Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 13
Variational Operator

by
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Hello, welcome to basics of finite element analysis course, this is the third week of this course and we will start this week with a discussion on variation, the concept of variation and variational symbol. And once we are dome with it then we will start with the figuring out details of weak formulation and different variational methods used in this particular context.

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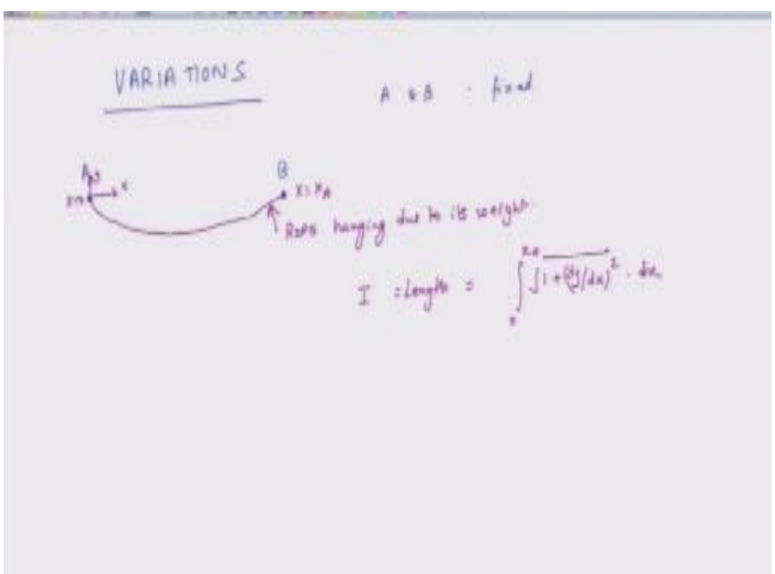

So we will discuss on, discuss variations. And I will again explain this by an example and then we will develop some mathematics around it. So suppose I have two fixed points, points A and B. So A and B are fixed. And between these two points there is a rope, this somewhat better rope. So this is a rope which is hanging due to its weight okay. And we have discussed let us say that we are interested in finding the length of this rope, length of this rope.

So that will be let us call it length and if this is my x- axis and this is my y-axis then length is equal to integral. So this is x is equal to zero, this is x is equal to x_A, integral of x to x_{A} , $(1+ dy/dx)^2$ square root hence dx. I mean this is the relation we had explained earlier right? So this I called in my earlier lecture as I of.

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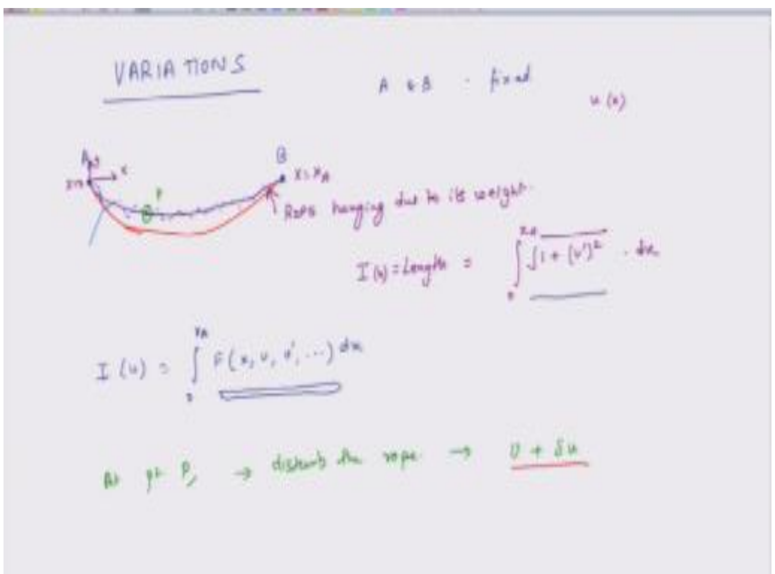

And I can replace this dy/dx, so suppose I call u which is the deflection of the rope at any point, so u depends on x, so I can call it as $(u')^2$. So this is I which is a functional and it depends on u, it depends on u, it depends on the shape of the rope, if the shape now it is hanging because of its own weight, so it has a certain thing but depending on the shape of the rope the length of the rope is going to change.

Now think about it, let us suppose I am looking at some point, let us say I am looking at this point, point P. So but before I go there I will rewrite this that I(u) is equal to integral of 0 to x_A and what is its I call this some function this thing F(x, u, u'). So now I am making, I just made it more general okay. So this we have done in the last lecture, so there is nothing new about it. Now we are looking at this function F closely okay.

So, so now consider that at point P and at point P, x is fixed x is not changing x is not changing. So at point P, I disturb the rope I disturb the rope by a small amount, so okay. I disturb the rope and because of that disturbance initially the deflection was u and it changed by a small, it varied by a small amount. So the new deflection of the rope is u plus small variation and I represent that variation as Δu, this Greek letter Δ small times u.

