Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 12
Functionals

by
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Good morning, welcome to Basics of Finite Element Analysis, this is the last day of the second week of this course and what we are going to discuss today is again we will do a concept review, and today we will focus on this terms called Functional and then if we have time we will also introduce another time called variation. Well both these terms functionals and variations they are heavily used in finite element analysis, so we will start with functionals and if we have time then we will start also the variational discussion.

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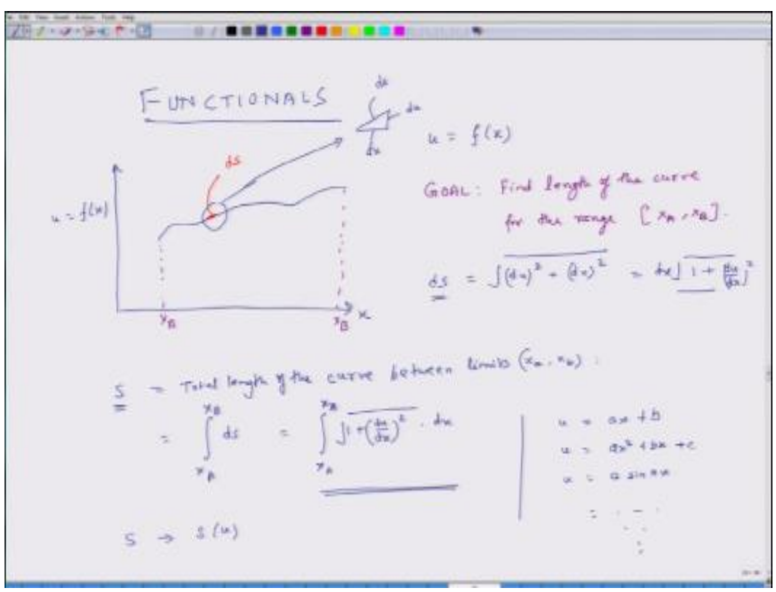

On variation and variationals, so that is what we are going to talk about, now consider so we will start with an example, so suppose I have this plot of a function f(x) and some function, so my y equals f(x) or I can because I am going to use this letter u, so I will say that u is a function of x and suppose I am interested in finding the length of this curve, so how will I do that? So I am interested, so suppose u is f (x) and I am and I want to find out the length of this curve so the goal is find length of the curve for the range X_A to X_B.

So what do I do basically let us say the, for a very small element, for a small portion of this curve let us say this length is ds, then I can write down that ds equals, so if I zoom in on this then it will be something like this, so this is ds, there is a change in y so it is du, and this is change in x, dx.

So ds = $(du)^2 + (dx)^2$ and I take the square root of that okay, and if I because this axis, u is equal to f(x) here. So if I take x out then I get 1+ du over dx whole square and dx is out, so that is the length of a very small element on the curve. So if I add up all these small elements on the curve then I get the total length.

So, s which is the total length of the curve, this equals, so this is the total length of the curve between limits, between limits x_a to x_b. So this is equal to, I can integrate x_a to x_b ds, and because I have calculated dx as this so it is x_a to x_b 1 + du over dx square times dx, okay. So that is the total length of the curve. Now to compute the value of s I have to know u, to compute the value of s I have to know u.

So u could be anything, u could be a straight line in that case u could be represented by ax+b, if it is a parabolic function then u would be ax^2+ bx+ c, if it is a a cosine function then u would be a sine Π x or some a sine Π x or some, some function like that and I can have infinite types of functions.

The point what okay, so the point what I am trying to make it is, make here is that the value of s in this case depends on the nature of u, the nature of u, so I can write this s, it does not depend on anything else right, because du over dx it also depends on the nature of u. So I can write s as s is a function of s(u).

If I change u then s is going to change. These types of integrals are known as functional. So I will make a general statement so what is, so, so.

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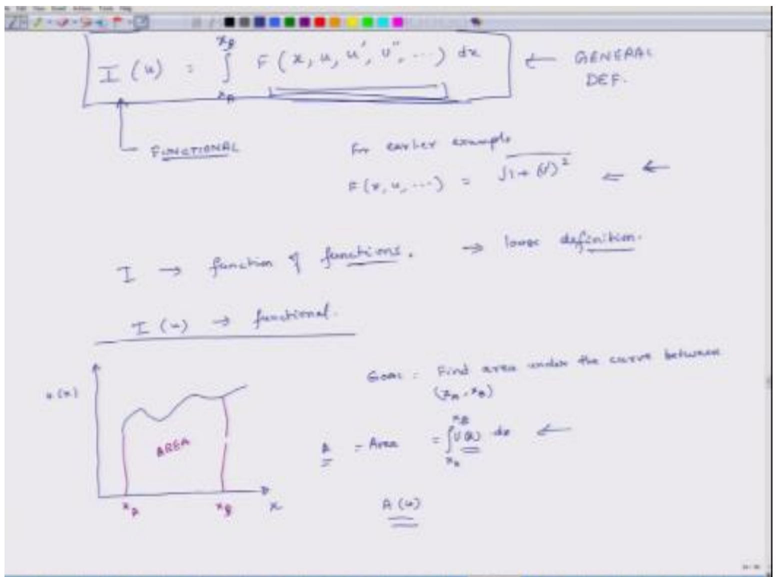

To make something general let us say there is an integral x_a to x_b and it is an integral of some function which depends on x, which depends on u, which depends on u′, which depends on u″, and if I integrate it then this thing, the integral of this entire expression is called functional okay.

So the example which we had discussed, so in the earlier example the one which we had discussed f for the earlier example, for earlier example f was defined as $1+(u')^2$ like this. So this is just an example but f this is a very generic statement that if there is a function which depends on x, u, and derivatives of u and if I integrate that function with respect to x over a limit then whatever I get out of that is known as a functional okay.

So in some sense, in a very loose sense I is nothing but a function of functions okay. What are the different functions, one function could be u, another function could be u′, another function could be and so on and so forth, but this is a loose definition okay. But the point to remember here is that when I change u the value of i changes so i depends purely on u, so that is why I say that i depends on u, it depends on u and this is a functional okay.

Another example of functional suppose I have another, so suppose I am plotting here u(x), so some function and on the x axis, this is the x-axis and suppose I am interested in finding this area, so this is the area, so suppose I am interested in finding the area under the curve okay, I am interested in finding the area under the curve, so the goal is find area under the curve between x_a to x_b so that is what I am interested in finding out.

