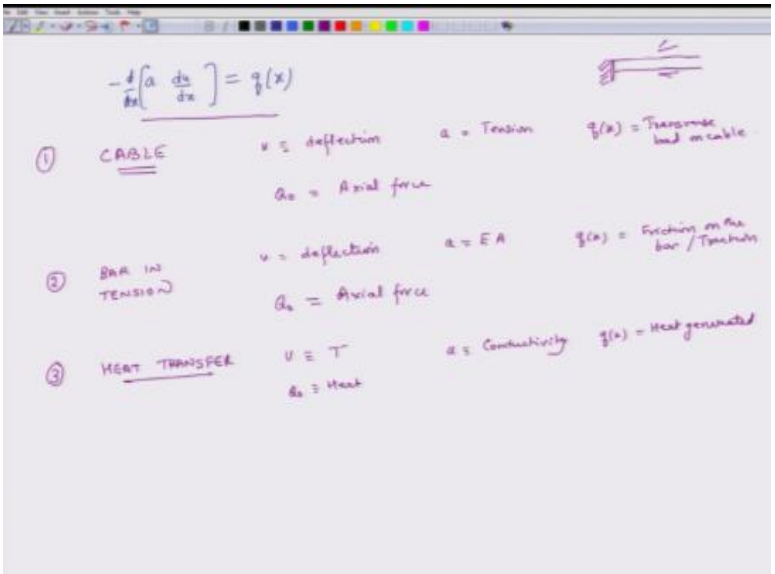

So u is deflection, a corresponds to a constant E which is young's modulus times cross section area of the bar okay, then this q corresponds to friction on the bar or some people can call it traction so what does it mean suppose I have a bar like this, let us say the bar is fixed here and I am applying some friction here, so that is what q_x means okay, and this Q, Q_0 is axial force okay so these two equations, these two cases are from the area of solid mechanics.

Now we will go to heat transfer, so this is heat transfer this is third area, so in case of heat transfer u corresponds to temperature which T, a corresponds to conductivity thermal conductivity okay, q(x) corresponds to heat generated, and q_0 corresponds to heat, so if we know how to solve one equation we can solve different types of problems that is the point what I am trying to make.

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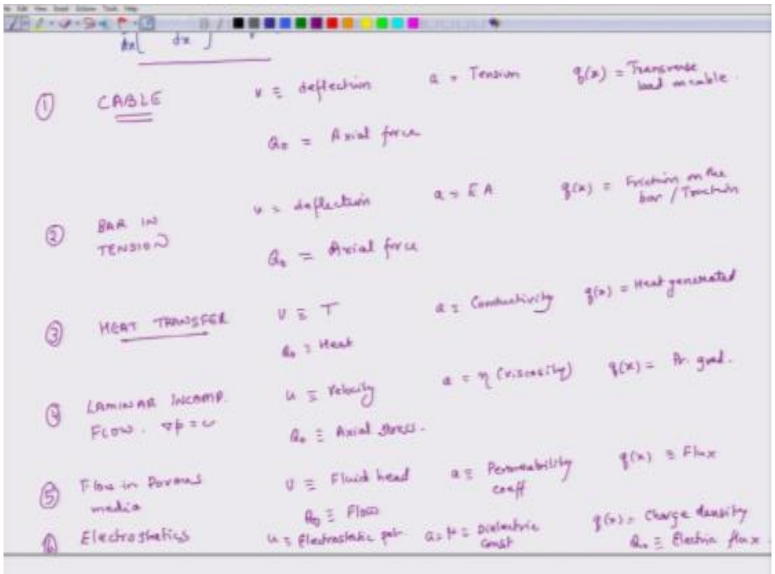

Another example it is this equation also works for one-dimensional laminar incompressible flow where in a channel where gradient of P is equal to constant ok, so here u corresponds to velocity, a corresponds to viscosity, q(x) corresponds to pressure gradient, and q_0 corresponds to axial stress. This is also good for flow in porous media in one dimension, in one dimension so suppose you have sand and you are putting water at one side and how this water moves in this sand the same question will help you understand.

