Kinematics of Machines

Prof. A. K. Mallik

Department of Civil Engineering

Indian Institute of Technology, Kanpur

Module – 12 Lecture - 3

We finished our last lecture with the contact ratio for a pair of involute gears. What is the expression of the contact ratio? Today, we start our discussion with the expression of the contact ratio for a rack and pinion drive.

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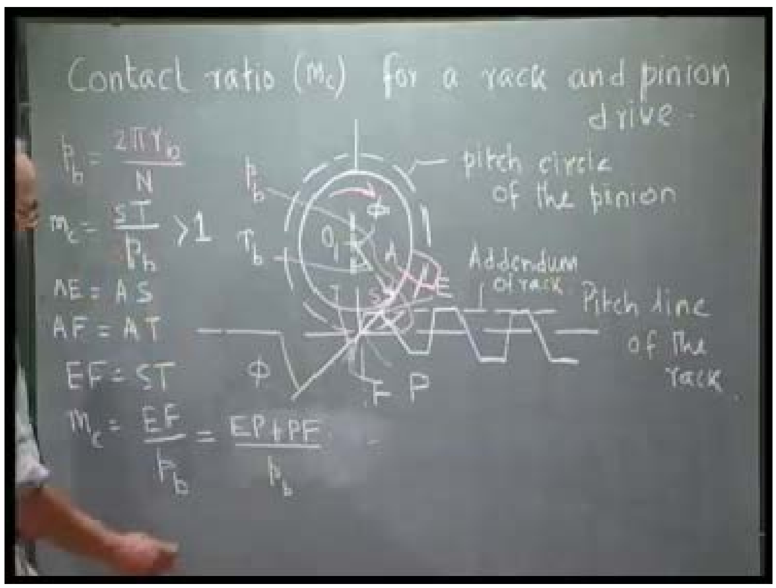

For contact ratio, we use the symbol $m_{c,}$ for a rack and pinion. Let us say, this is the pitch circle of the pinion. For the rack, as we know the pitch circle radius is infinity, that is the pitch circle gets converted into a straight line. As in the case of a pair of gears, two pitch circles are touching each other, hear again the pitch line of the rack is as a tangent to the pitch circle of the pinion. This is what we call pitch line of the rack and this is the pitch circle of the pinion. If we draw the base circle of the pinion, this point of contact between the pitch circle and the pitch line is the pitch point, which we denote it by P. From this pitch point, let me draw a tangent to the base circle. This line represents the line of action. Let this point of tangent be denoted by A, this is the centre of the pinion then this angle is the operating pressure angle phi, which is same as this operating pressure angle phi. The point of contact between the teeth of the pinion and the rack always moves along this line of action.

Let me talk of a particular tooth on this pinion, which is just beginning the engagement that means the contact with that particular tooth is just starting at this configuration. Let this be the tooth, which is coming in contact with the tooth of the pinion at this instant. Where does the contact start? Contact obviously starts at the addendum line of the rack that is, the highest point of the rack tooth. The rack tooth if I draw, this is the addendum line of the rack. The rack tooth is perpendicular to this line of action; let this be the rack tooth. The contact starts at this point between the pinion tooth and the rack tooth on the line of contact; this is the addendum line of the rack. If I complete the rack tooth, it will look something like this. This is the rack. At this beginning of the contact, this point let me call E. This pinion rotates in the clockwise direction, drives the rack to the left.

When the contact between this pair of teeth is lost? It is lost when the contact comes at the addendum circle of this pinion. Let this be the addendum circle of this pinion tooth; if I complete the pinion tooth it looks like this. When this addendum circle intersects this line of action, let us say this point I call F. This tooth has rotated and let us say this is the addendum circle and this point is F. How much is the movement of this tooth measured along the base circle, let me call this point S, this point on the base circle, and the tooth has rotated, and the movement along the base circle is from S to T. This is due to the rotation of the gear, this point S of this tooth has moved up to this point T.

To ensure that the teeth remain in contact the next tooth that is, the adjacent tooth on this gear, this point must get into beyond this point. This point must move beyond S such that this tooth comes in contact with the next tooth and this distance measured from this point to this point, let me mark it, the distance between these two points on adjacent tooth along the base circle, that is what we denoted as base pitch or p_b.

