BioMEMS and Microfluidics
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Lecture - 29

Hello and welcome back again to this twenty ninth lecture of bio microelectromechanical systems.

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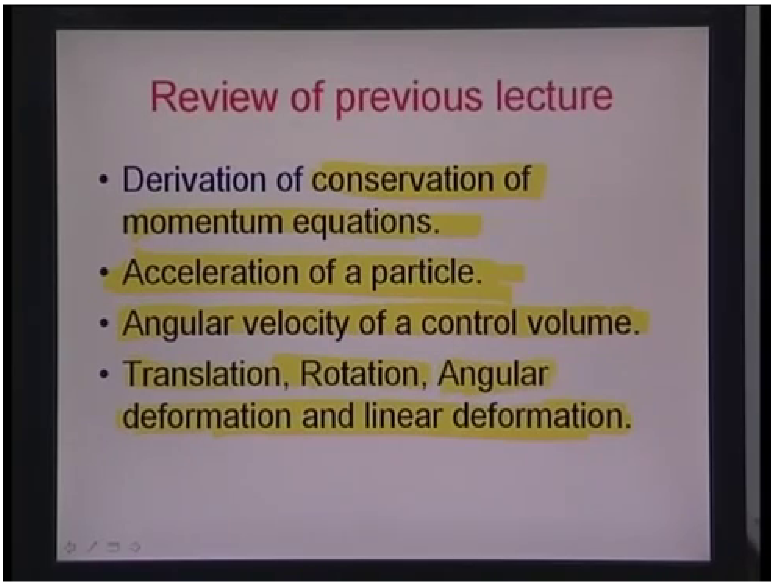

Let us quickly review what we did in the previous lecture. We talked about some of the steps towards derivation of the first Navier-Stokes conservation of momentum equation. And, basically, we discussed how we can represent the acceleration of a particle at a point p in a velocity v defined or varying with respect to the position coordinates and time. We also talked about how angular velocity of a certain particle essentially can be related to the average velocity of both sides of a control volume. And, we investigated the rotation case and the angular deformation case and found out that they can be represented as the variation of the y velocity in the x direction and the x velocity in the y direction respectively. And, we talked about these different kind of deformations that control volume; a cubic control volume would have including translation, rotation, angular deformation and linear deformation as the control volume moves along the path of fluid in a medium – in a certain medium So, today, let us just go ahead and try to complete what we left unfinished of the conservation of momentum equation.

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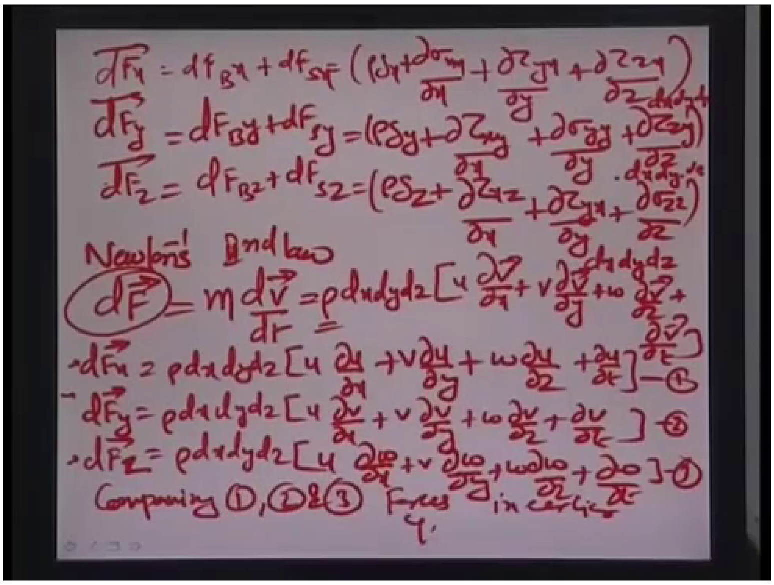

$$d\vec{F}_x = d\vec{F}_{Bx} + dF_{sx} = \left(\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dxdydz$$

$$d\vec{F}_y = d\vec{F}_{By} + dF_{sy} = \left(\rho g_y + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z}\right) dxdydz$$

$$d\vec{F}_z = d\vec{F}_{Bz} + dF_{sz} = \left(\rho g_z + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{xz}}{\partial x}\right) dxdydz$$

$$d\vec{F} = m\frac{d\vec{v}}{dt} = \rho dxdydz\left[u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z} + \frac{\partial \vec{v}}{\partial t}\right]$$

$$d\vec{F}_x = \rho dxdydz\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right]$$

$$d\vec{F}_y = \rho dxdydz\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}\right]$$

$$d\vec{F}_z = \rho dxdydz\left[u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}\right]$$

We got the several force components in the x, y and z direction respectively as the equations d fx equals dF_bx, which is the body force in the x direction times the force due to stresses in the x direction. And, we represent this as rho g x plus delta sigma x x by delta x plus delta tau y x by delta y plus delta tau z x by delta z. Similarly, d f y and d f z respectively – d f b y plus d f s y, which is rho g y plus delta tau x y by del x plus delta sigma y y by del y plus delta – tau z y by delta z. Similarly, d f z is the body force in the z direction plus the force due to stress in the z direction, which is equal to rho g z plus delta tau x z by dx plus delta tau y x by dy is delta sigma z z by dz times of dx dy dz. Actually the volume element is multiplied everywhere. So, this is also the same into dx dy dz. This is also the same times of dx dy dz respectively.

So, from the Newton's second law, if you consider this control volume that really d f and the amount of force that the control volume would actually try to incorporate or face is nothing but m – the mass of the control volume or delta m times of dv by dt; where, v is the velocity of the particle at a point p; and, it changes to v x plus dx y plus dy z plus dz at time instance t plus delta t respectively. We did this derivation as a matter of fact just before we started considering the stresses in the control volume when we talked about acceleration. So, therefore, in this particular case, you can represent really this as m d velocity vector with respect to time, which is nothing but rho dx dy dz, which is the elemental volume; rho being the density. We assume rho not to vary with t or it is essentially an incompressible case. So, times of u del v by del x plus v del v by del y plus w del v by del z plus del v by del t. And, if you actually physically resolve the different components d f x, d f y and d f z in this particular expression, then you are left with d f x vector is essentially rho dx dy dz times of u del u by del x plus v del u by del y plus w del u by del z plus del u by del t. And, let us call this equation 1.

Similarly, del f y – total amount of force is rho dx dy dz times of u del v by del x v del v by del y plus w del v by del z plus del v by del t; where, u, v and w are basically components of velocity vector – v vector in the x, y and z direction. And similarly, d f z vector is essentially rho dx dy dz times of u del w by del x plus v del w by del y plus w del w by del z plus del w by del t. So, that is equation number 3. So, you have these three equations as respectively d f x, d f y and d f z. And, we compare these to the forces obtained earlier; so, comparing 1, 2 and 3 with the forces, which we got in the earlier equations, which equates the body force and the force due to the stress.