So and remember this is at point P, where x is so x is not changing, x is not changing and I can disturb it in all sorts of ways. So I can similarly disturb all points on the rope in all sorts of ways, in one in one set of displacements could be I can disturb it like this. So I am disturbing it about with reference to the original configuration okay. So this is one variation, this blue line I can disturb it like this. So now I have amplified it right. This could be another disturbance okay.

So but the point is that I am disturbing it at a point P where x is fixed and the nature of this disturbance could be anything. The only constraint which I have to impose when I am making this disturbance is that at a and b it cannot violate the boundary conditions okay, that is very important.

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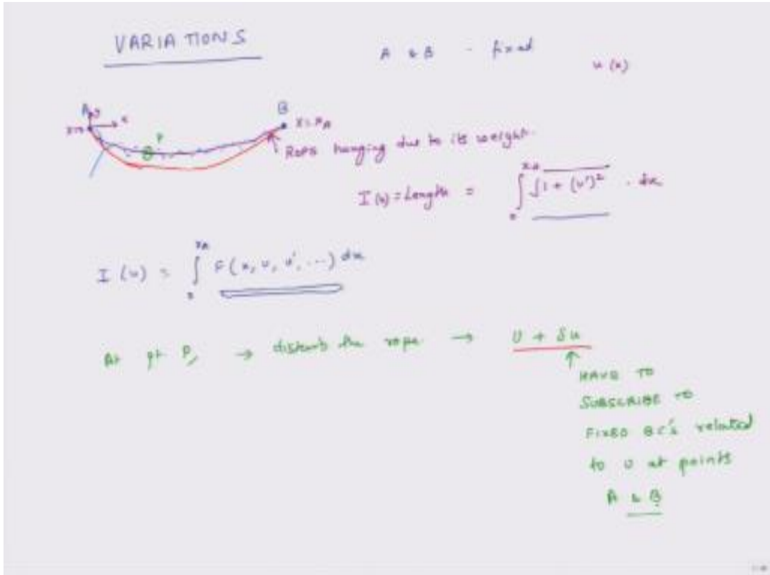

So, so these variations have to, to subscribe to, to fixed BC's related to u at points A and B, okay this is important. That they have to subscribe to fixed. So here the condition at A was zero, u is zero and at B also the value of u is zero. So the, what will be the variation of u at A? It will be zero, it can never be non – zero it can be non –zero at all other points but it cannot be non – zero at A and it cannot be non – zero at B.

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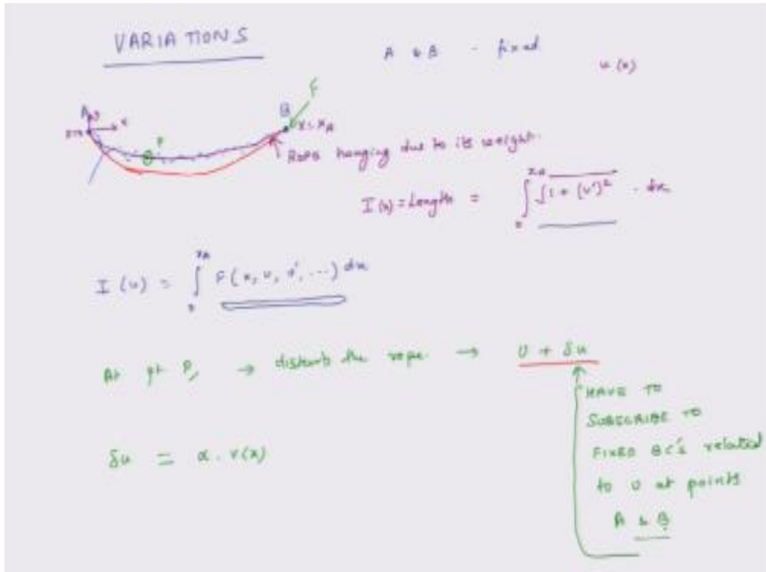

Because my boundary condition is like that, that the variation cannot violate those boundary conditions okay. So this is important to understand, so I say that δu represents what? The variation and I can express this δ u as some constant number α times a function of x which is v(x). Now this v could be anything, as long as it is consistent with the boundary conditions and it is not any boundary condition.

Suppose I have a force here F, this v does not depend on the nature of this force related boundary condition, agreed? But it will depend on the boundary condition which relates to u. Suppose instead of having a fixed situation here at point B at point B instead of having a fixed situation I am applying some force there. Suppose it is not rigidly fixed, in that case there could be variation at B okay.

There could be variation at B but if B is fixed then there cannot be variation, they are two different things you have to understand this, they are slightly two different things, okay. So it depends on boundary conditions which directly relate to u, okay. This directly relate to u atleast in this context.