The distance between two adjacent teeth along the base circle that means, from this point to this point, that is what we call p_b. An expression for p_b we have used last time, which is 2 phi r_b divided by N; where r_b is this base circle radius of the pinion, OA this is r_b, N is the number of teeth on the pinion. Contact ratio m_c was defined as: the movement of one tooth along the base circle, which is ST divided by the base pitch. As we said, this must be more than one so that continuous transmission of motion between these two tracks and opinion is ensured. To get the expression of this m_c, let us see the contact started at E and finished at F and these are involute profiles. So, AF is nothing but AE, AE is the string length which was wound on to this base circle and by unwinding I have generated this involute profile. Because it is an involute profile, the distance AE is same as the distance measured along the arc of the base circle AS. So, AE is same as AS.

Similarly, this is also an involute from the same base circle, this is the same tooth, AT measured along the base circle will be AF, that is, the length of the string which has been unwound from this base circle. AT is AF, length of the string AF is the length of the circular arc from A to T. If I subtract AF minus AE, I get EF and if I subtract AS from AT, I get ST, so ST is same as EF. That I can write again contact ratio is nothing but EF divided by p_b. (Not clearly Audible)I write this as, EP plus PF. EP is this distance, PF is this distance divided by p_b. I add PF plus AE plus EP.

EP I have written, EP I add AE to PF, and I again add EP, and subtract AE plus EP that is, AP. EP I have written here, PF I have written here, AE plus EP I have subtracted, AE plus EP which is nothing but AP divided by p_b.

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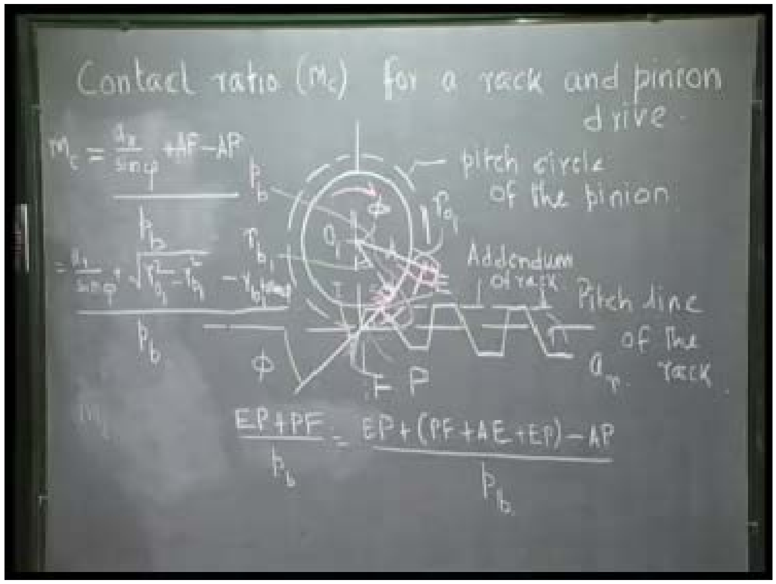

Finally, I get the expression of m_c contact ratio is EP. What is EP? EP is this height, which I can see, if this angle is phi and this vertical height is addendum of the rack, which I denote by a_r, a_r is the addendum of the rack. This EP is nothing but a_r by sine phi, where phi is the operating pressure angle, this is EP and PF plus EP plus AE is nothing but AF, whole thing is divided by p_b, this is AF and minus AP. A is this contact point between the tangent from the pitch point to the base circle, this angle is 90 degree. This is radius, this is tangent, so this angle is 90 degree and O_1P is nothing but the outer radius of the pinion. This is outer circle, this addendum circle, so O_1 to F is nothing but the outer radius of the pinion. AF is square root of outer radius square minus r_{b1} square.