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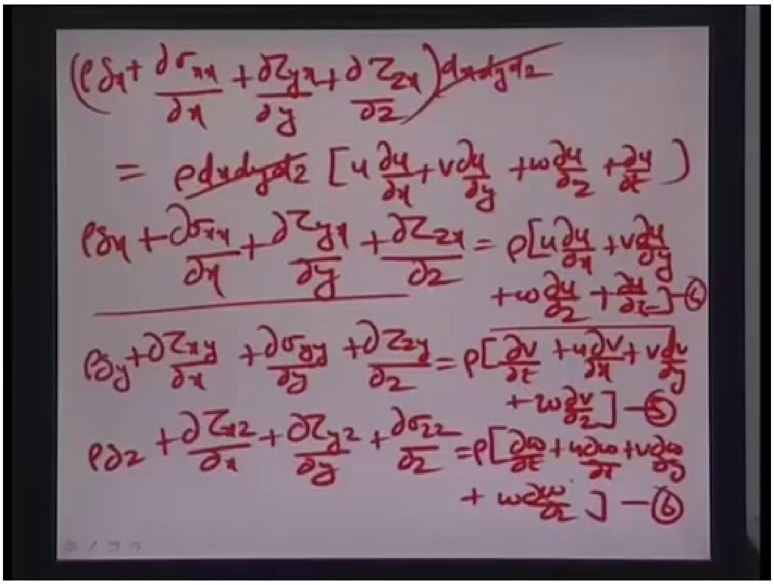

$$\left(\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dxdydz = \rho dxdydz \left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right]$$

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right]$$

$$\rho g_y + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}\right]$$

$$\rho g_z + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{xz}}{\partial x} = \rho\left[u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}\right]$$

So, basically, if you compare this the new set of equations which come out because of that would be essentially rho times g x plus del sigma x x by del x plus del tau y x by del y plus del tau z x by del z times of dx dy dz essentially equal to rho dx dy dz times of u del u by del x plus v del u by del y plus w del u by del z plus del u by del t of this time variation – the time component with respect to time is a separate entity altogether as you are seeing here as you made in the first assumption before. So this elemental volumes kind of cancel each other and we are left with straight equations – rho g x plus del sigma x x by del x plus del tau y x by del y plus tau del tau z x by del z is equal to rho times of u del u by del x plus v del u by del y plus w del u by del z plus del u by del t. So, that is what the x balance would be between the m a x that means the force in the x direction and the force due to the body force of the stress in the particular control volume in question.

Similarly, we will do the same kind of analysis in the y direction, the z directions respectively. So, the two equations that we get as a result of it; I am just writing down. So, this – let this be equation 4. So, similarly, we have equation fifth in the y direction as a comparison between all y forces as rho g y plus del tau x y by del x plus del sigma y y by del y plus del tau z y by del z equals to rho times of del v by del t plus u del v by del x plus v del v by del y plus w del v by del z. This is equation 5. Similarly, in the z direction, we have an identical result, where rho g z plus del tau x z by del x plus del tau y z by del y plus del sigma z z by del z is nothing but rho times of del w by del t plus u del w by del x plus v del w by del y plus w del w by del z respectively. So, this is equation 6. So, if you consider the values of the different shear stresses – tau x y, tau y x; similarly, tau z x and tau x z and tau y z and tau z y respectively in terms of its respective variations of the velocity components with respect to space components as we derived in case of rotation and angular deformation before. We will be left with a very simplified and straightforward equation.

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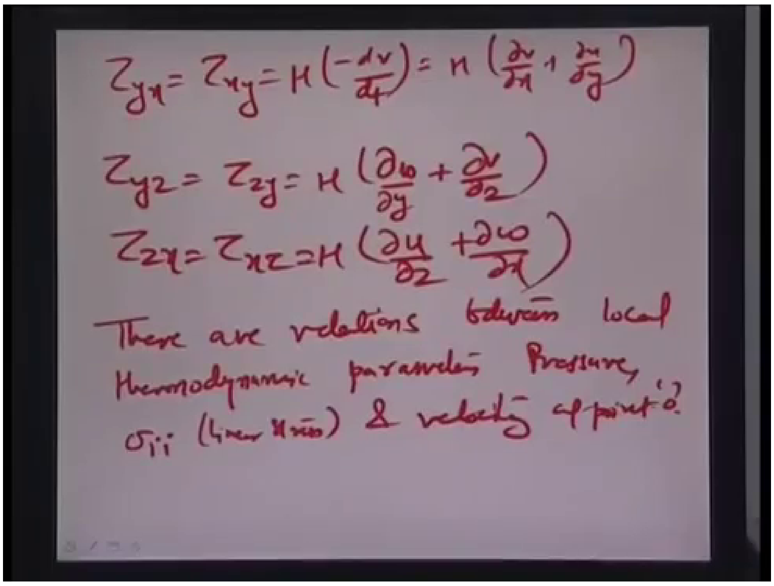

$$\tau_{yx}=\tau_{xy}=\mu\left(\frac{-d\gamma}{dt}\right)=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$

$$\tau_{yz}=\tau_{zy}=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)$$

$$\tau_{zx}=\tau_{xz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$$

So, let us say – as we have done before tau y x and tau x y causing angular deformation, can also be expressed as minus mu – let us say d gamma by dt, which is equal to mu times of del v by del x plus del u by del y. And similarly, tau y z equal to tau z x z y is same as mu times of del w by del y plus del v by del z. And similarly, tau z x equals tau x z. And, this we did actually in the last class or last lecture how this derivation happens again. du by del z plus del w by del x respectively. And, there are some other approximations that we need to make here, which comes from a thermodynamic pressure and the relationship between the thermodynamic pressure. The stress components be it shear or be it principle stress and the velocity; so, all these three link together; I am not going to actually derive these pressure-stress equations separately; it is an altogether separate topic.

But, I am going to assume the approximations, which are made in terms of relationships between the different stresses and the pressure, etcetera; and then, try to put this back into the equation in question and try to figure out what the final form of the Navier-Stokes momentum equations – conservation of momentum equations would look like. So, there are the definitely relationships between local thermodynamic parameters like pressure, sigma i i; this is linear stress; and, velocity at point o around which this control volume has been indicated. And, if you may remember, we indicated the control volume by defining a central location o on both sides of which the control volume extends dx by 2, dy by 2 and dz by 2 respectively with the plus and minus sign both.