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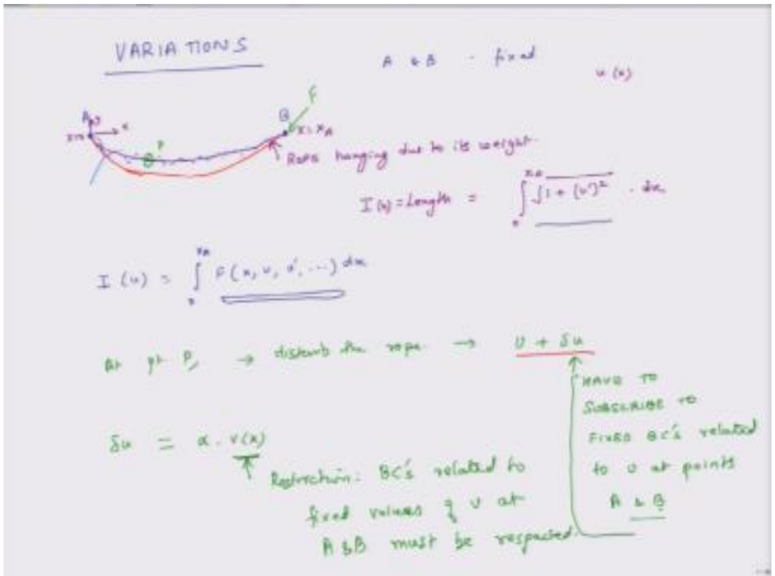

So our choice of v, can be general and the only restriction is that BC is related to fixed values of u at A and B must be respected, you cannot break those boundary conditions. This is very important to understand. Third point, so first point is variation at A and B in this case is zero because the value of u is going to be zero that is the first thing. Second thing is suppose at point B I remove the boundary condition that u is zero and instead I have I am holding it by some hand, okay I am holding it by hand. So I am applying some force, I am applying some force. Then in that case variation could be non - zero at point B, right?

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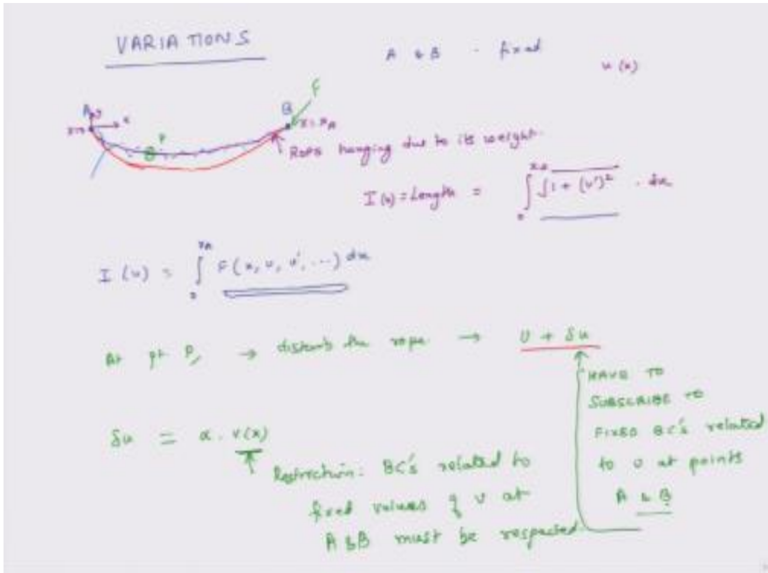

And the third condition is, that variation at A and B will still be zero even if the value of displacement at A and B is known but non - zero, what do I mean? Suppose the displacement at B is 5mm, displacement at A is 0 that we have understood and displacement at B is 5mm but it is fixed, I take the rope at point B and I move it 5 mm upwards and I fix it there. So displacement is 5m but what will be the variation at that point?

Is any variation possible at point B, I have fixed it at that point, so there is no variation possible at point B, okay. Variation can happen only if my boundary condition permits me to have that variation. So in this case I have moved the point B 5mm upwards and I fixed it there, so no variation is permissible at that place also, okay. If it was just the case of force my hand could move up and down.

In case I was just lifting the rope with hand my hand could move up and down the force is being applied but my hand. So variation would have been possible, but in this case I have moved my rope by 5mm up and I have fixed it, say I have glued it, or bonded it, or rigidly held, now I am holding it. So no variation is possible, so the third point is and this is a next level.

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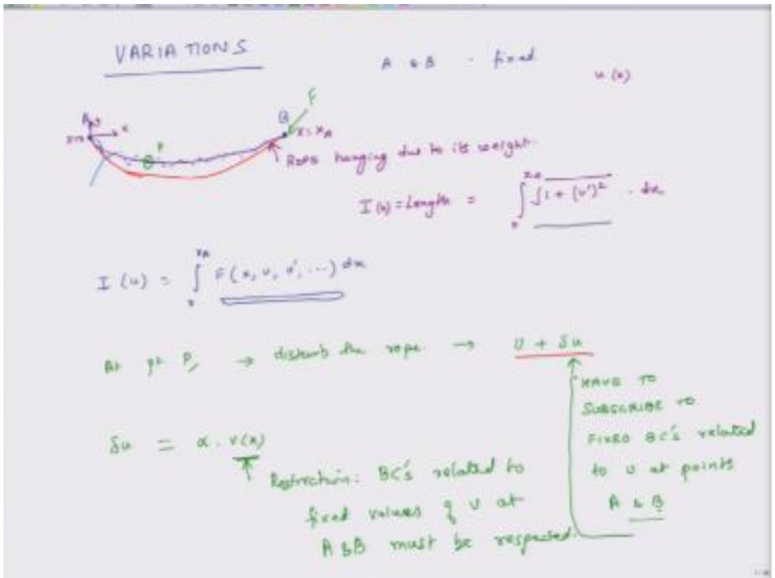

Another fine point that if A at point A, u is prescribed does not necessarily have to be zero but if u is prescribed then variation at u is not possible. Similarly if u is prescribed at B then variation at B is not possible and so on and so forth. At all other points in the in-between A and B u is not prescribed. So variation is possible and what kind of variation is possible?