So in that case u corresponds to fluid head, a corresponds to permeability constant, permeability coefficient and then q corresponds to flux and q_0 corresponds to flow okay, and the sixth case is electrostatics so here u corresponds to electro static potential, μ corresponds to permeability coefficient no actually I am sorry this is dielectric constant, and then q_0 q corresponds to charge density, and q_o corresponds to electric flux okay yes, okay. So you will see that there are one single equation if we know how to solve it.

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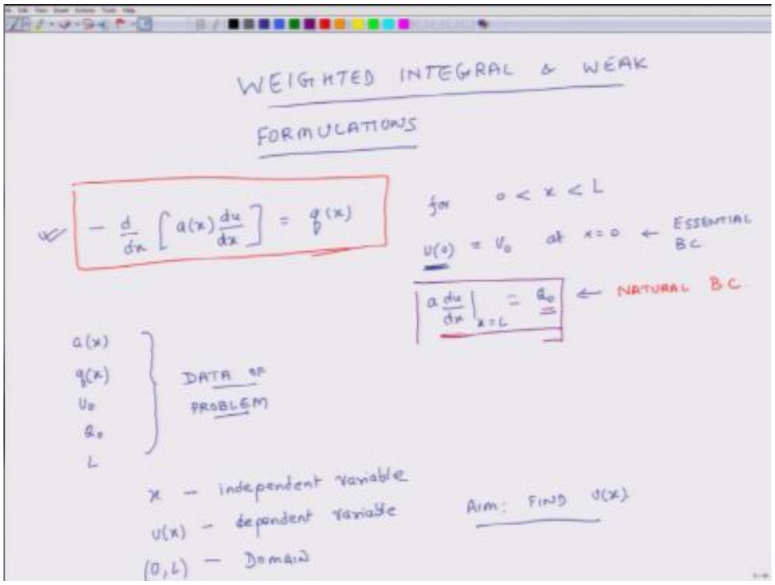

Which is this equation then it can solve a large class of problems okay, similarly we will see that in two dimensional situations there is one particular equation can address a very large class of problems.

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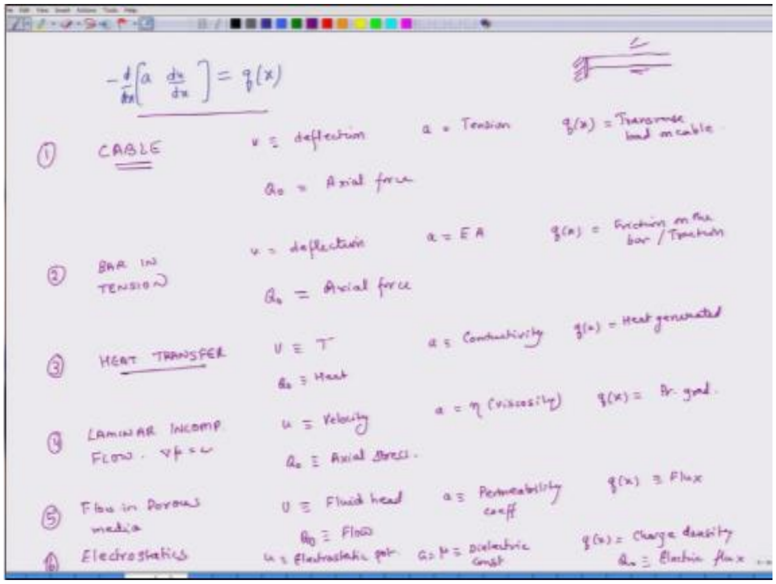

So this is there so now what we will do is we will with this understanding develop its weighted, first we will develop a weighted integral statement and then we will develop its weak formulation and then we will explain what is happening in this thing.