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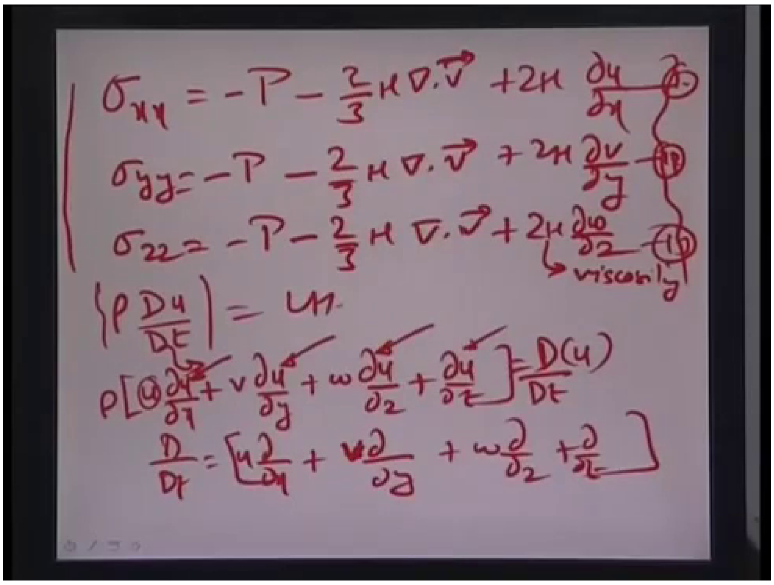

$$\sigma_{xx} = -P - \frac{2}{3}\mu\nabla\vec{V} + 2\mu\frac{\partial u}{\partial x}$$

$$\sigma_{xx} = -P - \frac{2}{3}\mu \nabla \vec{V} + 2\mu \frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -P - \frac{2}{3}\mu \nabla \vec{V} + 2\mu \frac{\partial w}{\partial z}$$

$$\rho \frac{Du}{Dt} = U$$

$$\rho \frac{Du}{Dt} = U$$

$$\rho \left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \right] = \rho \frac{Du}{Dt}$$

$$\frac{D}{Dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t}$$

So, here the relationships that come based on this argument are sigma x x equal to minus P – P is the thermodynamic pressure – minus 2 by 3 mu del dot v. v again is nothing but u i plus v j plus w k plus twice mu del u by del x. Similarly, you have sigma y y equals minus P minus 2 by 3 mu del dot v plus twice mu del v by del y. And, sigma z z equals minus P minus 2 by 3 mu del dot v plus twice mu. Give me a minute here; twice mu del w by del z. Mind you – this mu is essentially the viscosity. The relationship between shear stress and the velocity gradient with respect to are the perpendicular direction in the direction of flow. And, del v of course, is essentially nothing but again ratio between du by dx plus del v by del y plus del z by del x or del w by del z respectively. So, these relationships if we assume them, and we do not derive them and then put this back along with the stress vector that we have seen before the shear stress vectors here in equations let say 7, 8 and 9; we finally, get a form of the Navier-Stokes equations, which really is something that under the incompressible flow conditions are assumed to be true. So, the final form again.

So, I am going to write from here a substitution of these shear stresses and the relationships between the different principle stresses. Let us call these equations 10, 11 and 12 respectively. So, what was our earlier relation? Our earlier relation was between the M dv by dt and the forces due to the stress components, which came into being. And, here the relationship was really rho times of d by dt of u. So, basically, this du by dt here though is an operator, which we have designed in a very particular in a peculiar manner. So, as you know here, the left side of the equation already was rho times of u del u by del x plus v del u by del y plus w del u by del z plus del u by del t. So, we take this to be the operator d by dt of u; u being this variable here essentially. The other format – the operator d by dt is nothing but u del by del x plus v del by del y plus w del by del z plus del by del t. That is essentially what the operator is. So, we have defined this operator in this manner. So, the left side becomes rho du by dt.

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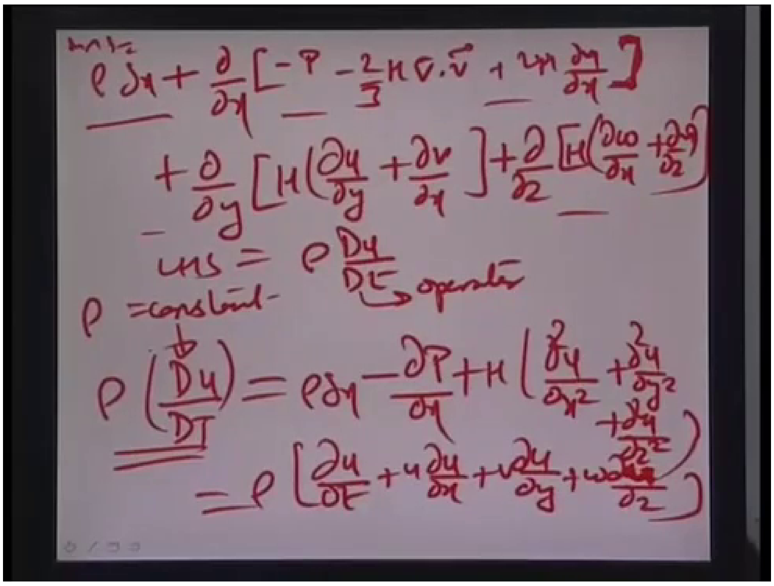

$$\rho g_x + \frac{\partial}{\partial x}\left[-P - \frac{2}{3}\mu \nabla \vec{V} + 2\mu \frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[-P - \frac{2}{3}\mu \nabla \vec{V} + 2\mu \frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial z}\left[-P - \frac{2}{3}\mu \nabla \vec{V} + 2\mu \frac{\partial w}{\partial z}\right]$$

$$\rho \frac{Du}{Dt} = \rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

The right side of this equation as we already know from previous examples becomes rho g x plus del by del x of sigma x x. And, sigma x x as you already know, comes from the pressure equation as minus P minus 2 by 3 mu del dot v plus twice mu del u by del x. That is what sigma x x is. So, d by d x of sigma x x plus del by del y of the equation was tau x – tau y x, which can be represented as from the angular deformation equation – mu times du by del u by del y plus del v by del x; and, plus we had d by dz of tau z x, which can again be defined from the angular deformation equation as mu times del w by del x plus del u by del z respectively. So, this is equated in general to rho du by dt; whereas, I told you this essentially is an operator. That is how you represent this equation.

So, if you solve this whole equation here on the right-hand side, this is of course the LHS;

this is the RHS of the equation. So, you are left with more particularly for incompressible flow if you assume the density is really constant, you are left with probably more appropriate form of equation, which is more like rho times of the operator d by d t of u is essentially equal to rho g x minus del p by del x; and, plus you are left with mu times of del 2 u by del x 2 plus del 2 u by del y 2 plus del 2 u by del z 2 respectively. So, this is in the x direction really. And, this again as you know is nothing but rho times of del u by del t plus u del u by del x plus v del u by del y plus w del u by del z. That is what the del operator or the d operator here – d by d t really is.