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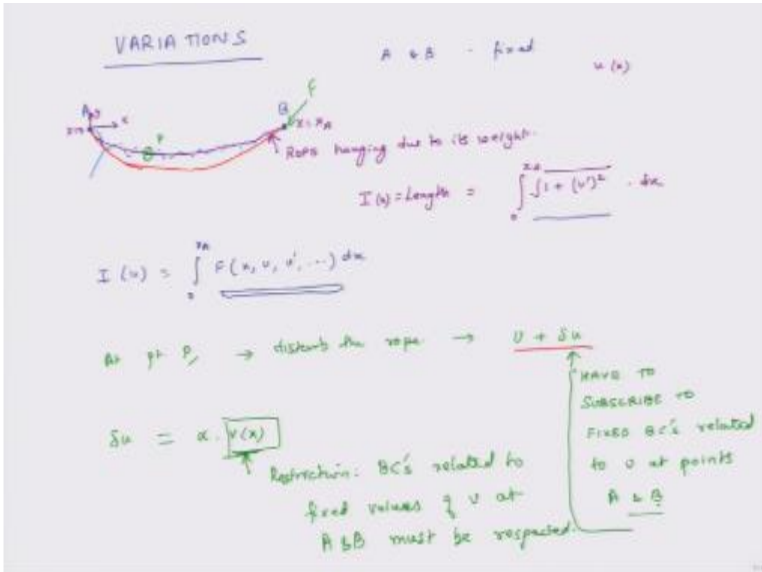

Any variation is possible, so I am not restricting v, any variation is possible. Variations are not possible only where u is explicitly prescribed it can be zero it can be non – zero but if it is prescribed then variation will be zero, this is very important to understand. Fourth thing I want in general that these variations and we will see it later why we want this? We want that the amplitude of these variations should be extremely small, okay.

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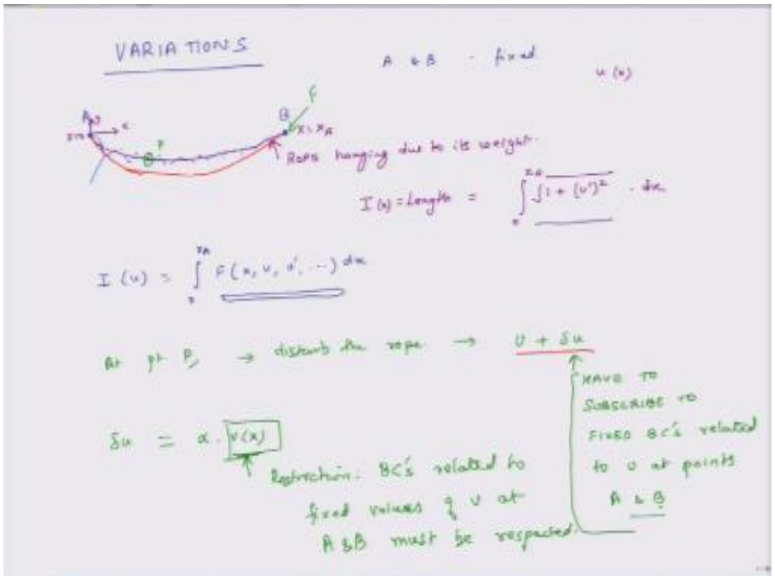

The amplitude of these variations should be extremely small, what is extremely small? I mean extremely, extremely, extremely small.

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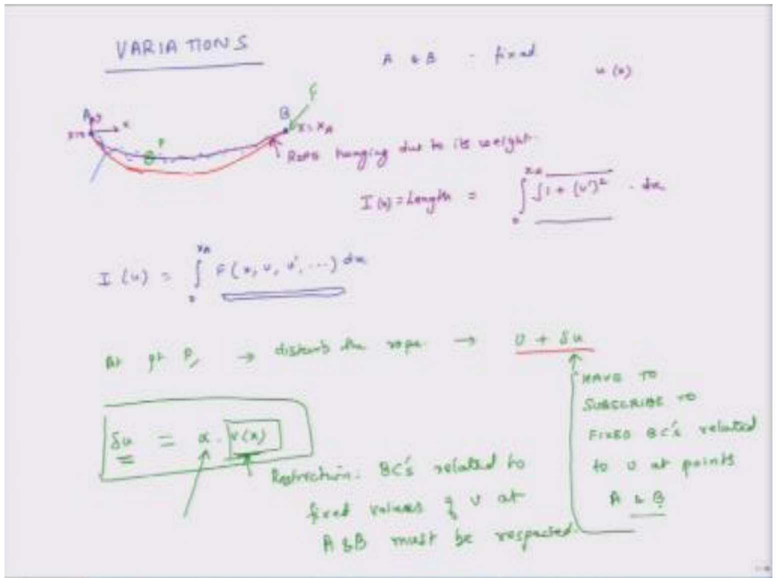

