Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 10
Gradient and Divergence Theorems

by
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Hello, welcome to basics of finite element analysis, this is the second week of this particular book course and today is the fourth day for this of this week. And the last class we had discussed the notion of weak formulation, and we have shown that how we can shift the differentiability operator from one function to the other function using integration by parts.

So we have done this exercise in context of 1d domains where integration was being performed in one dimension only. What we will do today is review two additional concepts one is the gradient theorem, and the second one is the divergence theorem. And then we will see how we can use these relations for gradient and divergence theorem to develop weak formulations in two-dimensional domain systems as well.

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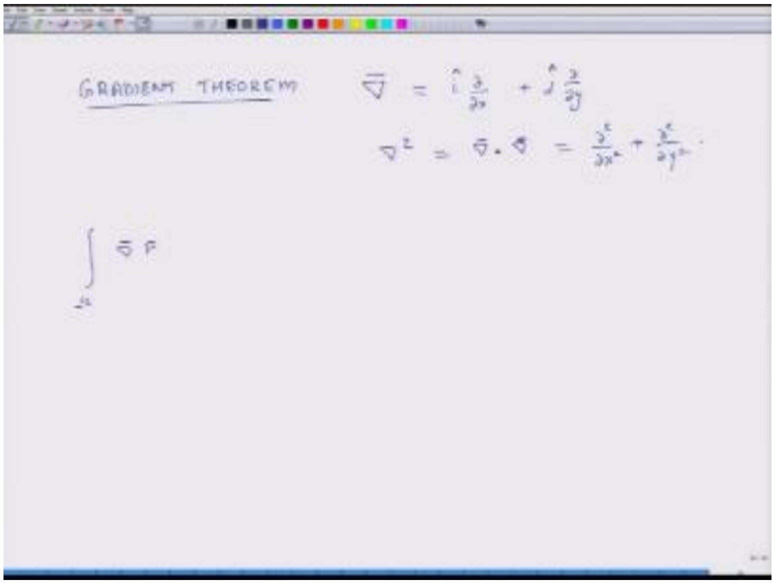

So the first one is the gradient theorem and in this theorem we use an operator, differential operator designated by this symbol and this is a vector operator, so it is associated to the direction and it is defined as ∇ over ∇ x times i + j times ∇ over ∇ 5. And then if I have an operator like this grad square, then this is basically the dot product or the inner product of two gradient operators. So this, so this is a scalar operator and this is what it in place.

Now first I will write the expression for gradient theorem, so what the gradient theorem says is, that if I take the gradient of some scalar function, so the function is not having any direction an example of a scalar function would be a plate, and we are measuring temperature in the plate, so the temperature does not have any sense of direction.

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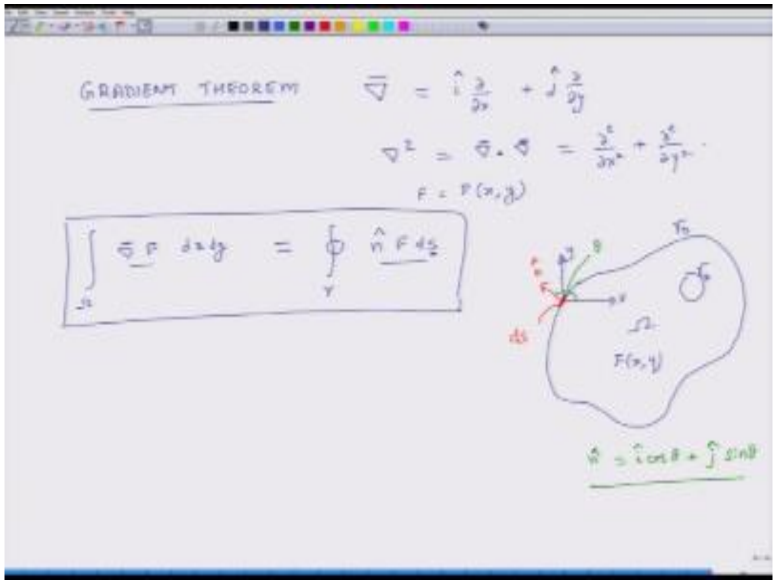

So this scalar function is there, and f is a function of x and y it is a 2d function. So here my domain is dx times dy, because I am hoped it doing the integration on a surface. So if I integrate the gradient of a scalar function on a domain that is a two-dimensional domain, then this is nothing but boundary integral and it is defined as this. So this is the gradient theorem we will explain this.