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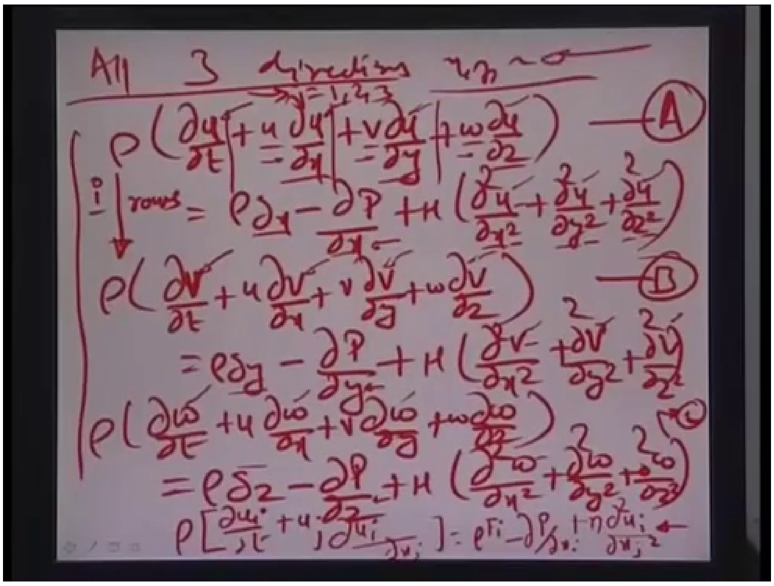

$$\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}\right]=\rho g_x-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left[u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t}\right]=\rho g_y-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left[u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial w}{\partial t}\right]=\rho g_z-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

So, in a nutshell, the Navier-Stokes equations then in all three dimensions – all three directions – x, y, z can be written as rho times of del u by del t plus u del u by del x plus v del u by del y plus w del u by del z equals rho g x minus del p by del x plus mu del 2 u by del x square plus del 2 u by del y square plus del 2 u by del z square. That is let us say equation A in the x direction. Similarly, you have rho del v by del t plus u del v by del x plus v del v by del y plus w del v by del z equals rho g y minus del p by del y now and plus you have mu times of del 2 v by del x 2 plus del 2 v by del y 2 plus del 2 v by del z 2. And similarly, this let us call as B equation in the y direction. And, in the C in the z direction, we call this equation C. So, del w by del t plus u times of del w by del x plus v times of del w by del y plus w times of del w by del z is equal to rho g z minus del p by del z plus mu times of del 2 w by del x square plus del 2 w by del y square plus del 2 w by del z square respectively. And, this we call as equation C. So, these are really the three directions of the conservation of momentum equation or Navier-Stokes second equation as you can see.

I would like to further kind of try to notate the two equations that we have formulated so far in terms of the conservation of mass and the conservation of momentum in terms of i's and j's. So, this is a generic notation, which can be used and extended to all the three dimensions. But then, essentially, the notational representation makes the equation much more look much more compressed. And, would essentially do a dimensional analysis on these equations. And, probably in the next slide, where we will see that, if I can translate the scale in question, where these equations are executed to the micron level, what is going to happen to both the conservation of mass and conservation of momentum equation. So, therefore, I would like to represent these equations – these all three equations by a notation.

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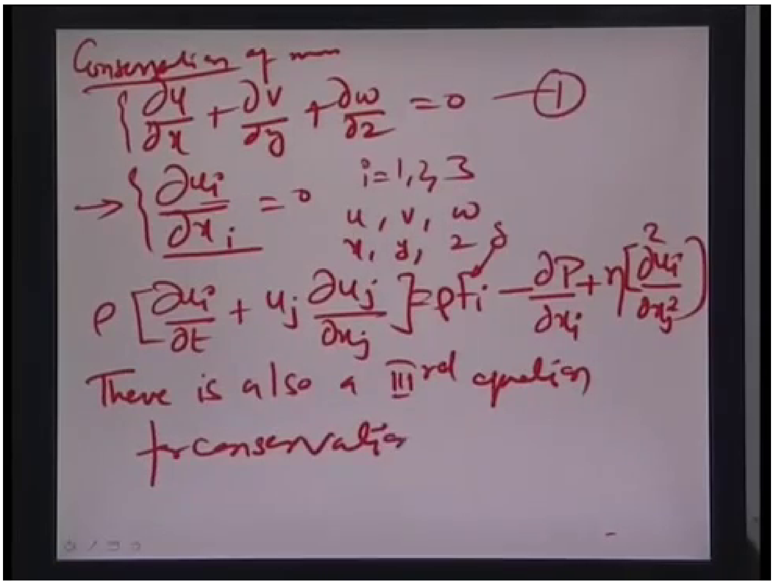

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial u_i}{\partial x_i}=0$$

$$\rho\left[\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_j}{\partial x_j}\right]=\rho F_i-\frac{\partial P}{\partial x_i}+\eta\frac{\partial^2 u_i}{\partial x_j{}^2}$$

Before that, let us actually write the conservation of mass again. So, conservation of mass as you know here is del u by del x plus del v by del y plus del w by del z equal to 0 for an incompressible flow essentially. And therefore, this I can notate in a little more appropriate manner as del u i by del x i equals 0. We assume that i's essentially are all the – i's essentially are all the… So, therefore, as you see here notationally, the i represents or i varies between 1, 2, 3 would represent u, v and w and x, y and z. So, this is a very straightforward equation that, del u i by del x I; that means del u by del x del v by del y del w by del z summation is equal to 0. So, this is a notational representation of the conservation of mass equation; it is a first Navier-Stokes equation; all right.

The other equation that we derived just about last slide can be notated as del… Or, let us actually do it here and then translate back the information. So, what is this equation really? If you see here there is a u component in all these operators in equation A. Similarly, there is a v component in all these operators in equation B. And similarly, a w component in C. And, what is interesting also is that, the x, y, z are varying in each of these equations; all right. So, if I notate all these u's, v's and w's as i; that means i varies in the direction of the rho's; and, j varies in the direction of the columns as if j is varying in the direction of the columns. So, I can notate this equation in a more appropriate manner as rho times of del u i by del t plus u j.

Now, as you see here, u, v, w are varying in the j directions; j varying between 1, 2, and 3 – meaning this u, this v, this w is actually corresponding to the j's. So, u j times of delta u i by delta x j. So, essentially, again as you see, the j is varying; wherever there is a variation in the columnar direction, it is j; wherever there is a variation in the row-wise manner, it is i. That is how you are subscripting both the variables. So, that is equal to rho times of f I; again i varies in the columnar direction. This is g x; this is g y; this is g z. All these in the columnar – in the rho direction. So, f i minus del p. And, let me just quickly read this right here. Let us mark C here. So, this is C. So, this is del p over del x i.

Again as you see here, in case of p, the subscript here varies in a row-wise direction in a row-wise manner. So, that is i – plus eta. And, essentially here you have plus eta times of del 2. And, you have a variation of u here as you see u, then v, then w is in the row-wise direction. So, this is corresponding to i. So, del 2 u i. And, you have in the denominator here, del x j 2 because x, y, z as you are seeing here is varying more in the columnar direction. So, that is what the representational – the notational representation of this particular three-directional momentum equation of Navier-Stokes is really like. So, we can write as rho del u i by del t plus u j del u j by del x j equals rho f i; f i is actually a representation of the body force or g – minus del pressure p by del x i plus eta – the viscosity – del 2 u i by del x j 2. That is how you represent the conservation of momentum equation. Now, there is also a third equation for conservation of energy.