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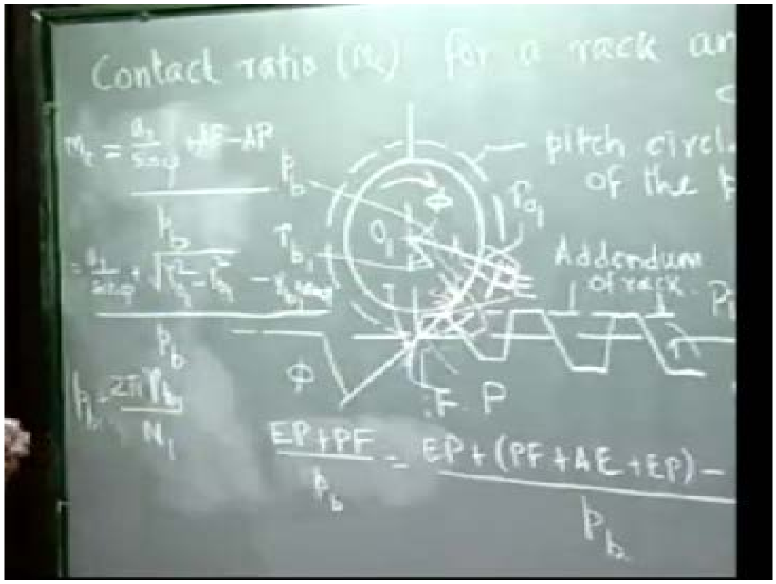

This I get, a_r by sine phi plus, this is r_{o1} square, that is the outer radius or the addendum circle radius, this is what I call r_{o1} minus r_{b1} square minus AP. This is phi, and this is r_{b1}, AP is r_{b1} tan phi, tan phi is AP by $r_{b1,}$ so AP is r_b tan phi minus r_{b1} tan phi divided by $p_{b,}$ for which I already wrote the expression. Finally, we have got the expression for the contact ratio for a rack and pinion drive involving the addendum of the rack, operating pressure angle, outer radius of the pinion, base circle radius of the pinion and the base pitch, which is $p_{b,}$ is equal to two pi r_{b1} divided by N_1, where r_{b1} is the base circle radius, N_1 is the number of teeth of the pinion. This completes our discussion on the contact ratio for a rack and pinion drive. The methodology is just similar to what we used for a pair of involute gear teeth profile.

We have finished discussion on contact ratio, let us start with a very important parameter for gear tooth geometry, that is known as interference and undercutting.

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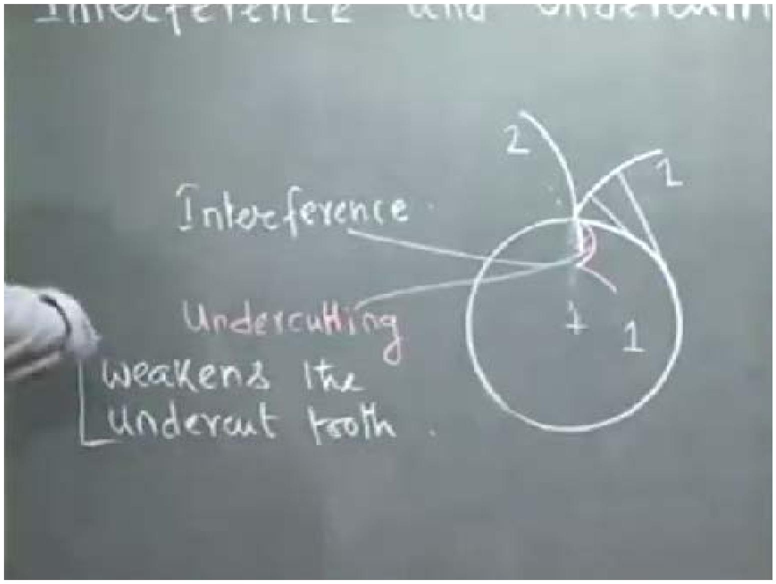

First, let me explain what do we mean by interference and what is undercutting. If we remember from the definition of involute profile, it is obvious, that the involute does not exist inside the base circle, it is from unwinding from this base circle outward, we created the involute profile. Let us say, it starts from this point A it starts radially, and as the string unwinds, we generate this involute profile. The involute profile exists only outside the base circle, does not exist inside the base circle. Suppose, we use very few numbers of teeth on a gear and the teeth size is very big, then it may so happen that the addendum circle of the meeting gear gets into the base circle of the other gear. The meeting profile, which is another involute for another base circle gets inside the base circle.