So what is the area, it will be basically u area, it will be u(x), dx integral x_a to x_b okay, so that is the area, and so let us call this area as A, and depending on our choice of u the value of this area is going to vary, so I can say that this area is a, it depends on u. If it is a, again based on the nature of this function u(x) area is going to vary, so A depends on u in this case, so it is a relatively simple, simpler functional where the dependence is only on u and not its derivatives.

In this case it was a little more complicated because there was this u′ and then I had to square it and take under square, you know the square root of all that stuff so it was a little more complicated, but this is the general definition of functional.

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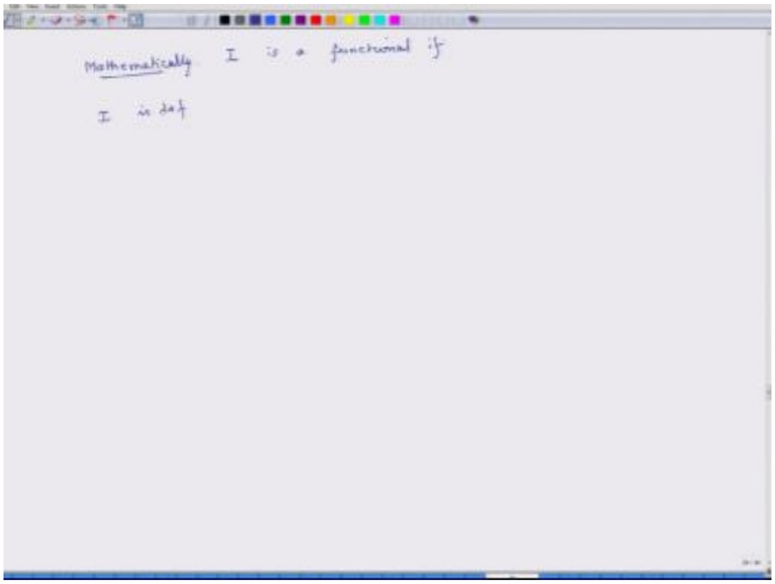

Mathematically i is defined, so mathematically I say i is a functional if.

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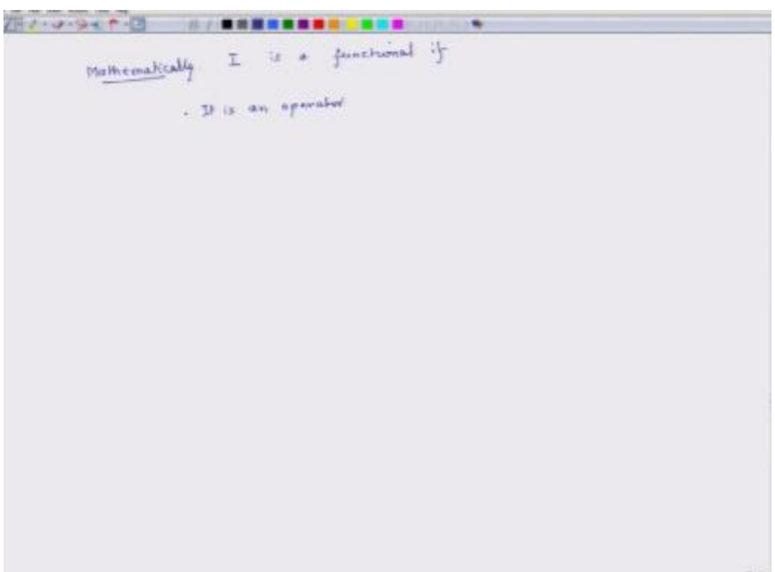

It does one important thing, so it is a functional if it is, it is an operator, it is as an operator so what is the operator?

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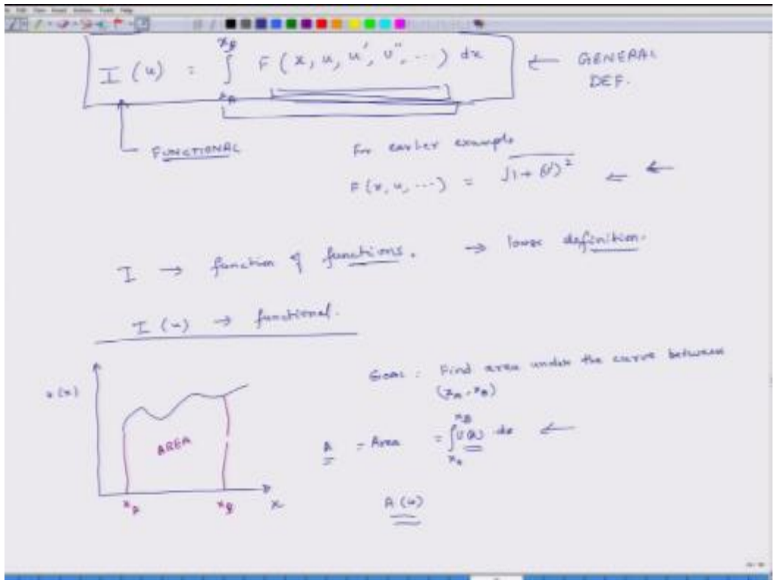

In this case this whole thing, this, this is the entire operation we are going to conduct right so that is why it is an operator.

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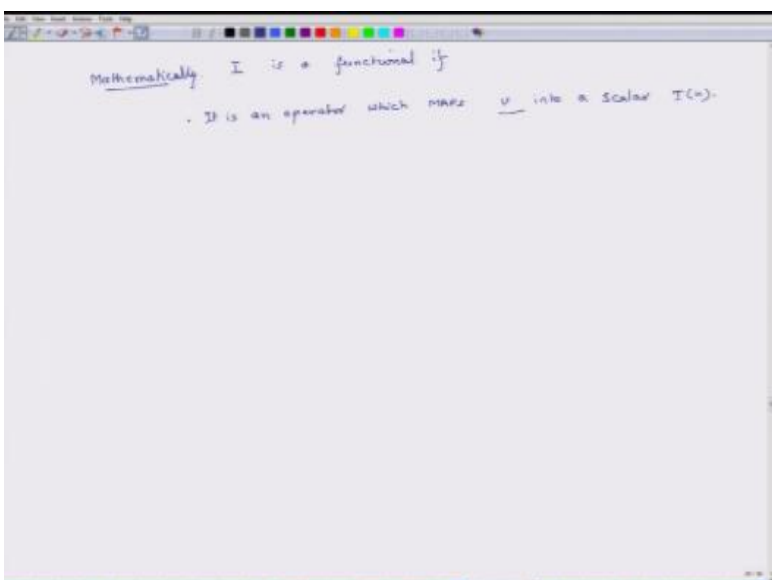

We are conducting an operation, so it is an operator which maps u into a scalar, that is i(u) okay, so i(u) will be a number, it will be a scalar, so it maps u which could be in vector space.