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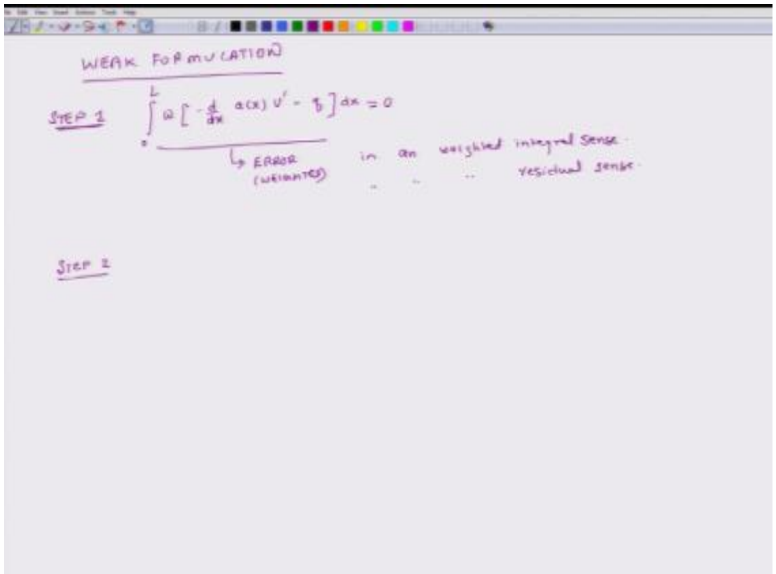

So weak formulation, so the first step time and we have discussed some of this earlier also but now we are going to expand on it is that we develop, we compute the residue of the system multiplied by a weight function and equate it to zero, so this is –d over dx a(x) times u´ -q and then I multiply it by weight function W and this is equal to 0, so this entire thing is the error and actually it is weighted error so weighted error in and weighted integrals since, so here we are completing the error over the whole domain not point-by-point because we are integrating this over the domain 0 to L, and some other times we also call it in a weighted residual sense.

And they mean the same okay, so the next thing we do is step two is that oh so there should be a dx here so the next thing we do is we integrate this entire expression by parts which we had done earlier we had shown that if I integrate this π parts I am able to shift

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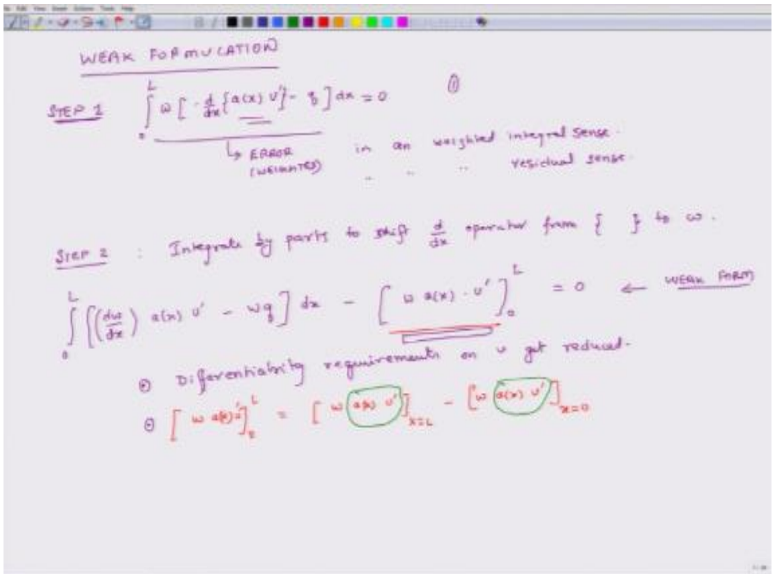

The differentiability operator so I am I think I miss this whole thing, differentiability operator from this term to w, so integrate bi parts to shift d over dx operator from this term which is in parenthesizes to w. So if we do that and I have explained that in the earlier class what we get is integral of 0 to L and then here I get dw over dx, this is a(x) u´ -wq, dx- and then I get the boundary term w teams a(x) times u´ 0 to L okay, and this entire thing equals 0, so this entire expression is mathematically same as equation 1 but this is the weak form, this is the weak form because our requirement of differentiability on u is only of first order and I have shifted that differentiability on to w.

So that is one so we will make some important comments here first is if we use this weak form differentiability requirements on u get reduced okay. The second and important condition to note is that we also get boundary terms, see this is a boundary term okay so if I look at the boundary terms what I get is w a(x) 0 to L this is the boundary term right, this is equal to w and I should have u´ here x u´ at x is equal to L-w a´, these are the boundary terms okay. Now in this we have this term so these are two boundary terms one boundary term is associated with the first boundary x=L other one is associated with x=0 and the one which I have put it in green the mathematical relation which is a(x) u´ is same as this boundary