So I do that by controlling this parameter α, I can make this α as small as possible, I can give a parabolic variation but I can make the amplitude of that parabolic variation extremely small by controlling this parameter α, I can give this para, you know this variation in blue color but the amplitude of this variation can be very small by picking up the right number and I actually make it infinitely small we will see it later, okay.

So that is the importance of this parameter α that is the importance of this parameter α. So the variation in u is designated as δ u, it can be represented so this is designated as δ u, it can be represented by any function v(x) as long as this function v is consistent with the boundary conditions related to u in this case, right? And the amplitude of these variations is we try to make it as small as possible.

By picking up some number δ which is extremely small, understood? So this is what is known as variation, this is what is known as variation, give another example.

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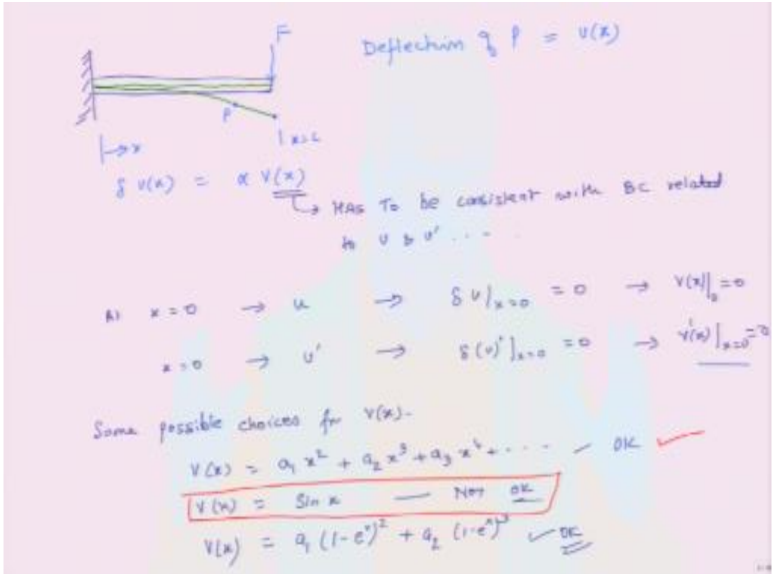

So suppose I have a beam, okay. And let us say this is my neutral access of the beam the midline the mid plane of the beam and I apply a force F, I apply a force here, okay. Then because of this the neutral access of the beam reflects by some amount, okay. And let us say I look at A point P and let us say the deflection of this point P is, deflection of point P is u(x) I mean typically we use the term w here I am using u(x), okay.

So now I am interested in figuring out the variation on u(x). So I will say variation in u(x) is what? It could be any function right? It could be any function v(x) multiplied by a parameter α. But this v has to be consistent with BC is related to u and u' and so on and so forth, okay. So now we have to see what kind of functions we can choose. What are those restrictions? We know that at x = 0.

What is the value of u? Zero, so what will be the variation of u at x = 0, zero. Also at x = 0 what is the slope? Slope is zero it is a face can tell you, which means the slope of the variation also should be zero, this means v(x) at 0 should be 0 and this means v(x)' at x = 0 should be 0 okay. Is there any boundary condition here at, so suppose this is x = x. So let us say this is x = l, so any such boundary condition is needed at the free end?

Is there any restriction on choice of which is dictated by the conditions at the free end? It is not a free end, it is a force there, there is no right? So the variation could be non – zero at the other end. I will not call it free end because there is a force being applied right? So there is a force being applied. So this is there, so we will pick one or two functions to see what kind of functions we can pick.

One function could be some possible choices for v|(x), v represents the shape of the variation, v represents the shape of the variation and α represents its amplitude. So some choices, so v(x) could be $a_1x^2 + a_2x^3 + a_3x^4$… this could be one choice right? At x= 0 does it meet our condition? At x = 0 v should be 0 it meets our condition. And it should meet the other condition also, its derivative should be 0 at x = 0, it meets the condition.

So this is okay, let us look at another function v(x) = Sin(x) does it meet the first condition? It meets the first condition, does it meet the second condition? Yes or no, it does not meet the second condition because the derivative is Cosine(x) so it is not okay. So you have to be careful what kind of functions you choose? Okay. So that is there, one more function, $a_1(1-e^x)^2 + a_2(1 - e^x)^3$` okay.

This is the third function, some possible. So this is not okay, this is okay. What about this guy? Third function, at x = 0 it is v_0 or v is not 0. It is 0, at x = 0 is it derivative zero, zero. So this is also okay. So this is an acceptable function for representing the variation, okay. So these are, so this is an overview of the variation in the system. So again I will summarize this whole thing that the first thing is that when we are picking up variational functions.