As we see, there is no involute of this gear say, this is gear number 1 and this is 2, the gear number 1 does not have any involute inside this base circle. Whereas, the involute of the other meeting gear is getting into the base circle so, there is no conjugate profile. One can extend this involute of gear 1 may be radially, even then there is interference. As we see, this tooth will interfere with this tooth because, this tooth at once to occupy this position, whereas this tooth is lying there, this is called interference. One solution to this interference is to remove some portion of gear 1 - we do not have the involute profile of gear 1 anyway - so let me cut this portion from gear 1. Then as we see, this gear tooth is not interfering with gear 1, the tooth of gear 2 is having a space inside gear 1, though there is no conjugate profile to maintain contact. However, the conjugate action maybe taken care of by the other pair of teeth in engagement, the continuous motion is transmitted maintaining the constant angular velocity ratio, but this tooth is not transmitting any power because this is getting into this portion and this is what is called undercutting. We have avoided the interference by undercutting the tooth of gear 1, inside the base circle. But this is not a very satisfactory solution, because this undercutting weakens the tooth of gear 1, undercutting which weakens the undercut tooth and it is here at the root of the tooth where failure takes place if we have weakened the tooth near its root, that is not a very desirable solution, only if it is a must, then only we should undercut the tooth.

Next, we shall discuss how to ensure that that there is no interference and how to avoid interference by ensuring some minimum number of teeth on a gear. We cannot have the number of teeth on a gear less than a minimum number to avoid interference -that will be our next topic of discussion. As we have just now shown the possibility of interference between a pair of gear teeth, because involute does not exist inside the base circle, not so satisfactory solution was to undercut the tooth.

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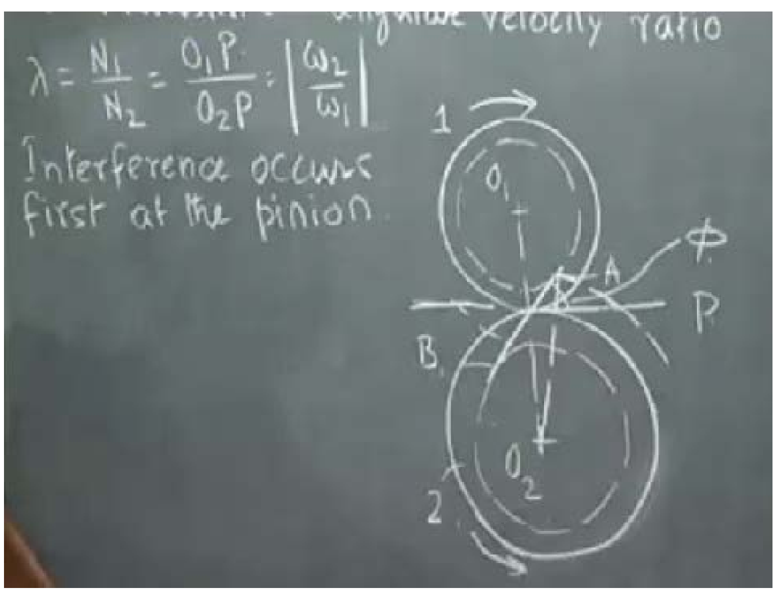

But a better solution is to have a minimum number of teeth on a gear to avoid interference. We have already seen how the interference took place first of all, because the addendum circle gets into the base circle. We talk of a minimum number of teeth to avoid interference to transmit a particular angular velocity ratio - we call it lambda - which is defined as $omega_2$ by $omega_1$ which is less than equal to 1, that means $omega_1$ is more than $omega_2$. When we transmit continuous rotation from one shaft to another at the constant angular velocity ratio, let us forget about the sign for the time being, I defined the angular velocity ratio only by the positive number and I define it in such a way, that it is less than equal to one, that means the higher speed is at the denominator. For example, lambda is 8 by 9. To transmit a constant angular velocity 8 by 9, I can have 16 and 18 teeth or 24 and 27 teeth or 32 and 36 teeth. The question is, what must be the minimum number of teeth to transmit angular velocity is 8 by 9, can I use 8 and 9 teeth to transmit constant angular velocity 8 by 9. The question we are trying to answer is -what is the minimum number of teeth such that interference is avoided while transmitting a particular angular velocity ratio which is given to us, the value of lambda? Towards this end, let me first draw the two pitch circles. These are the two pitch circles $omega_1$ is the higher speed, so this is gear1 which rotates at a faster speed and this is gear2 which is rotating at a slower speed. Suppose, this gear is rotating this way, and this gear is rotating this way, these are the pitch circles.