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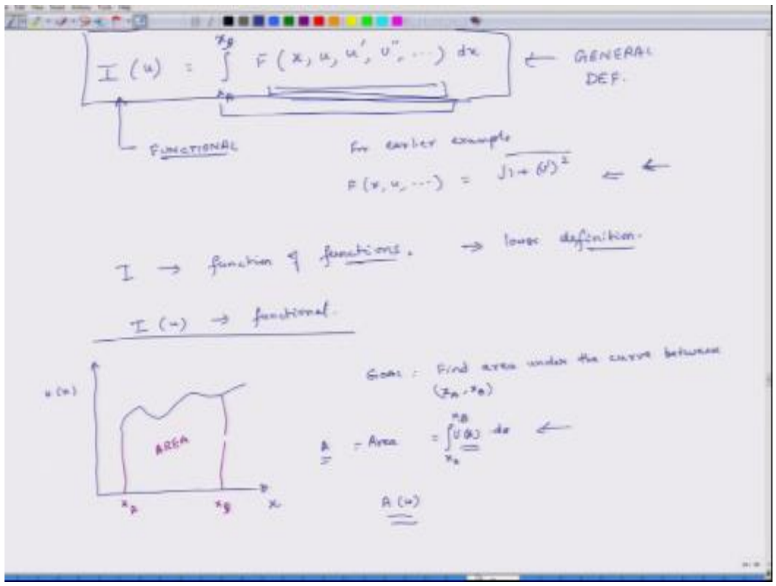

And it maps it into a scalar quantity. Now in context of this particular course we will be dealing with a lot of functionals but they will belong to an important category called linear functionals.

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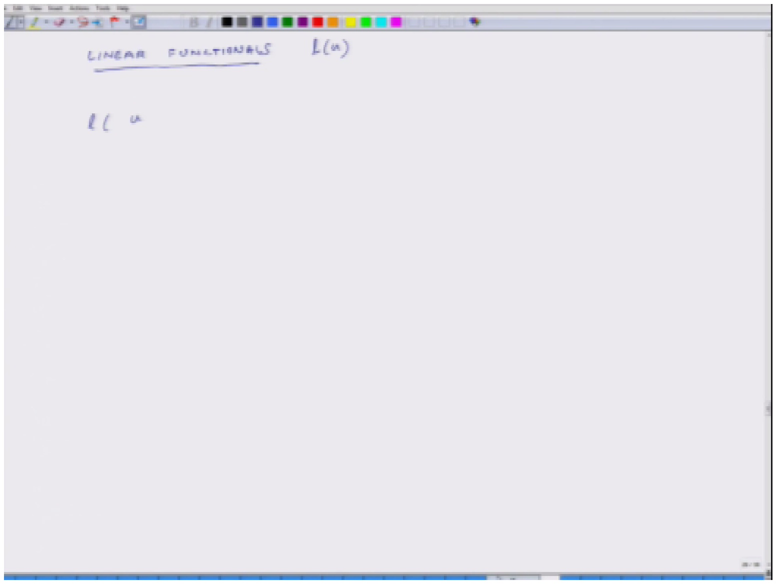

Linear functionals, and we will write them as L(u), this is the terminology we will use. So I can say this is the, so a linear functional could depend on some function u and it can also,

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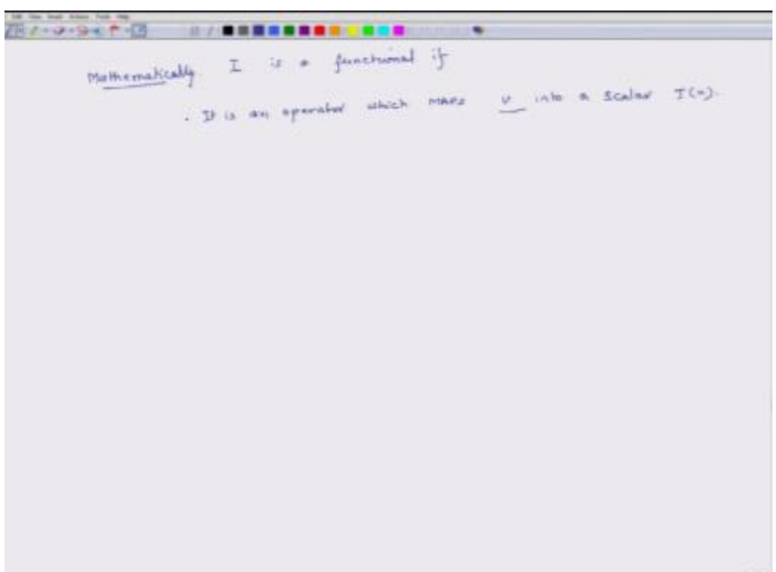

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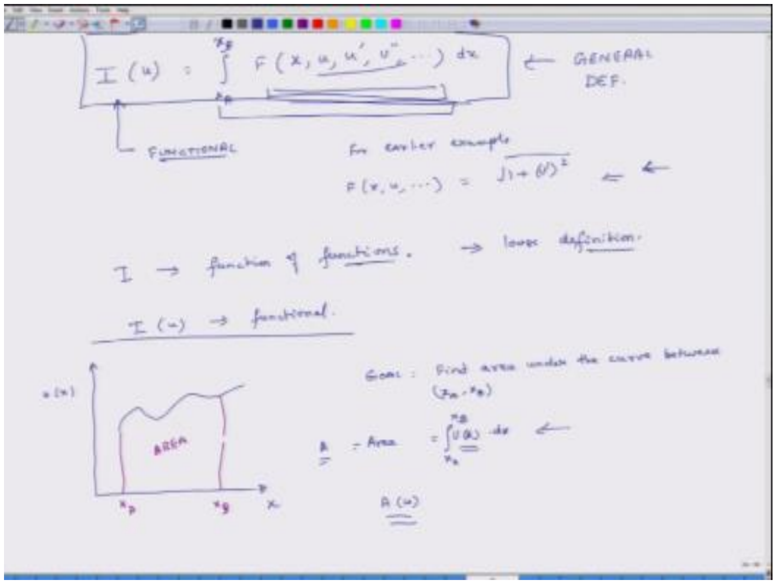

So one thing before I want, before I explain this linear functional is, that in this case the dependence is only on one function u and its derivatives, but there is no rule that functional cannot depend on only one function, it can depend on several functions, it could be a function u, another function v and so on and so forth okay.