So what we have seen is that we had developed an approach through which we can depict variations in the function u, u could represent something physical for your deflection of a beam or temperature and we have figured out that how do we correctly choose the variational functions related to maybe a function u or v or w whatever and the only restriction which is required atleast at this stage we are going to describe is.

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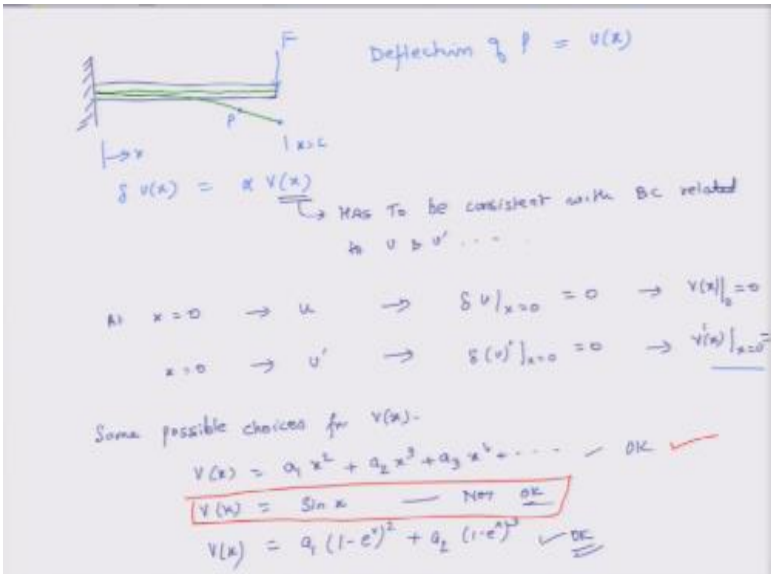

That the variation should be of a nature so that the boundary conditions associated with the variable u and its derivative if needed are not violated, if we do not violate these boundary conditions then our choice of the variation is appropriate.

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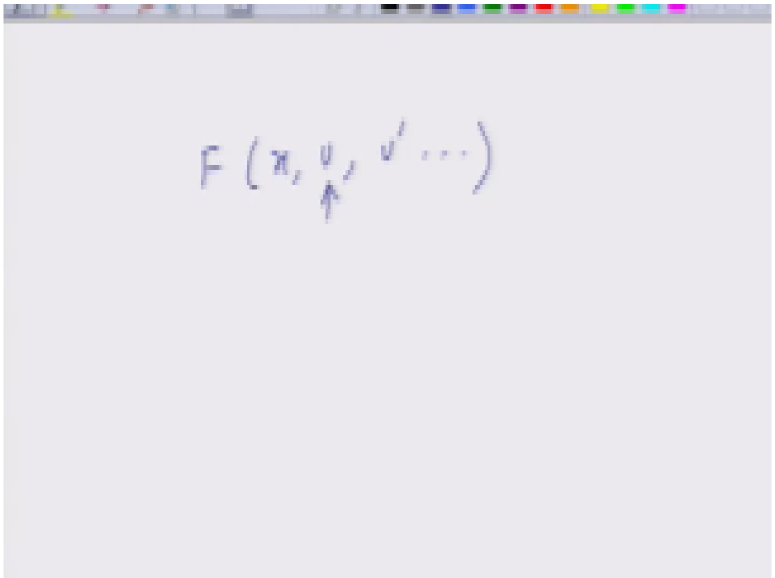

Next thing, earlier we had seen that F is dependent on u and its derivatives and so on and so forth right? So if u is changing if there is a variation in u, then there will be a variation in F also, okay.

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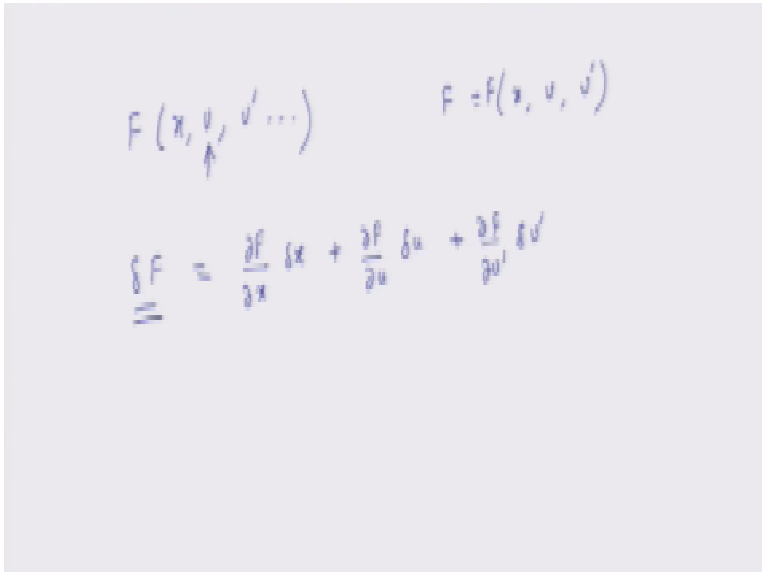

So that variation in F which occurs due to variation in u which occurs is known as variation of F and it is designated like this, okay. And this I can calculate it, how? So suppose F is, depends on (x, u and u') F depends on x, u and u' then the variation in F is, this is the mathematical thing, how do I calculate the total variation in F? First I change x by a small amount and see the variation in F and I measure that number. Then I change u by a small amount and I again measure the change.