So suppose this is my domain, this is the domain, this is boundary maybe there is another inside boundary, so this boundary A, this is boundary B so what the gradient theorem says is, that if I take the gradients of F is present in this domain at all points. Then what it says is that the gradient of this function F let us say it is temperature function. So a gradient of temperatures and it was always – if it is a temperature gradient it will have two components x partial in x and partial in y.

So these gradients if I integrate over the whole domain that integral is equal to this function. So normal so this is the boundary so s represents a small portion of the boundary. So this is my ds okay, and the normal is this. So if let us say this is my x direction, this is my y direction and normal is in this direction and associated with the normal is a direction and let us say this angle is Θ.

So this n is equal to cosine of Θ i times j times sine of Θ. So what it means is the physical interpretation of this is, that the integral of gradient of F in the whole domain is equal or equivalent to whatever is happening on the boundaries. So I can look at it in two ways, if I know what is happening inside the whole domain I can calculate what is happening on the boundary, or the other way to look at it is, if I know what is happening on the boundary, then I can figure out all the details of the stuff which is happening in the domain. This is what – this is the most important take away from this.

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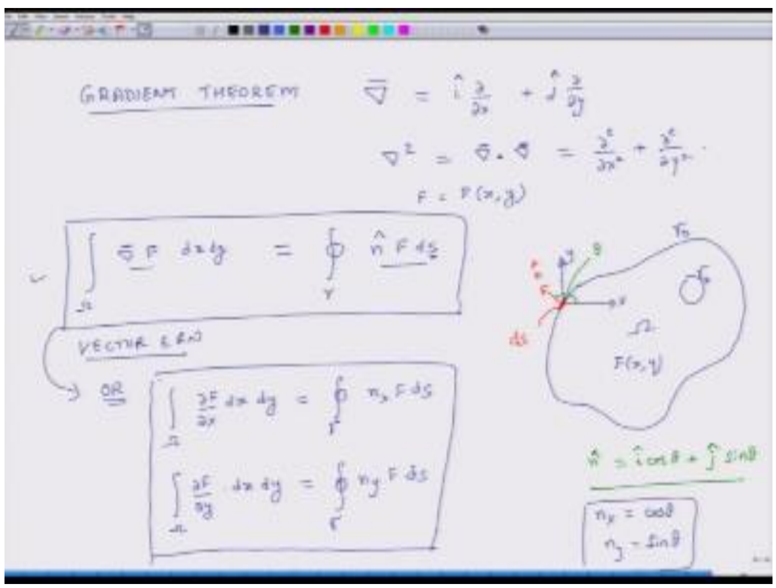

Now this is a vector equation, so it had, because this gradient operator is a vector, so this -- I can break it up into two separate equations. So I can alternatively write it as -- I can write it as – so integrate. So this is the other this is the other form of the equation this is in vector from, this is a scalar form of the equation. And here nx is equal to cosine of Θ and y is equal to sine of Θ. So this is what the gradient theorem says, so the gradient theorem deals with scalar fields.

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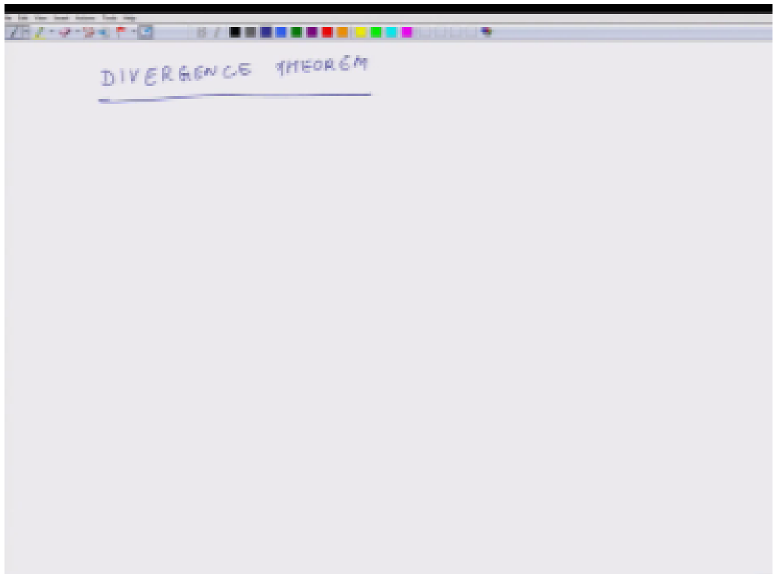

Then there is another theorem known as the divergence theorem. So divergence is conducted in context of vectors.