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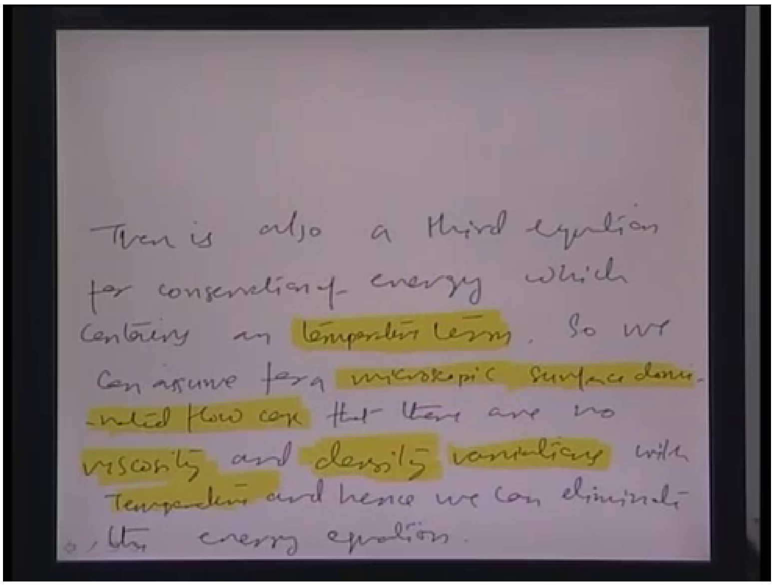

But, essentially, it contains a temperature term; that is the only difference that this particular equation has. And so, therefore, as in this particular scale we consider in the microscopic particularly surface domain, our flows are mostly dominated by the prominence of the surface over the volume. And, there are effectively not much change in the viscosity and the density; we still assume continuum base properties at this particular scale – the micron scale at least. And so, there are no variations in these properties with temperature. So, therefore, really the energy equation is not needed as far as the micro scale flows are concerned. What I would be more worried about at this stage is that, how we can scale down the momentum equation.

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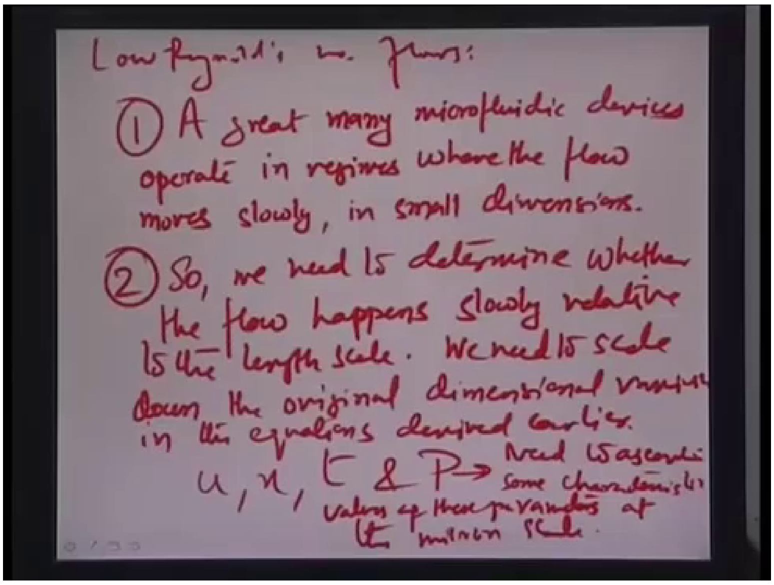

So, in the scaling equation, we had assume the following presumptions. Number 1 is that, all flows are low Reynold's number flows. So, why we actually try to take a low Reynold's number flow is that, a great many microfluidic devices operate in regimes, where the flow moves slowly; that is number 1. Number 2 – in small dimensions; that is number 2. Say for instance, you are talking about a very thin piece of channel or a very thin size of the microchannel, which is defined by for the lithography.

So, there the channel thickness is defined by the film thickness. And, the film thickness could be anywhere between let us say 20 microns all the way to about hundred microns or so. Now, hundred microns is effectively the diameter of a human hair. So, you can consider that, what is the effective volume through which this flow would actually flow. And, it is very obvious to assume that, the flow rates typically would be a few microliters per minute; that means the volume discharge through this thin sample is really really low. And so, you are packing the molecules in a smaller volume and plus secondly moving them in a very very slow manner, which is constraint by the geometry. And therefore, most of the cases, the flows are typically laminar in nature. And, low Reynold's number is obvious conclusion out of all these, because Reynold's number is nothing but rho v d by mu; where, velocity v or this dimension d – length dimension d, whichever is smaller makes the overall Reynold's number very small.

So, effectively, we really need to determine whether the flow happens slowly relative to the length's scale that we are considering. And therefore, we really need to scale down the original dimensional variables in the earlier two equations – the conservation of mass and conservation of momentum. So, one of the reasons why dimensionalization, non-dimensionalization is preferred at these and many other applications is because scaling down will ensure that, you do not have any absolute physical parameters like density, viscosity, then length scale, time scale, etcetera. So, what you instead have is a ratio. And, the ratio is a comparative to certain feature size or a certain parameter size, which is generally prevalent at the scale at which you are non-dimensionalizing the particular equation. So, this is the method, which is used or be it Lennard-Jones potential, be it a microfluidics; be it MD simulations. Just to ascertain that, you are essentially using non-dimensional variables at the scale that your experiments are all supposed to be.

So, the equation would be a good estimate of that particular scale when instead of a dimensional form, you use it in a non-dimensionalized way compared to – in comparison to parameters at the particular scale of the experiment. So, therefore, in this particular case also, how do we do that, we first of all find out what are the variables, which are effectively there in all the Navier-Stokes equation. So, you have velocity u as one variable, space coordinate – x, y, z – whatever u call. So, this is the length variable. There is a time variable. And then, there is a pressure variable p. And so, therefore, we need to ascertain some characteristic values of these parameters at the microns scale for non-dimensionalization.