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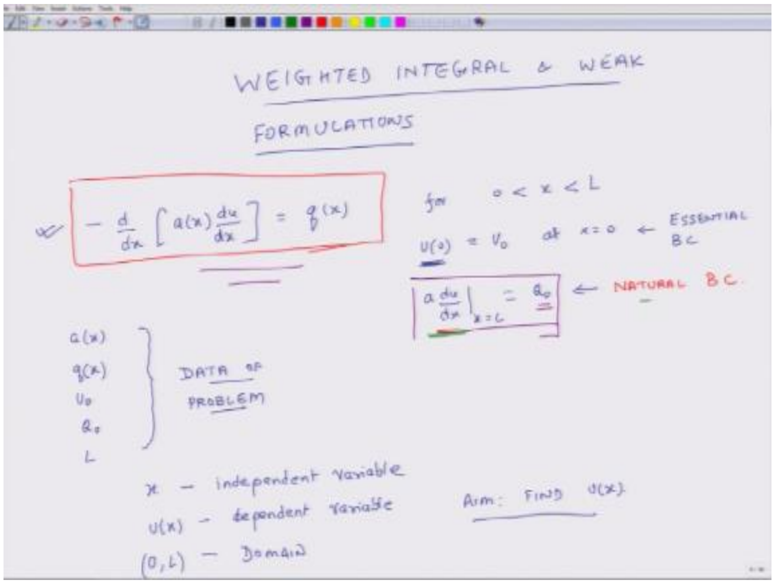

Term ad over dx okay so when we integrate by parts so this is my natural boundary condition

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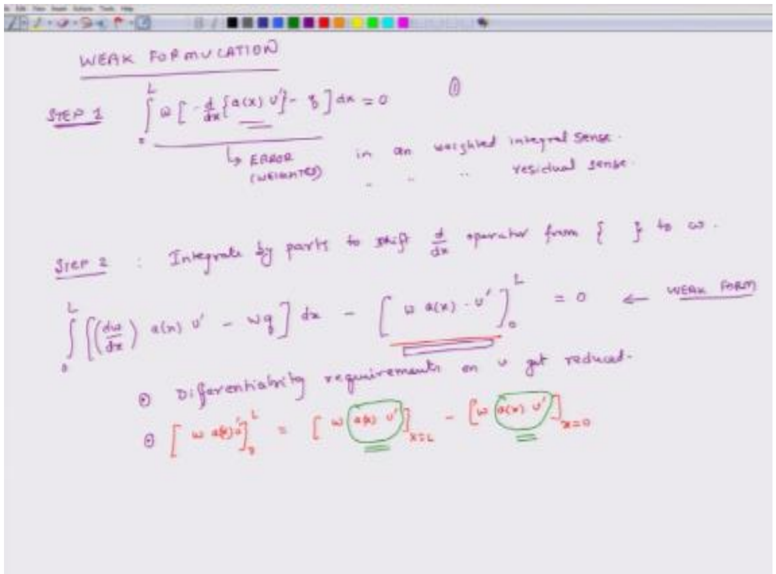

And what we see is that when we integrate this equation in part by parts I get boundary terms and one boundary term which is the natural boundary term comes out naturally or by itself in the process, so I do not have to worry of enforcing this boundary term when I make a choice of what kind of u should be chosen okay. If you remember earlier we said that we can choose u as a sum of C_j $Ø_j$ different C_j and $Ø_j$ right and those things have to satisfy the boundary conditions but because this boundary term comes out by itself, so in that case I do not have to necessarily worry about enforcing this boundary term by proper choice of c_j and $Ø_j$ because it will get naturally implemented. So the only thing now I have to worry about is that my