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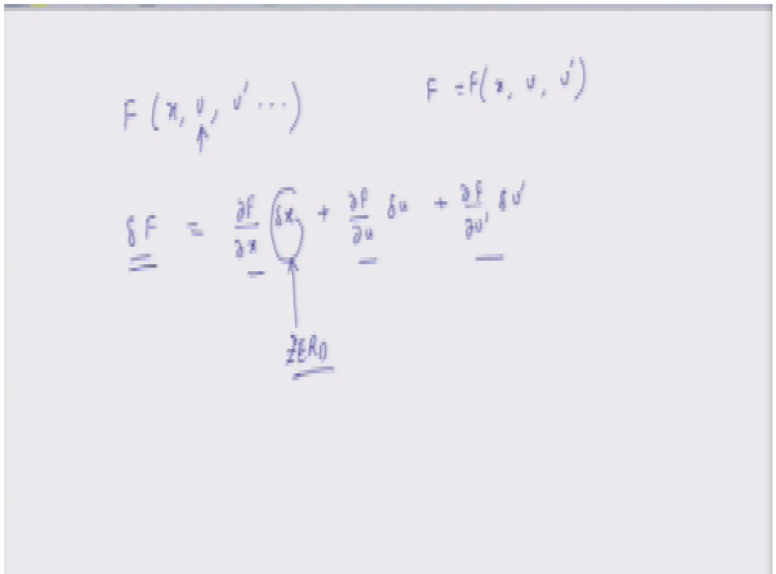

So these are those individual variations x ascribable to changes only in x only in u changes only in u', but this term is zero change in x, why?

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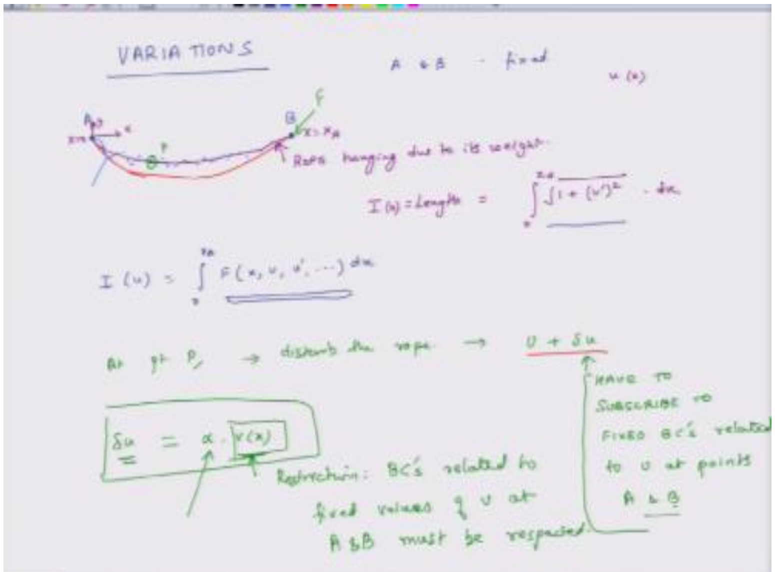

Because when we were discussing variations what did we say? That we are looking variations at a given point, right?

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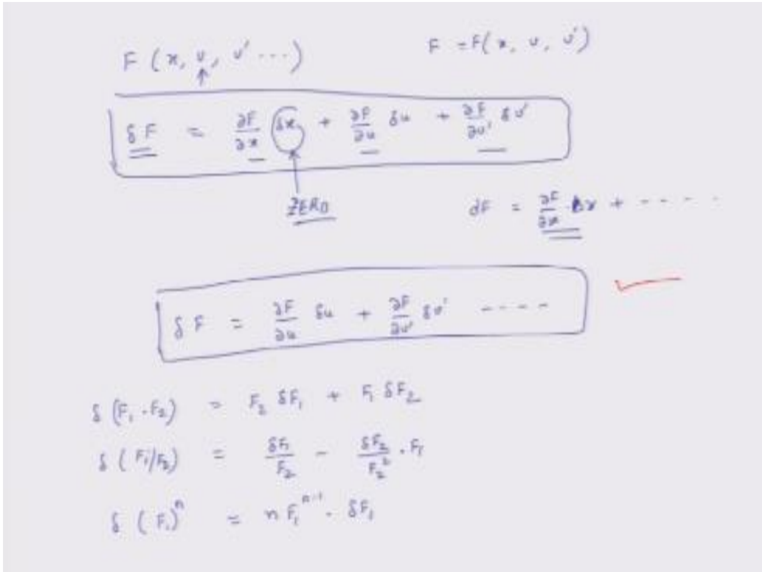

So, so this equation is very similar to the equation dF = ∂F/∂x Δx like that but this term does not exist when we are talking about variations, this is important to understand. So variation in F is equal to ∂F with respect to u times variation in u plus ∂F with respect to u times variation in u' and so on and so forth.