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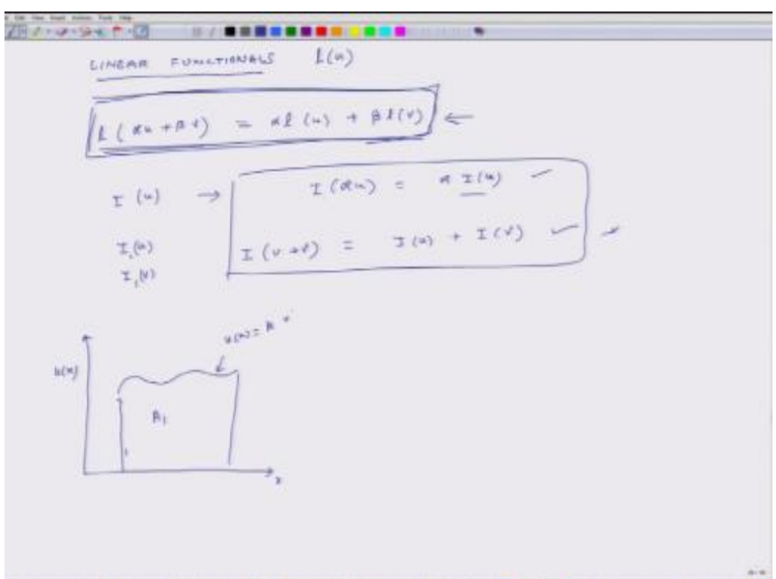

So the property of a linear functional is that it is a linear functional then and I will explain this, so this is the property which linear functionals obey. Basically this is the last superposition what does it mean, that suppose you have a functional, suppose you have a functional i and it depends on u okay, suppose there is a functional i and it depends on u, then the first rule which this functional has to obey is that if I multiply u by a factor alpha then its value will be same as alpha times i(u), this is the first thing.

So this statement is a combination of two different statements, this is the first thing okay, that if I have a functional which depends on u then if I multiply instead of i, instead of u I use alpha u as the input then the value of this functional will be alpha times i(u), this is one thing. The second thing is if I have a functional u and a functional of u and there is and then I also compute it for another functional, the function v right, so this is i_1 this is i_2, so I compute two different values for I, i_1 and i_2 and then what I do is I compute i for u+v, then this will be equal to i(u) + i(v).

So this is the second and both these conditions, so if it meets these two conditions then it is a linear functional and what I have done is I have combined both these conditions in this statement okay, so this is a, that is the definition of a linear functional to illustrate something, this thing, so suppose there is a function u(x) okay, and I am interested in finding the area below it, so I will call it A_1 and okay, and let us say this curve u(x) is represented by some constant k times u. Excuse me,

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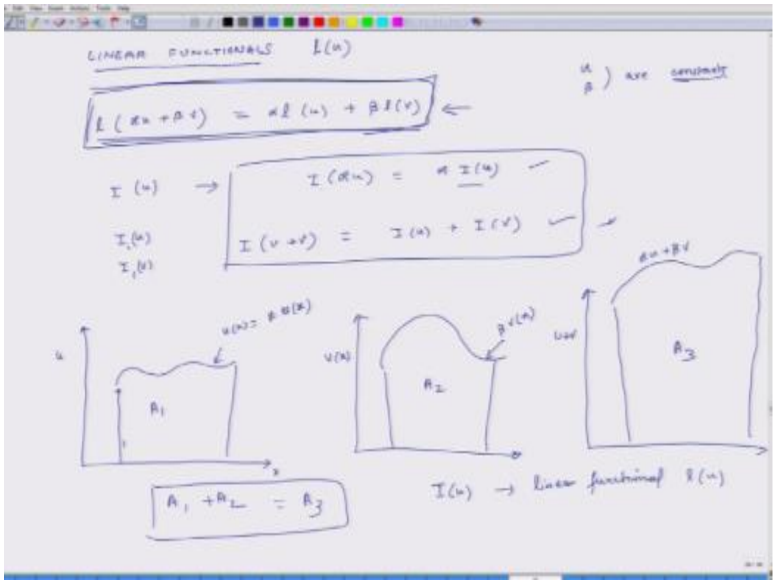

So this is some constant times u(x) okay, and I have another function and let us say here I have v(x) and here also I am interested in finding this area under the curve and let us say I call this area A_2 okay, and let us say this function is represented by, I have okay so instead of k I will use alpha, instead of k, I will use alpha because I have used alpha here. So here let us say this is beta of another beta times another function v.

So I did not say this alpha and beta are constants okay, so if I add these two areas then I get A_1+A_2. Now I create another functional, functional which corresponds to the sum of these two individuals functions okay, so this is another functional which corresponds to the sum of these two functions so here what I do is I am going to plot u+v okay, so I get some third curve.

I get some third curve and this curve is represented by α times u+β times v okay, so how do I generate this curve, I basically get this point from here, I take individual points and I add up the values for all the points so I get my total sum of those two curves. Then and now I am interested in finding this area A_3 okay. So I integrate this whole curve and I get this A_3 curve, and what I will find is that A_1+A_2 will be A_3.

So what, in that case what do I say that the functional i(u) which represents the area under the curve is a linear functional, that is l(u)it is a linear functional understood, so that is what it means. Because I can add these if the $A_1 + A_2$ did not come out to be same as A_3 then it would not have been a linear functional, it would not have been a linear functional.