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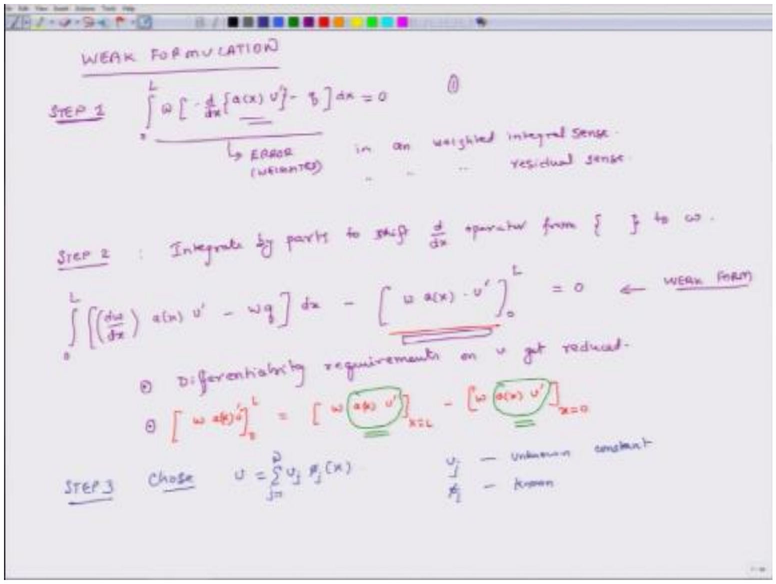

Essential boundary teams are enforced okay, so I will explain that, so in a step three we choose u so this is we have to now assume some form of u, so we choose u to be u_j $\O_j$ (x) j is equal to one to n okay this is what we choose and we and how while we are making this choice u_j is unknown constant, and $\O_j$ this is the shape function right this the shape function and because we are making a choice if whether it is a cubic function or square function or linear function we have to make it, so it could be in sinusoidal it could be even sinusoidal, there is no reason that it has to but polynomials are easy to integrate and differentiate so we

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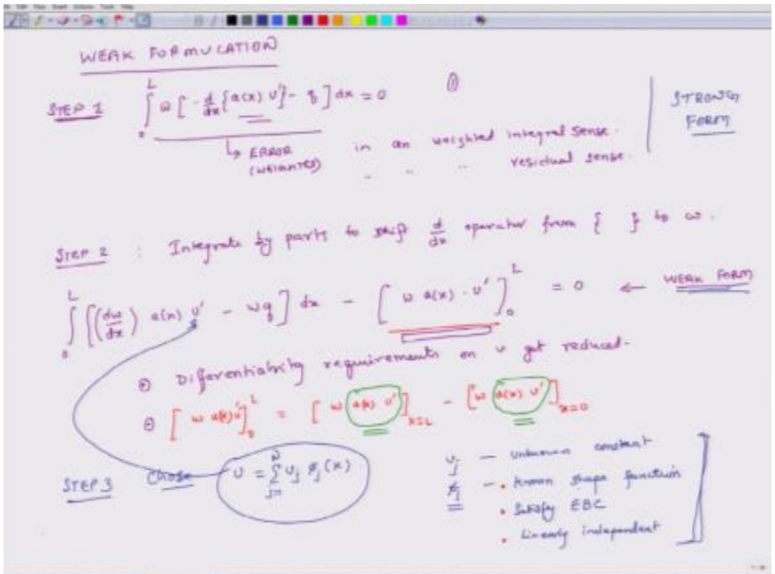

$\varnothing_j$ is known shape function and in this case if we our domain is from 0 to L then $\varnothing_j$ should be such that it meets the, so it is known shape function and it has to be of a particular type that it should satisfy essential boundary conditions it need not, it need not satisfy the natural boundary condition because the natural boundary condition through this mathematics is itself coming out and we can put the value and when we solve this equation I can replace ex, suppose ex u´ is equal to three suppose the boundary conditions such that a(x) u´ at x is equal to equals three then I can replace this by that number three okay, so I do not have to worry my choice $\varnothing_j$ has to be such that it should meet essential boundary conditions, we do not have to worry that it also should meet natural boundary conditions because that compliance is automatically taken care of in the weak form.

Not in the strong from so by the way this is known as strong from, so in the weak form it is automatically taken care of so I have more flexibility in choosing the nature of $\varnothing_j$ or πj so that is there, so one is that $\varnothing_j$ is known, second is about picking $\varnothing_j$ is that it has to satisfy essential boundary conditions, and the third criteria which we have to enforce is that it has to be linearly independent, what does that mean, so suppose I take $\varnothing_j$ as x for instance, for instance if I take $\varnothing_j$

as x and for first shape function I pick it pick as $\emptyset_1$ = x then the next shape function should not be a linear combination of $\emptyset_1$ so it could be x^2 that is fine.