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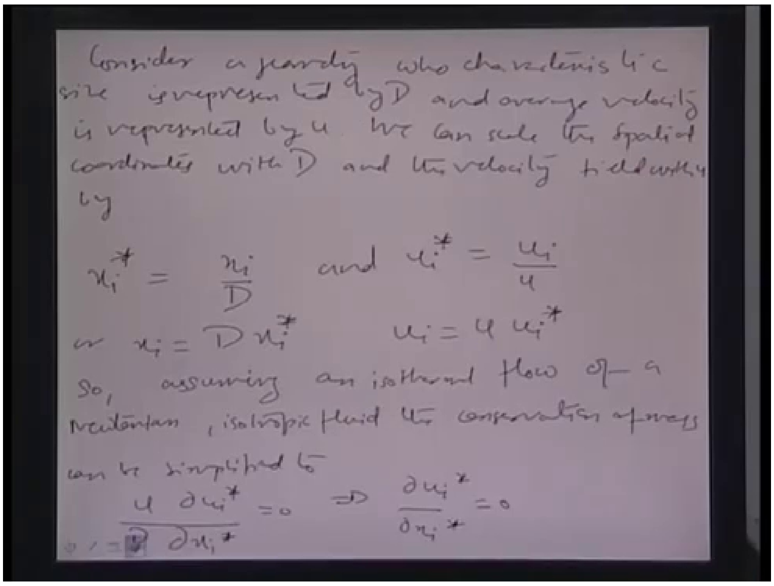

$$x_i^{\square} = \frac{x_i}{D}$$

$$x_i = x_i^{\square} D$$

$$u_i^{\square} = \frac{u_i}{u}$$

$$u_i = u\,u_i^{\square}$$

$$\frac{u\,\partial u_i^{\square}}{D\,\partial x_i^{\square}} = 0$$

$$\frac{\partial u_i^{\square}}{\partial x_i^{\square}} = 0$$

So, essentially, let us consider a geometry, So, we now, consider a geometry whose characteristic length is let suppose d and whose characteristic velocity is u. So, we represent everything in terms of d and u; time scale automatically follows suit. And, as we will see the density and the pressure, etcetera will also be represented in terms of all these quantities. So, essentially, that is what we will be trying to scale down. And, we make these non-dimensional numbers and call them with the or notate them with the subscript star like x i star, u i star, so on and so forth.

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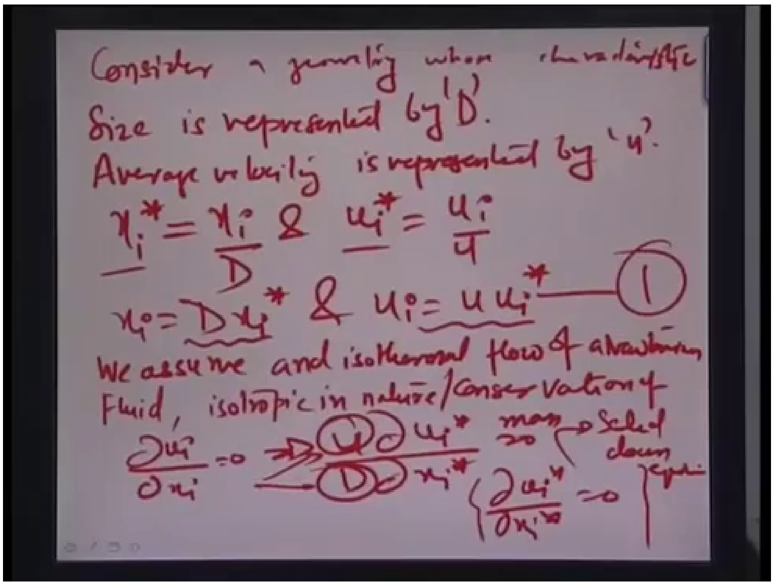

So, basically, now, if you look at let us say the scaling; so, we consider a geometry whose characteristic size is represented – is represented by – of the quantity xi. So, it is the quantity D; so, it is represented by the quantity D. Similarly, the average velocity is represented by u here. So, average velocity at that scale is represented by u. So, therefore, this number xi star, which is actually a dimensional – non dimensional number is exactly equal to xi by D. And similarly, u i star – the non-dimensional velocity number is equal to ui by u. So, therefore, that is how you represent these dimensionless numbers. The idea is to convert the Naiver-Stokes equations – both the mass as well as the momentum conservation equations into these quantities with subscripts star, so that equation is kind of a scale down model into the microscale for applications.

So, now, from these two equations, we can further derive that xi definitely can be represented as D times of x i star. And similarly, u i can be represented as u times of u i star. So, assuming an isothermal flow of Newtonian isotropic fluid – the conservation of mass is essentially very very simplified as D u i by d x i equal to 0. So, we assume an isothermal flow – isothermal being incompressible because there is no variation in density, with temperature, etcetera. All these flow at a constant temperature. So, this is often Newtonian fluid; which means again that, the shear stress is proportional to the rate of change of velocity with respect to the cross direction. So, du by dy proportional to tau essentially; so of Newtonian fluid. And essentially, which is also isotropic in nature. So, isotropic in nature means there is no non-homogeneity or inconsistency problem dimensions within the density or viscosity. They are all homogenous; they are all uniform across the whole medium. So, we assume these three conditions.

So, the conservation of mass equation then can be really represented as you saw earlier as D u i by d x i equal to 0. So, we try to now represent or put these different quantities here which have been formulated here. And, let us say the ((Refer Time: 38:51)) equations 1. So, we are left with u del u i star by d del x i star equal to 0. Or, in other words, del u i star by del x i star equal to 0. These two being characteristic numbers representing velocity and dimensions, they kind of remain constant; so, they can be taken outside the differential here. And so, therefore, del u i star by del x i star is 0. So, this is the scale down equation – scale down equation. So, the formulation of the scale down equation in case of conservation of mass is really not very critical; it does not go and change. It is just a ratio of the u i star number with respect to the d x i star number again. However, the changes would occur when you look at the Naiver-Stokes second conservation of momentum equation. Or, significant changes would occur, which can be interpreted. And, some of the properties – essential properties of the microscale of flows would really come out if you scale down the second equation from Navier-Stokes. So, let us do that.

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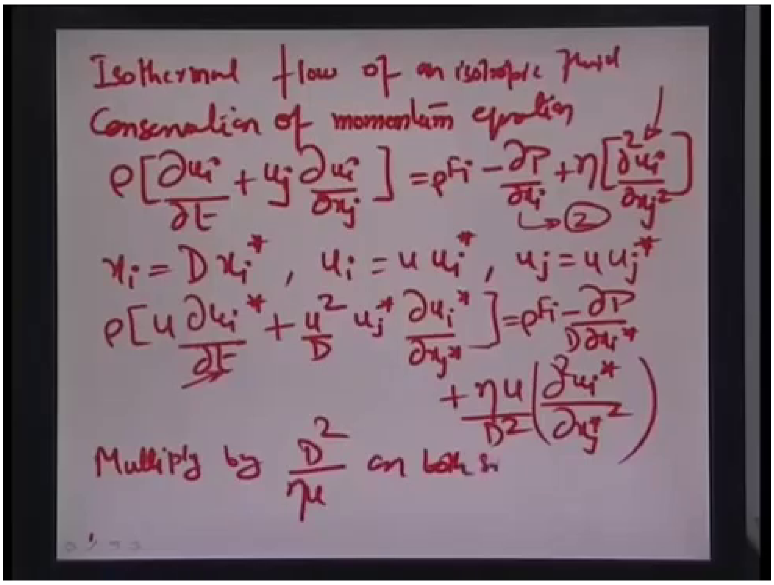

$$\rho\left[\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right]=\rho F_i-\frac{\partial P}{\partial x_i}+\eta\frac{\partial^2 u_i}{\partial {x_j}^2}$$

$$x_i = x_i^{\square} D$$

$$u_i = u\,u_i^{\square}$$

$$u_j = u u_j^*$$

$$\rho\left[u\frac{\partial u_i^*}{\partial t}+\frac{u^2}{D}u_j^*\frac{\partial u_i^*}{\partial x_j^*}\right]=\rho F_i-\frac{\partial P}{D\,\partial x_i^*}+\frac{\eta u}{D^2}\frac{\partial^2 u_i^*}{\partial x_j^{*2}}$$