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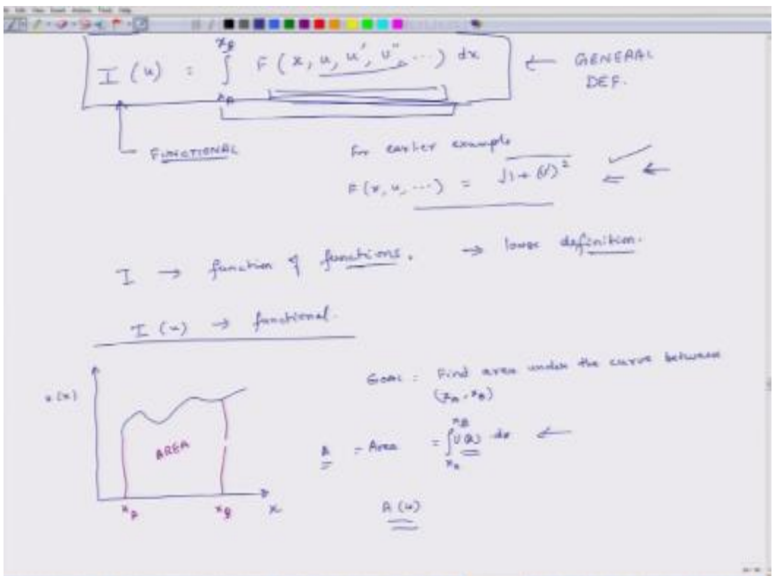

This functional, which we have discussed earlier is not a linear fractional, it is not linear functional why if you take one curve what, what does what does this.

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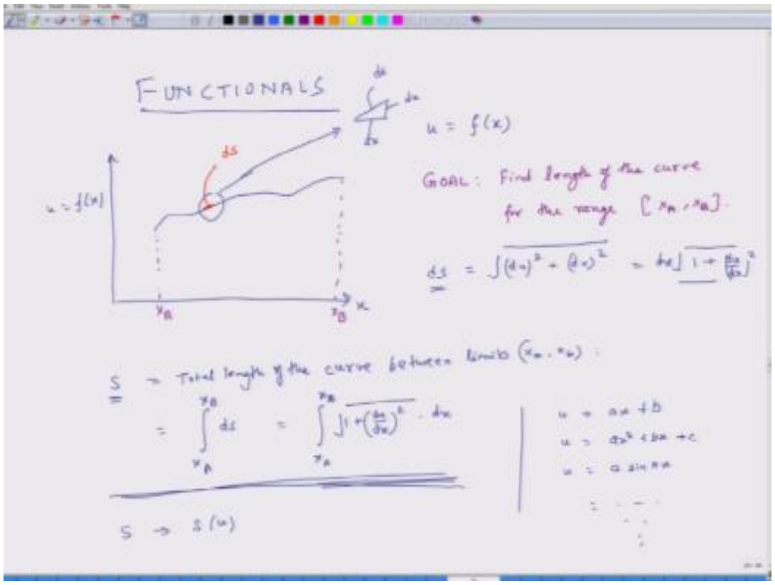

Yeah this functional this represents the length of a curve, if you take one curve and add to it another curve, the individual lengths of the two individual curves will not be same as the length of the sum total of those two curves okay, it will not be same as sum total of those two curves. So this is a different type.

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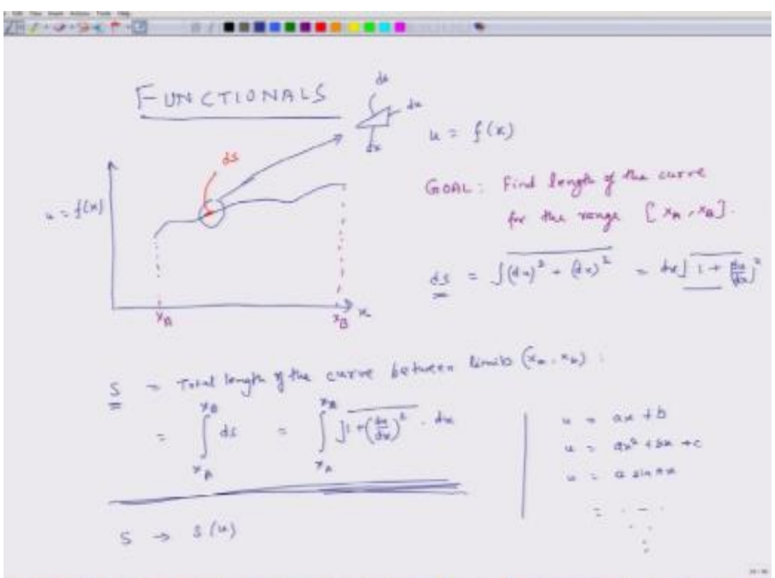

Of function, but it is definitely not linear. So now we have discussed what is a linear functional, and next what we will do is we will talk about a functional.

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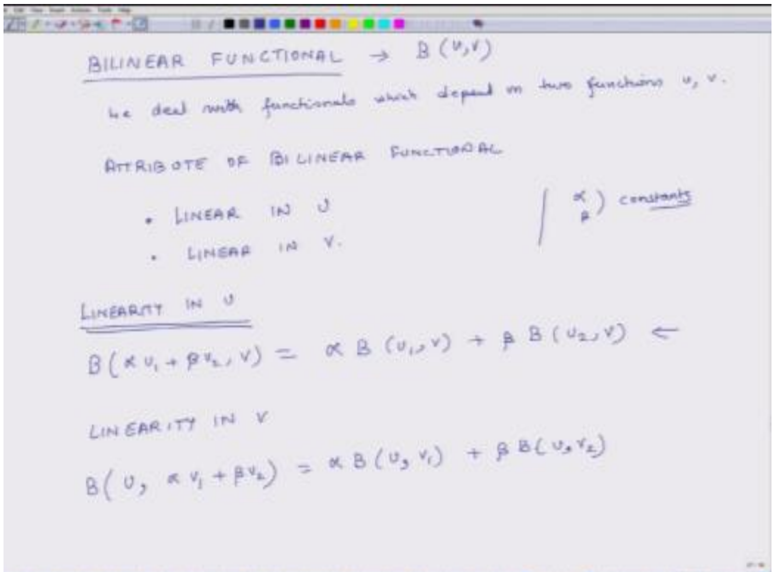

Which is called bilinear, bilinear function, so in such a situation we are here, we deal with functionals which depend on two functions, u and v, u and v and they could be mutually independent they would be two functions which could be u and v okay, this bilinear functional is represented as B standing for bilinear and the two functions which we are talking about are v and u okay.