But it cannot be 2X because it is a linear multiple okay, so the third shape function could not be $x + x^2$ it has to be linearly independent function, so I have I can pick as many functions as I want but they should be mutually linearly independent, so that is a very important criteria when we are making this thing, so if we choose $\emptyset_j$ as this and then what I do is I plug this thing back into the definition of u okay, so then what I get is.

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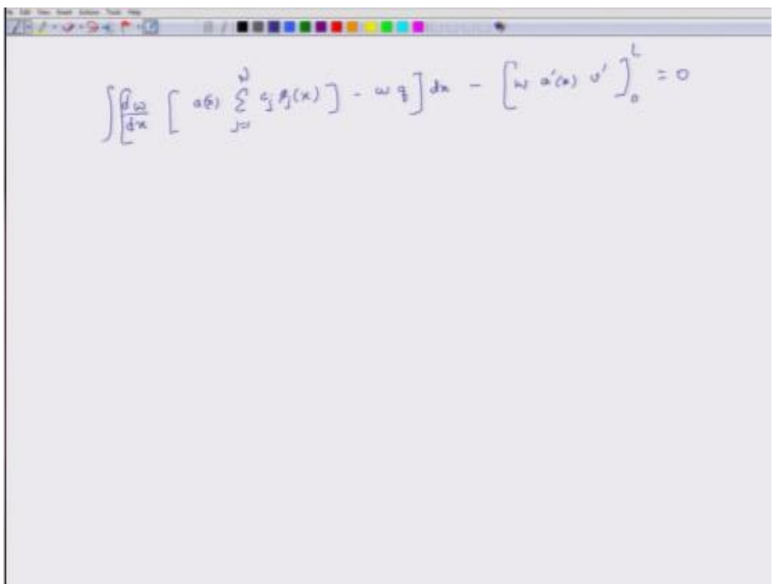

I get, okay so this is the equation I get and I am here summing up j =1 to n okay, so $\emptyset_j$ is known, C_J is not known, there are n values of these unknowns, and W is also right now not known but it is something weighting function right, it is some weight function and to know

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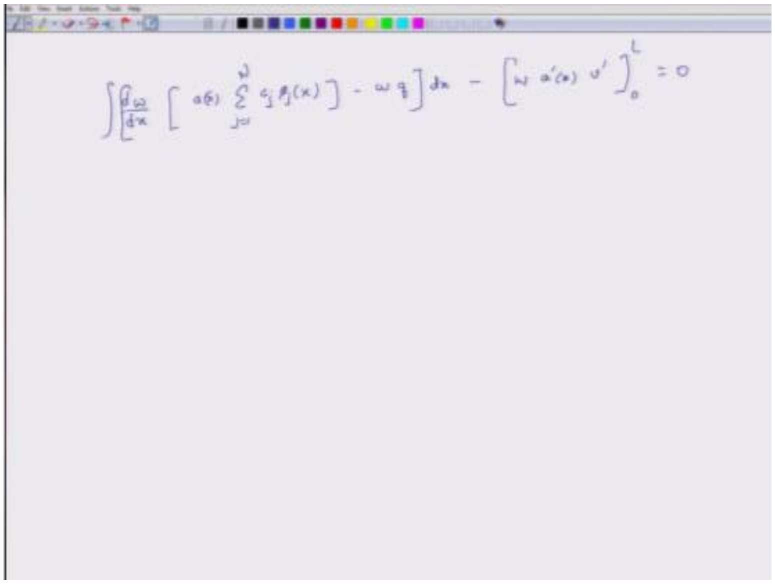

These n unknowns I can pick n types of n different weighting functions and I will get n different equations, I can solve them and I can get the value of c_j okay. So that is the overall thinking underlying the weak formulation so what we will do is we will continue this discussion in the next class Ø´ okay