Let me draw the two base circles, this point is O_1, this point is O_2, this is the pitch point which is called P and if I draw the common tangent to these base circles, that is the line of action AB. This is very clear, because this is a smaller gear AP is smaller than BP this is a larger gear, which is rotating at a slower speed. Lambda is N_1 by N_2, where N_1 is the number of teeth on this gear and N_2 is the number of teeth on this gear, which is also same as O_1P divided by O_2P.

The question is, that addendum circle of either gear should not get into the base circle of the other gear and as we know, the contact point is restricted to move on this line of action that is line AB, so which addendum circle will first get into the base circle. Addendum circles are of same size -both the gears teeth are of same size, they have the same addendum. It is obvious, that the bigger gear addendum will first get into the base circle. If this is the addendum circle, if this touches A the addendum circle of this gear, which has exactly same addendum will still pass through below point B, it will not get into the base circle. The point of contact always lies on this line, so I am only interested in seeing whether the addendum circle of this line intersecting AB beyond A or beyond B. Interference obviously occurs first with the pinion when the addendum circle of the gear just passes through the point A. Because, even at that time the addendum circle of the pinion does not pass through B, it is still to interfere with this gear.

First point to note interference occurs, first at the pinion, that is the smaller gear, and we are considering as if the interferences has just started. From this diagram, we should be able to find out, what is the minimum number of teeth that is N_1 to avoid interference that is the task. This common tangent to the pitch circles, this is the point P, and this angle is the operating pressure angle phi, this distance O_2A which is nothing but the radius of the addendum circle of gear2.

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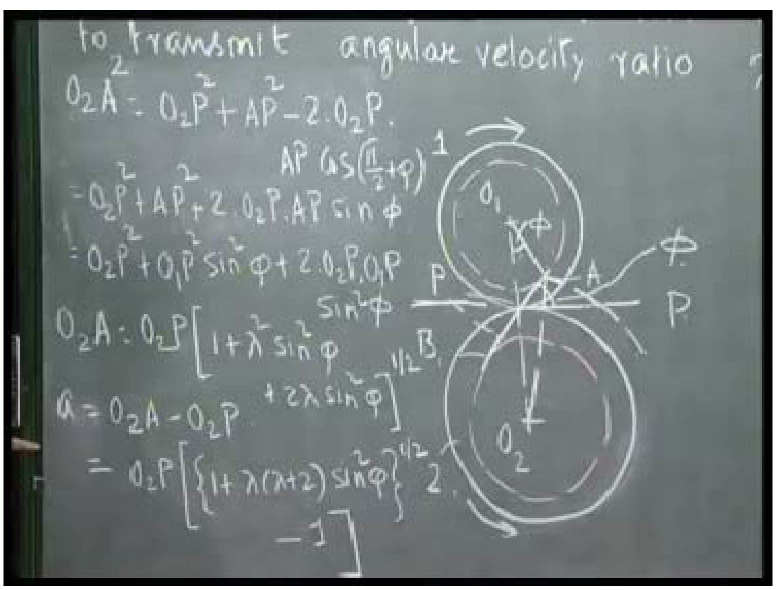

Let me first write an expression for this O_2A. We consider the triangle O_2PA, this point is P. O_2A square is O_2P square plus AP square minus twice O_2P into AP into cosine of the angle between O_2P and AP, which is pi by two plus phi –using the triangle law. This, I write O_2P square plus AP square plus twice O_2P into AP sine phi and this angle if we remember, we have used it number of times this angle is also phi. What is AP? It is O_1P sine phi. This, I can write O_2P square plus O_1P square sine square phi plus twice O_2P and for AP I write O_1P into sine phi, so another sine phi, so I get sine square phi.

If I take O_2P common from this expression and take the square root, I get O_2A is O_2P into, from here I get 1, here I get O_1P square divided by O_2P square and if we remember lambda, this was equal to O_1P by O_2P. The speed ratio is nothing but the pitch circle radius ratio. If I take O_2P square, I get lambda square sine square phi and from here I get one lambda, so two lambda sine square phi to the power half. I have taken O_2P common from these expressions, I get O_1P by O_2P, which gives me this lambda, here I get O_1P square by O_2P square, which I get lambda square and then I have taken the square root. O_2A minus O_2P, O_2A is the addendum radius O_2P is the pitch circle radius. The difference of this is nothing but O_2A minus O_2P, which is the addendum. If I use this expression of O_2A, I get this as, O_2P one plus lambda into lambda plus two sine square phi to the power half minus one. This is the expression for the addendum.