So what is the attribute of bilinear functional, first thing is it is linear in u and it is linear in, in v. So if it is linear in u, if it is linear in u, so actually I am going to just change this it is u and v, just to be consistent. So if it is a linear in u then $B(\alpha u_1 + \beta u_2, v)$ equals $\alpha B(u_1,v)+\beta$ of $\beta(u2,v)$ that is the, that stabilization the linearity in u. So, a bilinear functional has to be both linear in u and linear in u,v so if I am having u itself is a sum of two functions u_1 and u_2.

Then if I compute the value of that functional for u, u alone and u_2 alone and I add those two up it should be the same as the value of the functional which is calculated using u_1+u_2, that is what it means okay. So this is the condition for linearity in u, and the condition for linearity in v is that so it is a very similar formula so here instead of u_1 and u_2 we will have v_1 and v_2, and let us say alpha times v_1+ β times v_2 equals alpha times $B(u, v_1)+\beta$ times $B(u, v_2)$ okay.

Once again alpha, beta are constants, so a bilinear functional where first thing is it depends on two functions, two independent functions okay, two independent functions and both these functions could be functions of X and.

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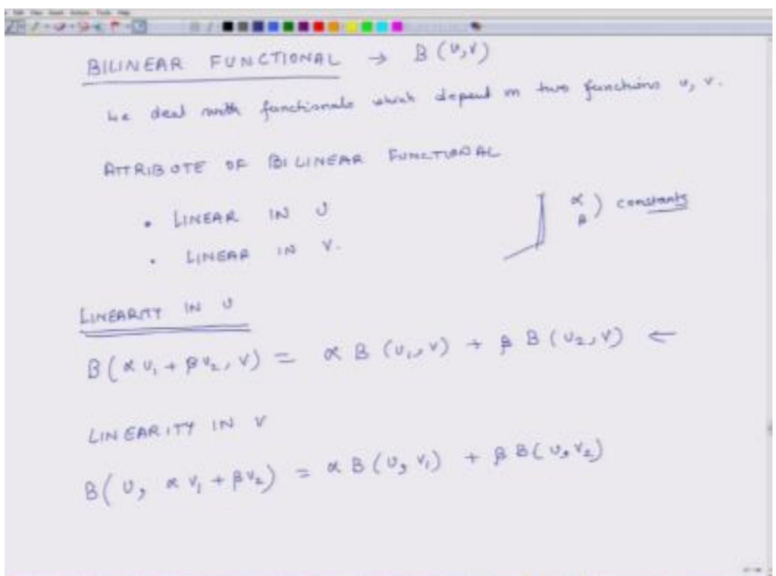

It will be linear if it is linear in u and linear in v. It has to satisfy both these constants, both these conditions okay, and then.

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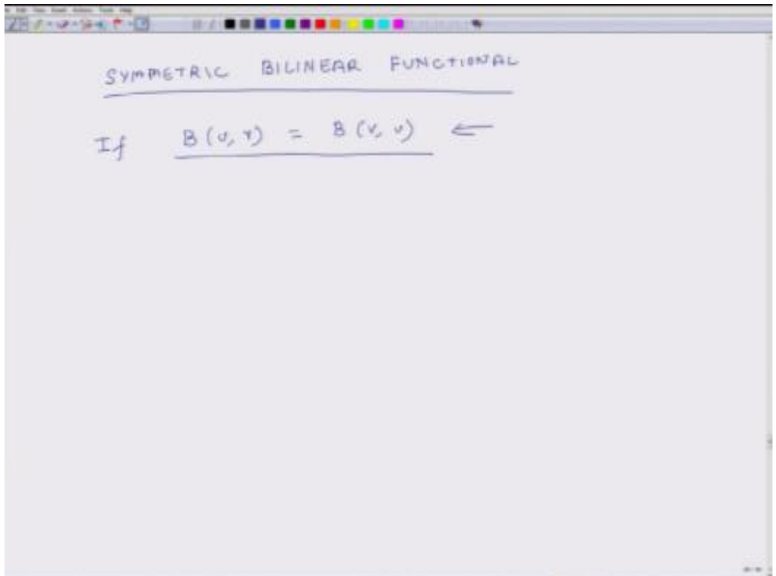

There is a special category of a bilinear functional and it is symmetric, symmetric bilinear functional. So what is it, it is symmetric if B of u and v is same as B of v and u. We will give examples for that, so then it is the condition for symmetry is that if you have a bilinear functional depending on two functions, mutually independent functions u and v instead of u you put v, instead of v you put u, the form of the functional should be same.

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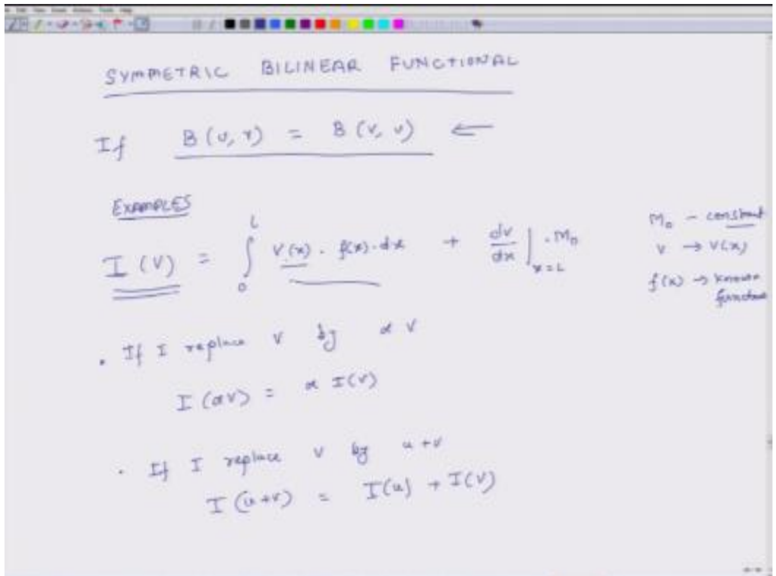

The value of the functional should be same, so then it would be a bilinear functional. Examples, so let us consider i(u) is equal to integral 0 to l of x times f(x) times dx+ dv over dx, x is equal at x equals l times some constant M_0 okay. So here M_0 is constant, v is a function of v(x) and this is a functional. So I should have written it here as v, I should written it as v, f is known function, f is known function so I cannot vary f, I cannot vary f, v is something let us say we are interested in finding out.