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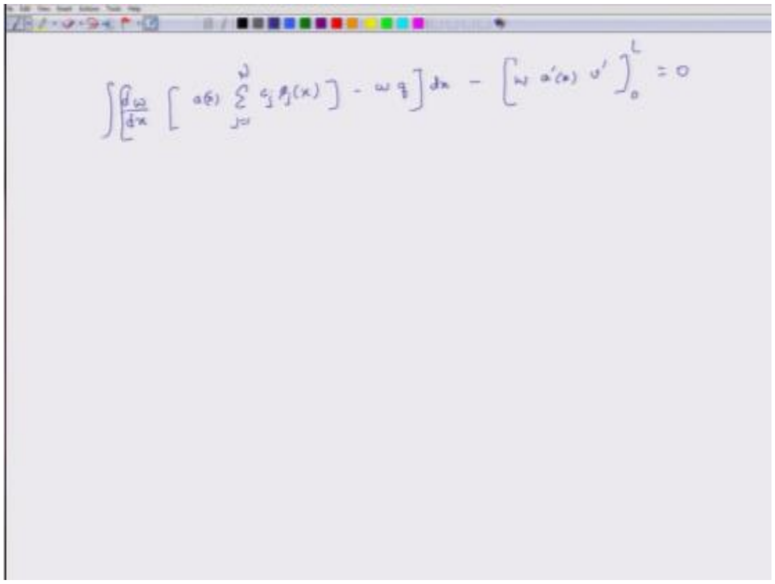

Yeah there is a prime thing and

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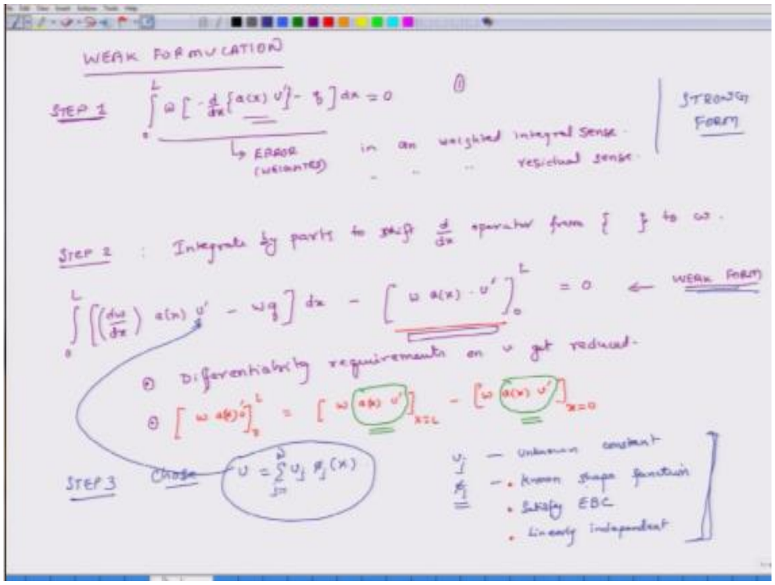

Yeah

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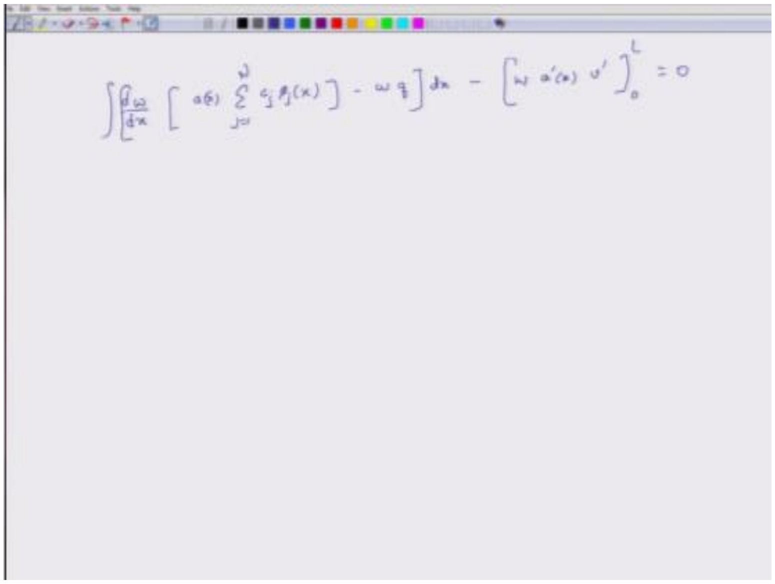

So that is right so this is a differential of $\varnothing_j$, so that concludes our discussion for today and we will extend this discussion also tomorrow, so look forward to seeing you, thanks.

<u>Acknowledgement</u>
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