But the x related term does not come. So this is an important equation and we will use this relation very frequently when we do finite element formulations, then we will also write down a few other laws that suppose related to variations then if there are two functions F1 and F2 and I have to calculate the variation of in their products then very similar to the way we do for differential calculus.

It is nothing but F2 times variation in F1 plus F1 times variation in F2. Similarly variation in F1 over F2 is equal to variation in F1/F2 minus variation in $F2/F2^2$ times F1. This is the rule which is very similar to that which we use for derivatives. And if I have $F1^n$ then the variation in that is nothing but n $F1^{n-1}$ times variation of F1 and the last and very important relation is.

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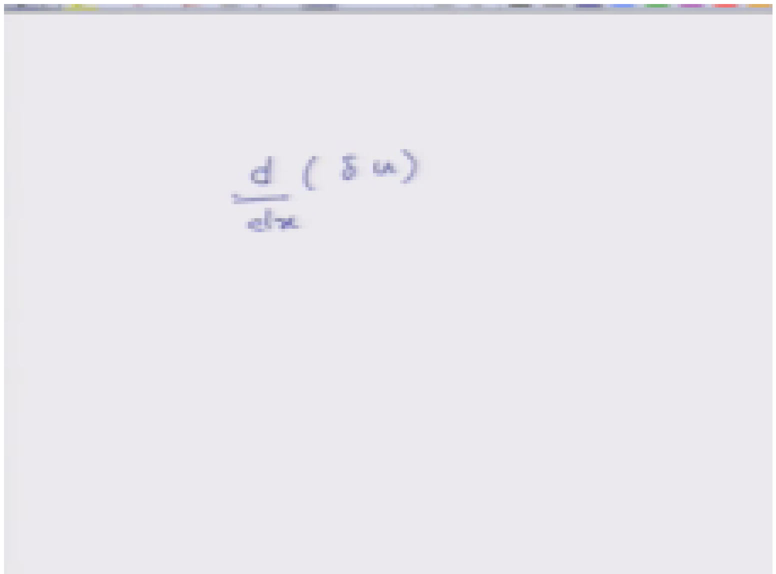

That suppose I have a function and I take its variation and then I differentiate the variation, so d/dx. So what am I doing? In this case what I am doing is I am having a function u, I pick up an appropriate variation which is consistent with boundary conditions directly related to u and then I am differentiating that function.

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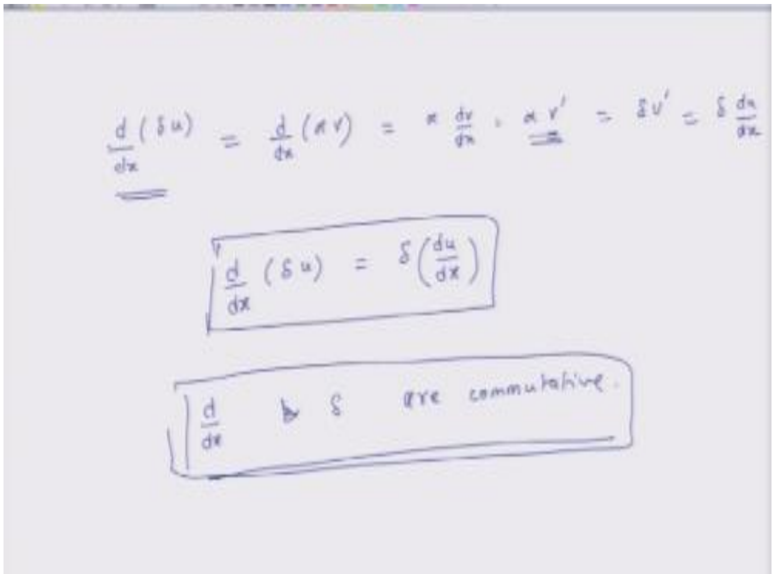

So this is equal to d/dx and what is variation of u? It has a shape associated with function v and an amplitude, right? This v could be anything as long as it is consistent with boundary conditions. So I am trying to figure out how do I calculate. So my goal is how to calculate this thing? So this is equal to α dv/dx = α times v' is equal to, now what is v'? α times v' is what? This is nothing but variation in u', right?

And this equal to dx, so I can write it, write that d/dx is equal to variation in du/dx. Which means that d/dx, see d/dx is an mathematical operator and variation, this is also an operator it is an operator, right? Are commutative, so either you do variation and then take the differential or you take the differential and then take the variation you will end up with the same answer. So they are commutative.

So depending upon your ease you can do this first and do that later or vice versa. So this completes our discussion on variations and starting tomorrow we will start discussing weighted integral weak formulations and all so all the stuff related to this broad theme of topic, thank you very much.

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