So, we assume an isothermal flow of a isotropic fluid. So, the conservation of… So, we have a… So, we assume an isothermal flow of homogeneous fluid – an isotropic fluid. So, conservation of momentum equation as we saw earlier can be represented in terms of… If you just go ahead and look into the equation – the momentum equation that we derived before, it was rho del u i by del t plus u i or u j rather del u i by del x j equals rho f i minus del P by del x i plus eta del 2 u i by del x j 2. That is what conservation of momentum equation was really in terms of the notational form. Now, if you want to go ahead and substitute the different values of the non-dimensional numbers here; we have again two numbers as you may just recall, last slide we did this. So, x i is D x i star.

All velocities whether it is u i or u j essentially u u i star; similarly, u j; we are just talking about a scale. So, essentially, this is a representative quantity; u is a representative velocity at the particular scale of interest. So, whether it is a subscript j or i; whether it is a change in the columnar fashion or the row fashion, the corresponding dimensioned number or non-dimensional number will really not change because of that. And therefore, the relationship for j also holds valid. You have this u j as u – u j star; where, u j star is the dimensional number in the or the variation as j varies in the columnar manner; j is equal to 1, 2, 3. We already talked in details about this notation – notation if you may recall when we were trying to notate the whole set of the conservation of the momentum equation in Navier-Stokes derivation.

So, just substituting this back into the equation here; let us say this equation was equation number 2. We are left with condition, where rho times of u del u i star by del t; we have not yet characterized or we have not yet changed the time dimension. That we will be doing in the next step. So, plus we call it u square by D; and, this should be equal to really u j star times of del u i star by del x j star. That is how the ((Refer Time: 43:29)) can be written. And, it is equal to rho f i again minus del pressure p by del x i star into D. That is how you characterize this – plus eta. And, you call this u divided by square of D. As this notation represents here, it is del x j square. So, it is D square times of x j star square with del. So, you have del 2 u i star here and you have del x j star square here. And, what comes out of the equation is the eta u by D square. So, this kind of clear at the stage what this is about. And, let us just do a little bit of algebraic manipulation here; we multiply this equation by D square by eta u on both sides.

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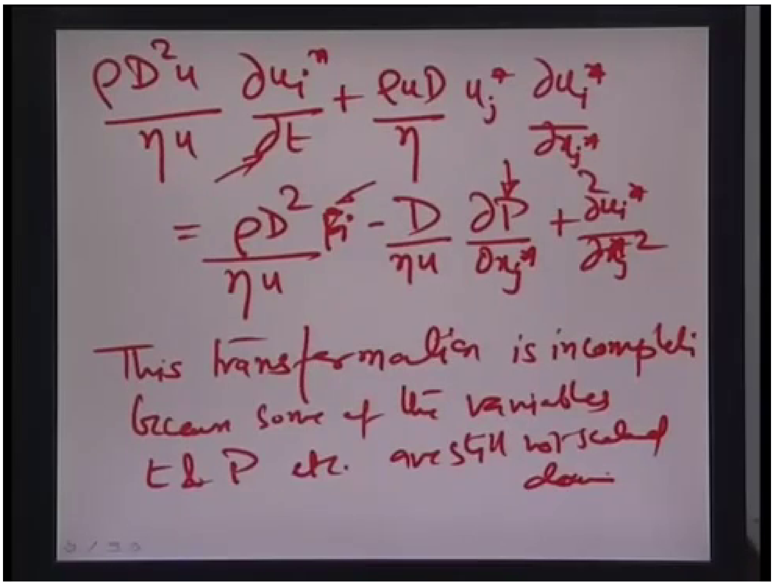

$$\frac{\rho D^2 u}{\eta u}\frac{\partial u_i^{\square}}{\partial t}+\frac{\rho D u}{\eta} u_j^{\square}\frac{\partial u_i^{\square}}{\partial x_j^{\square}}=\frac{\rho D^2}{\eta u}F_i-\frac{D}{\eta u}\frac{\partial P}{\partial x_i^{\square}}+\frac{\partial^2 u_i^{\square}}{\partial x_j^{\square 2}}$$

So, we are left with now, rho D square u by eta u del u i star by d t – del t plus rho u D by eta u j star times of del u i star by u del x j star. And, here we have rho D square by eta u f i minus f i minus del by eta u del p by del x j star plus del 2 u i star by del x j star square. So, that is how u can write the non dimensional form of this equation. Although this is not a complete non-dimensional form again; there are certain quantities here; I will like to illustrate like t here, pressure p here or the force f here, which is still in the old domain. And, you have values here – absolute value of these forces. And, somehow we have developed a mechanism out of whatever parameters we have now to find out if we can really scale down these numbers or scale down these particular parameters by comparing it to a parameter of same type at that particular scale. So, let us actually go ahead and transform. So, we will say that this transformation is incomplete because some of the variables like let us say temperature, pressure etc are still not scaled down. And, so for that, or for doing that, you would go ahead and actually try to see how we can represent these variables.

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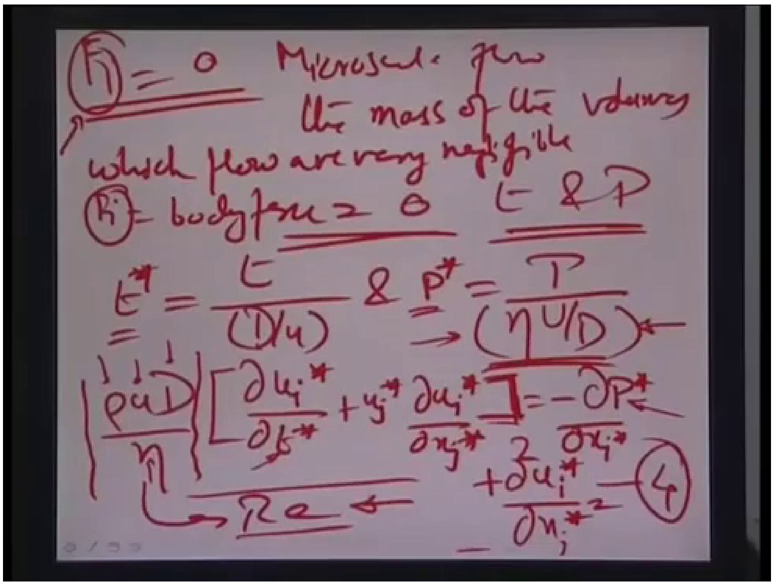

$$t^{\square} = \frac{t}{D/u} \wedge P^{\square} = \frac{P}{\eta \frac{u}{D}}$$

$$\frac{\rho u D}{\eta}\left[\frac{\partial u_i^{\square}}{\partial t} + u_j^{\square}\frac{\partial u_i^{\square}}{\partial x_j^{\square}}\right] = \frac{-\partial P^{\square}}{\partial x_i^{\square}} + \frac{\partial^2 u_i^{\square}}{\partial x_j^{\square 2}}$$

$$\frac{\rho u D}{\eta} = \Re$$

Now, for all practical purposes, the only quantity which may differ a little bit is f i force, which is related to the body force. Now, when we are talking about micro scale flows, the mass of the volumes, which flow really are very very negligible. Therefore, the whole business about f i or the body force is also negligibly small; we can neglect it to 0. So, we need really quantities like t and p to be scaled down in order to ascertain whether we can complete scaling down of the Navier-Stokes equation at the particular scale of reference. So, let us actually figure out what this scale would be.