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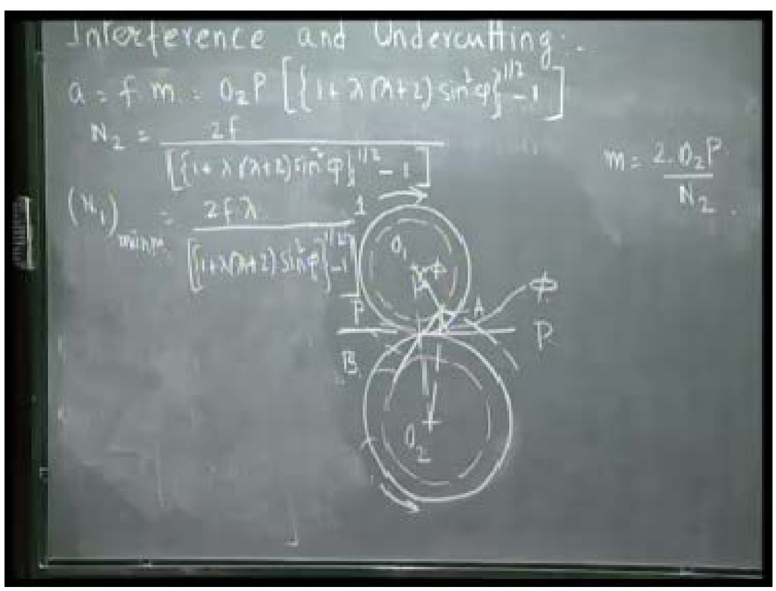

Addendum is always expressed as some fraction f of the module. If we remember, for the British standard for 20 degree pressure angle, we said f is normally one, but I keep it as f, which is of the order of one, A which is f into m is O_2P into this expression, which is one plus lambda into lambda plus two sine square phi to the power half minus one and what was the module? Module was 2 into r_P divided by number of tooth. Module by definition is pitch circle diameter by the number of teeth, number of teeth of the second gear, pitch circle diameter of the second gear. This O_2P, I can write as mN_2 by 2, if I write mN_2 by 2, mm cancels, I get N_2 is O_2P, I am writing mN_2 by 2, so 2f divided by this expression.

We have considered the situation where interference just started with the pinion and it will get this expression for N_2 which means if I multiply by N_1 by N_2, which is nothing but lambda. N_1 minimum tooth on the pinion, we get 2f lambda divided by this expression, the denominator remains the same. 2f lambda divided by 1 plus lambda into lambda plus 2 sine square phi, this whole thing to the power half minus 1. This is the minimum number of teeth required to avoid interference, while transmitting an angular velocity ratio lambda and lambda as we have defined, is always less than 1. This is how we get the minimum number of teeth to just avoid interference. N_1 at least should be this much, if N_1 is more, then teeth will be smaller, so there will be no problem. This is just, when the interference starts so N_1 should be more than this N_1 minimum.

Just now, we derived the expression for the minimum number of teeth that must be on a pinion so that interference is avoided while transmitting a constant angular velocity ratio lambda.

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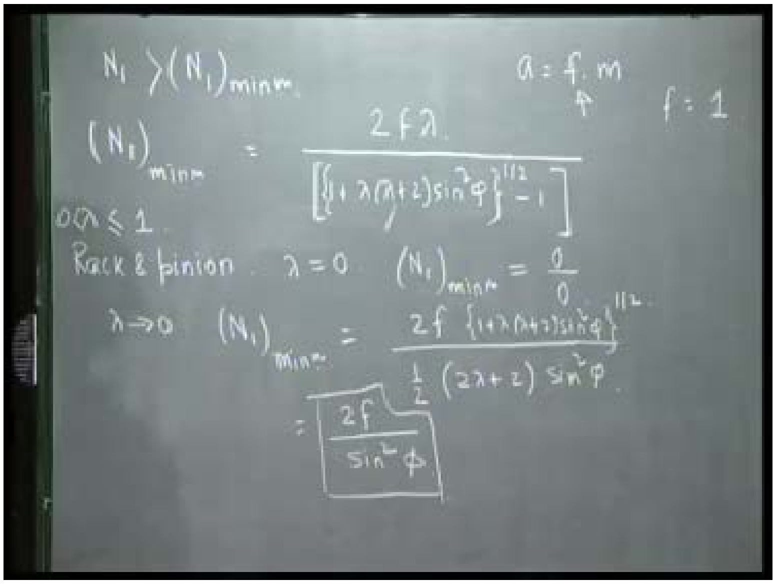