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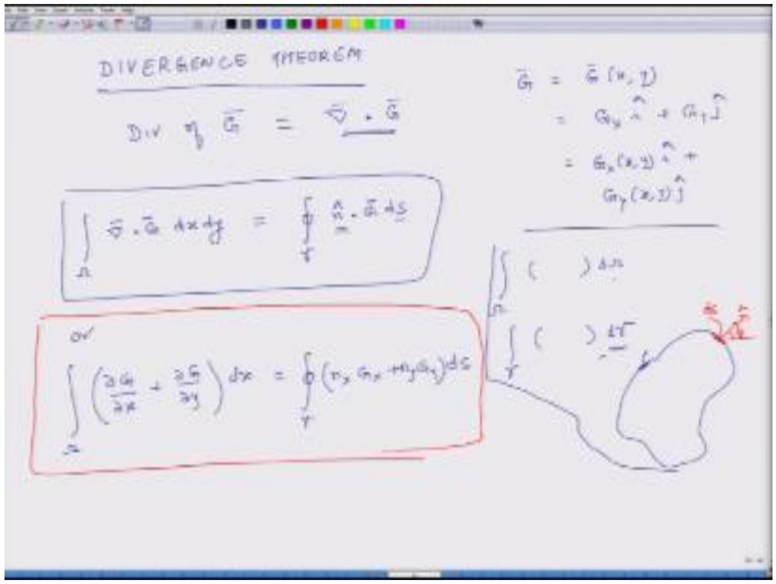

So the definition is divergence of a vector G is nothing but, the inner product or the dot product of the gradient operator and the vector which we are dealing with. Now here G could be dependent on x and y. So you can write it as G_x + G_y there G_x could be a function of x and y and G_i would also be a different function of x and y. So it is not that G_x is only dependent on x and G_y is only different on y okay.

And its divergence is the inner product of the gradient operator and the vector G. So what the divergence theorem says is, that divergence of G when I integrate it over the domain is nothing but, the line integral or closed integral of the unit vector on the boundary times G ds okay. So we have defined what these entities are, what is the normal vector, unit vector in the normal direction to the boundary, what is ds and all that stuff.

So we will not do that again, one thing I wanted to mention is that in lot of literature you may see that there is a domain integral, so some function will be here and then they will write it as this d Ω. If it is a 2d domain then this d Ω may mean dx and dy for Cartesian reference system. If it is 1d domain then this d Ω may imply dx so this is one thing.

And similarly the boundary integral may be written as some terms in parentheses times d Γ and this d Γ is basically whatever is a boundary this d Γ it could be points. So in case of point the integral is not valid if it is a 2d domain then d Γ corresponds to d, you know this small line element if it is a 3d integral then it will corresponds to a surface, so it is a 2d surface element.

So this somewhat little bit divergent about terminology. So this is the divergence theorem, and I can also write it as -- so this is a scalar equation. So my equation for the gradient theorem was a vector equation, here we have a scalar equation and this it can be written as partial derivative of G with respect to x plus partial of g with respect to y dx is equal to Γ integral on the boundary times nx, G_x + ny, G_y times ds.

So again my ds there is a small line element here, this is my normal, this is my Θ. So these are the two forms for the divergence theorem, and then we had also seen the gradient theorem. So what this divergence theorem shows is again something that basic message of the divergence theorem is also fairly similar to that what we sign context of gradient theorem.

Now gradient theorem is applicable to scalar fields here we use this in context of vectors, but here also the basic messages same the if you have a vector function over a domain then it is the divergence of that which can be expressed as partial derivatives of its x and y components, they are dependent on whatever is happening on the boundary whatever is happening on the boundary and vice versa.

So these are two theorems one deals with scalars, other deals with vectors and these two theorems, and that is why these two theorems are, so they relate the processes happening on the boundary to something and inside the boundary. Now the reason we talked about these theorems is that we will use these theorems to again shift the differentiability operator from one function to other function.

So that is the context in which we will apply these theorems. So we will continue this discussion in the next class and we will terminate our discussion at least in context of this particular class here itself, thanks.

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