We can change, so this is so that is why I am saying that i depends on v alone it does not depend on f, where f is known so I know but it, the v if I change v I cannot change f it by but I can change v, I can change v so if I change v then i changes so that is why i is dependent on v alone okay, and then M_0 is also a constant okay. So this is there, now if I replace v by alpha times v right what will I get I will get this condition i(α v) =is equal to α times i(v) also if I replace v by u+v, then I get i(u+v) is equal to i(u)+ i(v,) I mean I can do that right, so both these conditions are satisfied which when the conditions were

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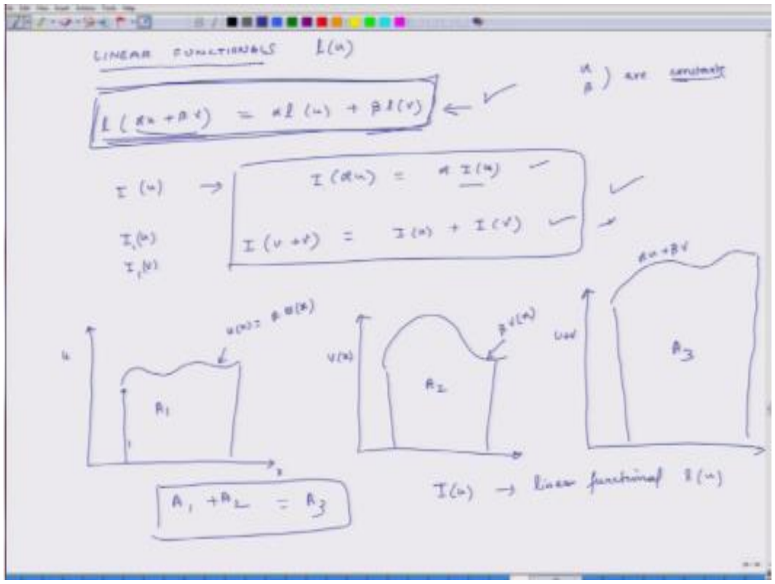

Linear functional, I can satisfy both these conditions, so I can say that bits are linear functional. I could have directly instead of doing those two conditions tried this thing, the first condition which is a combination and it would have satisfied that also.

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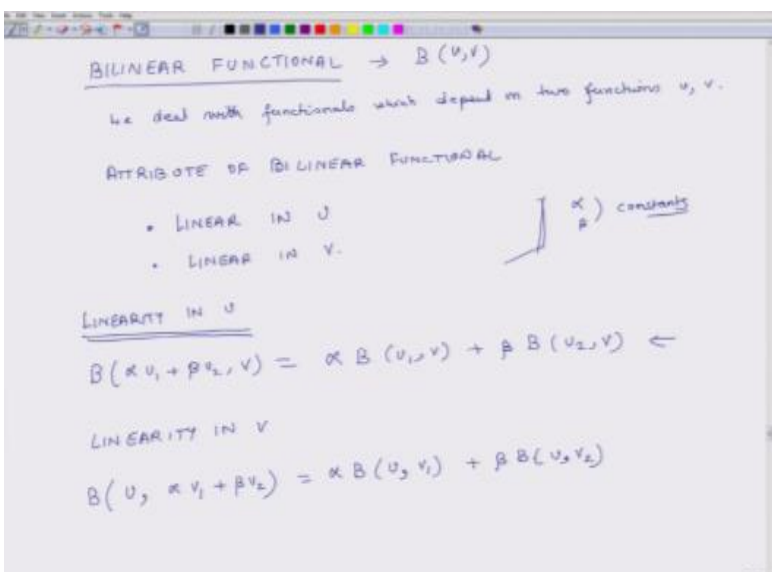

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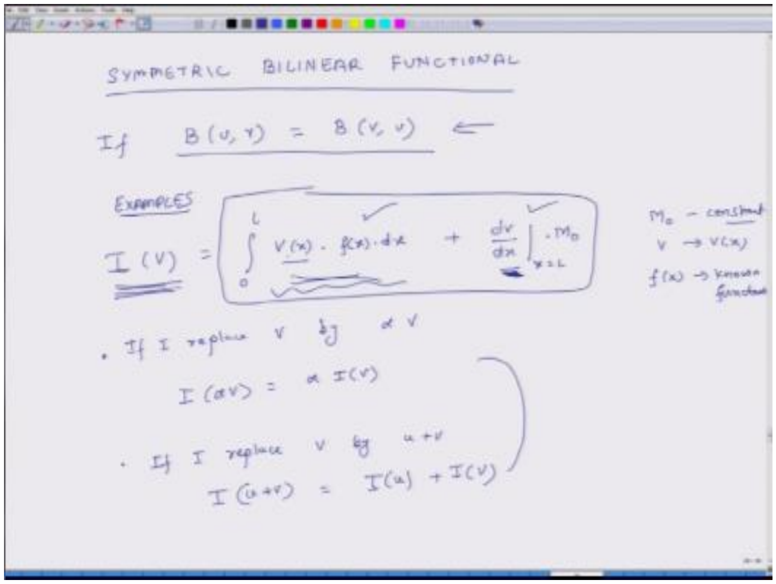

Okay so and I have to use that condition also in this term okay, because I am not defining i as just only this thing I am defining i as the entire sum of this thing okay. So I should substitute make these operations in this term and also in this term, and then see whether i is meeting those conditions or not okay.

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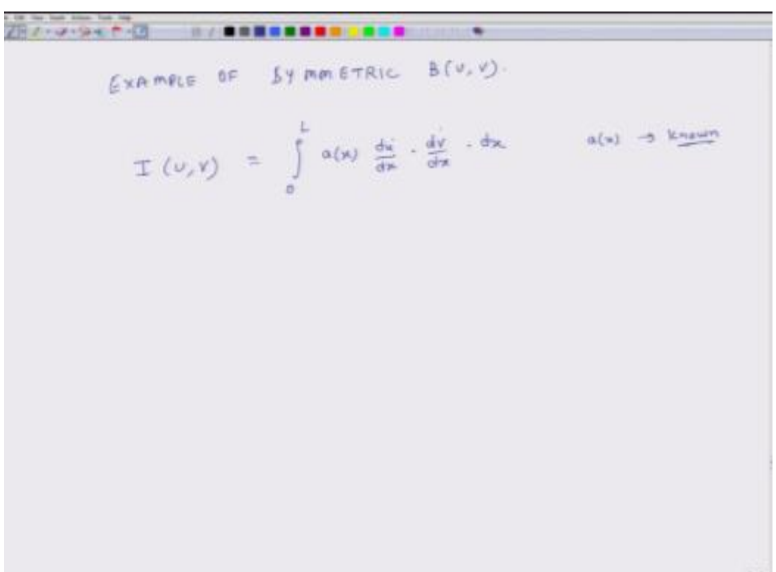

So that is here, it is another example, example of symmetric B(u,v), so this is an example of symmetric bilinear functional. So let us say that i(u,v) wells 0 to L a(x) du of dx times dv of dx times dx okay. And a(x) is known okay, so again if a(x) is known these are the things we can, I can change u and v right, I cannot change a, for instance a could be the cross-sectional area of a rode along the length.