Number of teeth on a pinion must be greater than $(N_1)_{minimum}$ square, $(N_1)_{minimum}$ was given by 2f lambda divided by 1 plus lambda into lambda plus 2 sine square phi to the power half minus 1; where lambda by definition is between 0 and 1. We always define lambda as less than 1, that is the slower angular velocity divided by the faster angular velocity and phi is the operating pressure angle. What is f? f is addendum is expressed away constant fraction of the module, it is this constant fraction is f; f is normally 1 unless otherwise stated. From here let me try to drive the expression for the minimum number of teeth that must be there on the pinion while it is in contact with a rack and pinion. If it is a rack and pinion, then what must be the minimum number of teeth on the pinion to avoid interference, for rack and pinion, as we see lambda is 0. Because the angular velocity of the rack is 0, rack has only translational velocity. Lambda is 0, $omega_2$ by $omega_1$ is 0.

If I put lambda equal to 0 here, then I get 0 and this is 0, so 1 this is 0 by 0. For this $(N_1)_{minimum}$ from this expression is standing out to be of the form 0 by 0. That is, now nothing problematic as we know, we can always take the limit, let us take the limit of this as lambda tends to 0. For rack and pinion, I can get $(N_1)_{minimum,}$ by taking the limit of this expression as lambda tends to 0. Because it is of the 0 by 0 form, I can use L'Hopital's rule, that means differentiate the numerator with respect to lambda, denominator with respect to lambda, then put lambda equal to 0 and see the limiting value. That way, if I take derivative of this with respect to a lambda and the numerator I am getting 2 f and the denominator if I difference with respect to lambda, I get half then this goes at the top 1 plus lambda into lambda plus 2 sine square phi to the power half and here I am getting lambda square is 2 lambda plus 2.

If I differentiate the derivative of the numerator and derivative of the denominator, this is what I get: Half into this to the power minus half, then while differentiating this, I am getting sine square phi. Sine square phi lambda square gives me 2 lambda and 2 lambda gives me 2 and now I put lambda equal to 0. If I put lambda equal to 0, this goes to 0, I get 2f and here if I put lambda equal to 0, then 2 cancels, we get sine square phi. This is the minimum number of teeth on the pinion which engages with a rack and without interference. This expression of course we could have got geometrically very simply by drawing the rack, just undergoing interference with the pinion that is what we are going to do next.

We derived the minimum number of teeth on a pinion that is required to avoid interference with a rack, which we did analytically from the expression that we derived earlier for a pair of involute gear and a pinion.

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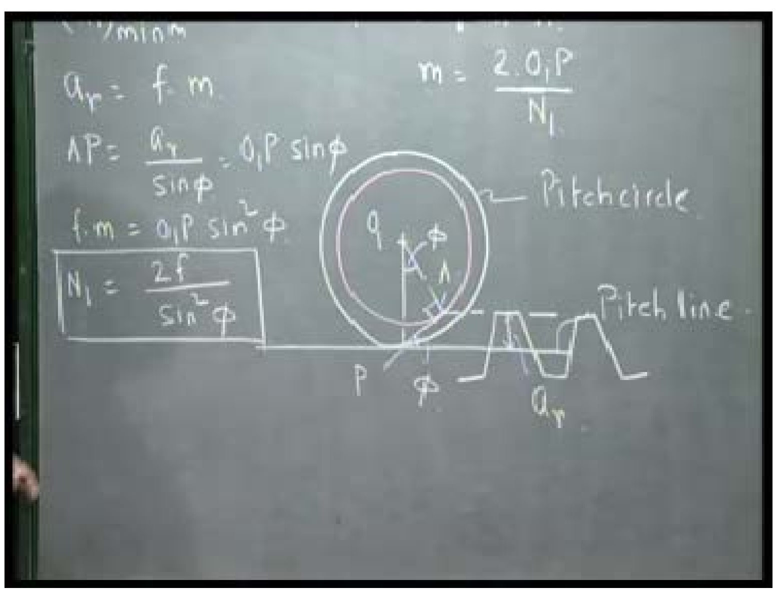