So, t star would definitely be equal to the time in this particular scale t divided by a time equivalent in the scale that we are considering. And, we already have the corresponding scaled or scaling parameters for the micro scale of D and u for the length and the velocity respectively. So, D by u definitely would give an idea of what kind of time scales would be appropriate for the scaling question or for the microscale at which this equation is being scaled down. And therefore, we can represent t star a quantity, which is equal to t by D by u.

Similarly, for the pressure, p star can be the ratio between p. And, as we know, eta does not vary because eta is actually scale independent property; it is the viscosity of the fluid. And, it is same across all scales whether it is micro, nano or till it goes to a level, where continuum is destabilizes. But, we are talking about the microscale, where we still the continuum holds true. And so, eta essentially, viscosity or mu, whatever you call remains kind of fixed across all these different scales wherever the continuum is still maintained or established. So, n… So, therefore, this pressure unit as you know is same as that of shear stress. And, shear stress is nothing but eta u by D; the rate of change of velocity with respect to the separation distance in the perpendicular direction. So, therefore, when we are talking about the scaled parameters, that is good to assume u by D times of eta to be the corresponding shear stress, which is a required for separating such flows or layers of such flows. And therefore, this can be considered equivalent to the kind of pressure scale. So, p star again becomes p by eta u by D.

Now, if I put all these derivations back into our equation here, which we formulated just about a minute back, we will be left with the something like rho u D by eta times of del u i star by del t star plus u j star times of del u i star by del x j star minus you can call it not minus, you can finish that bracket here – equals minus of del p star by del x i star plus del 2 u i star by del x j star – del x j star square. Let me just write this little more clearer manner. So, this is del p star minus del p star by del x i star plus really del 2 u i star divided by del x j star square. So, that is what essentially the relationship would be in a totally totally scaled down manner.

Now, you have t star here note, which is the kind of non-dimensional analog of time t. And, you have p star here, which is the non-dimensional analog of p star. And, essentially, the f i here – the body forces, which was termed about towards the right-hand side here is neglected, because we consider in microscale flows, the volume are the masses involved to be too low for the gravity effects to be significantly effecting the flow. So, therefore, f i for all practical purposes in micro scale is 0. But, something very interesting is happened in this equation. Let us call it as equation 4. So, what is interesting here is that, this term is nothing but the

Reynold’s number; the rho u D by n. And, this is the kind of characteristic Reynold’s number at this scale that you are considering, because u, D, rho and eta – rho and eta of course do not change across the scales till the continuum is established. And, u and D are the scaled velocities and the length dimensions at the scale that you are questioning or concerned.

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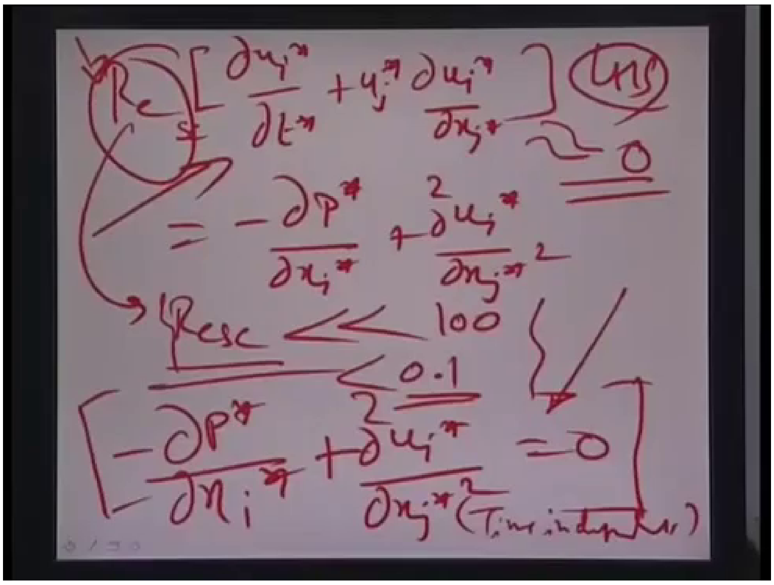

$$\frac{-\partial P^{*}}{\partial x_i^{*}} + \frac{\partial^2 u_i^{*}}{\partial x_j^{*2}} = 0$$

So, therefore, it can be very appropriate to assume that, the Reynold’s number at the particular scale – I call it R e sc times of del u i star by del t star. All these different components of the equation on the left side – del u i star by del x j star is equal to minus del p star by del x i star plus del 2 u i star by del x j star square. Now, as we know that, the Reynold’s number at the scales that we are looking at is really really small; it is very very less than… I mean almost always less than 100 and very often less than 0.1. Reynold’s number is very low. And so, therefore, the contribution coming from the LHS of this equation is kind of overshadowed by the smallness of the Reynold’s number itself. And therefore, the LHS vanishes away. You can say that, this is very very negligibly small and it is 0. And therefore, the Navier-Stokes equations finally turn around into minus del p star by del x i star plus del 2 u i star by del x j star square is equal to 0.

So, this is a very important goal that we have establish here; that if you scale down the conservation of momentum equation at the scale of the Reynold’s number being very small – micro scale; you immediately find out that, the equation becomes time independent – time independent. And therefore, there are certain effects and situations in the microscale, which becomes very very prominent; where, time no longer matters. I mean things like mixing, etcetera – if you just consider mixing by the means of just mass transport; that mixing actually becomes insignificant at the microscale just because if you have two flows, which you are timing in together on to a chip and they go side by side for a little bit; and, if you want to reverse them back in time, it should be able to extract the flows as it is back and mixed. So, therefore, this is the very very important conclusion out of scaling down the Reynold's number. So, we are towards the end of this particular lecture. I would like to kind of take on from here and the next lecture and try to show you some of the observations and conclusions that we can have from this scaling approach, which essentially starts the domain of microfluidics; and then, probably go over some of these fluidic devices like mixers, valves, pumps, etcetera in little more details; and, privacy – how they can be applied to BioMEMS platforms.

Thank you.