That is not going to change but u and v we do not know we can try different solutions for u and v and can see whether it will meet our needs are not.

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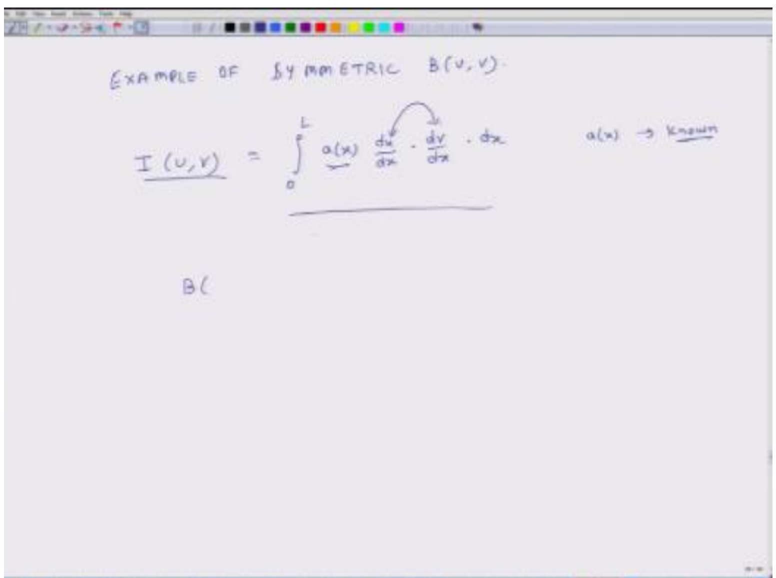

So that is why, so i depends only on u and v, I mean of course it depends on a but that dependence does not change, because a is not changing okay. So i is depends on u and v, and if I replace u and v with each other this entire expression does not change, so that is why it is first thing is it is bilinear because.

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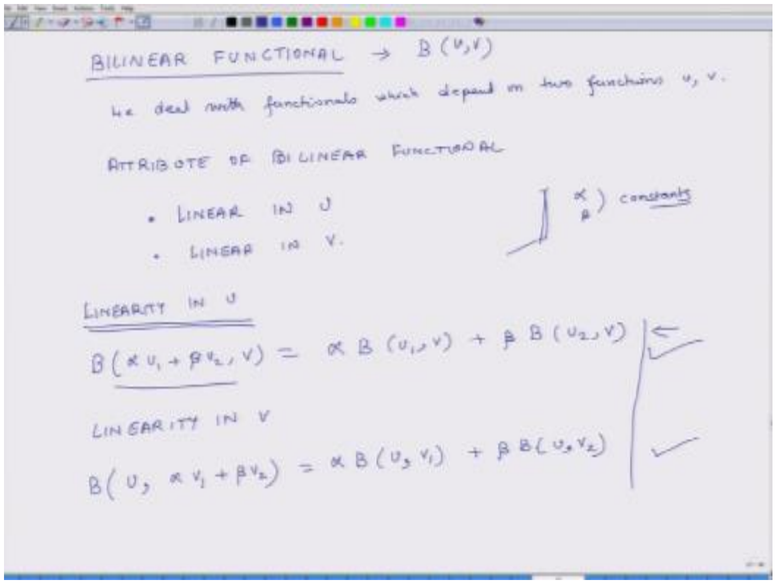

If I use this condition what is the condition, what is the condition for bilinearity, that it should be linear in u and linear in v. So I, but to verify whether this condition is satisfied the first condition is satisfied and I will say it satisfies the first condition, and I will also see that it satisfy the second condition. So that will establish that it is bilinear.

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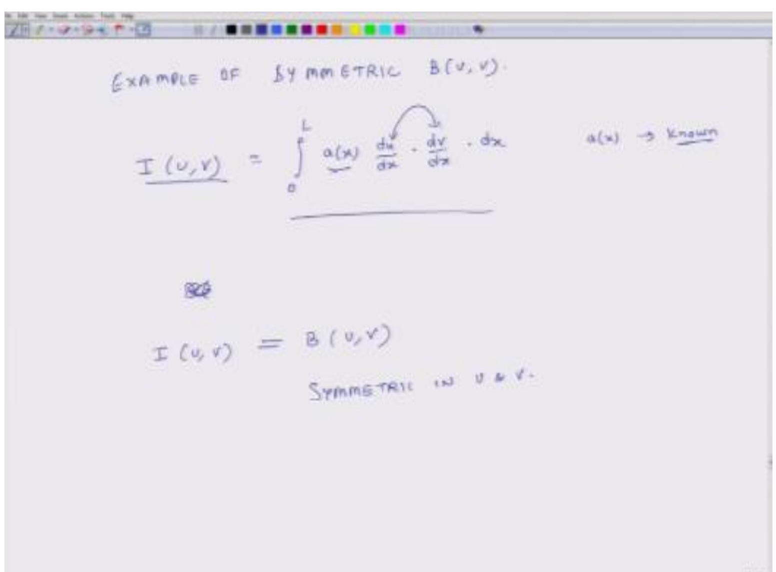

And then I will go further and I will replace u with v and v with u, and then I see that the form of i does not change, so in that case so because of these two reasons i(u,v) is first thing is its bilinear, and it is also symmetric, it is also symmetric in u and v. So this closes the discussion on functionals, we will not cover the idea of variations and variationals in this lecture because of lack of time. But starting next week we will start with discussion on variations and then we will start looking in detail at weak formulations. So thank you very much and have a great weekend. Bye.

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