Let me now get the same expression of the minimum number of teeth geometrically. $(N_1)_{minimum}$ on the pinion for a rack and pinion to avoid interference. Towards this end, let me say this is the pitch circle of the pinion; this is the pitch line of the rack and this is the base circle of the pinion. From this pitch point P, if I draw a tangent to the base circle, this angle is phi, this is $O_{1,}$ this is A. If the addendum line passes through the point A, if this is the addendum line of the rack that is the rack tooth look something like this. If the addendum line intersects this base circle just at A, that is when the interference starts and we can use this diagram for getting the minimum number of teeth on the pinion. This is the rack of the addendum a_r which we write as f into m and this angle is the operating pressure angle phi. AP is a_r sine phi just when the interference starts, AP is a_r divided by sine phi. AP is also O_1P sine phi, this angle is 90 degree this is the base circle radius and this is the tangent, this angle is phi, pitch circle radius is O_1P, and AP is O_1P sine phi, which tells me a_r which is f into m is O_1P sine square phi, m is nothing but 2 into O_1P by N_1, pitch circle diameter by the number of teeth, that is the definition of the module. m I am writing twice O_1P by $N_{1,}$ O_1P cancels, N_1 is 2f by sine square phi. This is the same expression, that we got earlier by taking the limiting value from the expression of $(N_1)_{minimum}$ for a pair of involute gears.

Next we shall solve an example to show the application of this expression that we got for $(N_1)_{minimum}$ how do we determine the minimum number of teeth or the number of teeth that must be used to transmit a particular given angular velocity ratio lambda.

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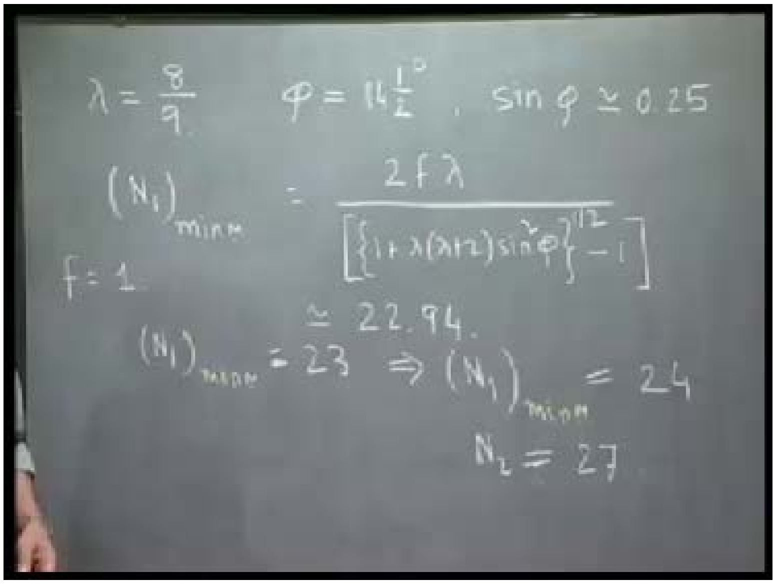

Let me now solve an example, for a pair of involute gears to transmit an angular velocity ratio lambda as 8 by 9. Lambda is given to be 8 by 9 and the operating pressure angle phi just as an example, let me take 14 and 1/2 degree which means sine phi will be 0.25. If we remember the expression for $(N_1)_{minimum}$ was given as 2f lambda divided by 1 plus lambda into lambda plus 2 sine square phi square root minus 1. The most usual value is 1, so let me say f is taken to be 1. If I put f equal to 1, lambda equal to 8 by 9 and sine phi equal to 0.25, $(N_1)_{minimum}$ from this expression transfer to be 22.94. But, number of teeth cannot be 22.94, it has to be an integer, so $(N_1)_{minimum}$ looks like it should be 23. If N_1 is 23 and to transmit 8 by 9 angular velocity ratio, N_2 will turn out to be non-integer which means, this will imply N_1 has to be a multiple of 8, such that N_2 can be a multiple of 9 and they will be both integers, $(N_1)_{minimum}$ is actually 24 and corresponding N_2 is 27. If I use 24 and 27 teeth, I can maintain a constant angular velocity 8 by 9 and without interference. This is what we mean by minimum number of teeth to avoid interference.