Kinematics of Machines

Prof. A. K. Mallik

Department of Civil Engineering

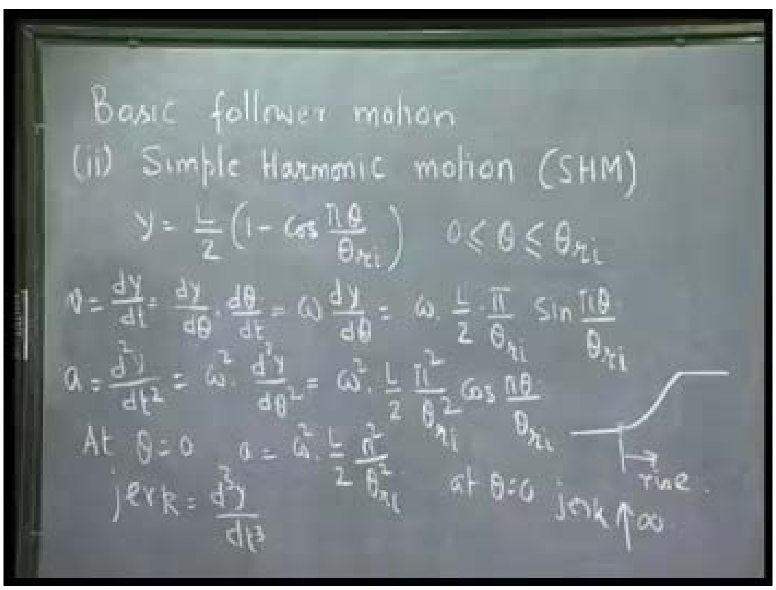
Indian Institute of Technology, Kanpur

Module - 10 Lecture - 3

Today, we will continue our discussion on basic follower motions. In the last lecture, we have already discussed the simplest type of follower motion which we called uniform motion. We also explained the difficulty associated with such simple motion that is of infinite acceleration and deceleration of the follower at the beginning and end of the rise.

Today, first we take up a second type of motion, which is known as simple harmonic motion.

(Refer Slide Time: 00:52)



In short, we call it SHM. During this type of motion, the displacement of the follower y can be expressed as L by 2 into 1 minus cosine pi theta divided by theta$_{ri}$, where L is the lift of the cam; y is the displacement of the follower, for cam rotation angle theta y is given by this, where theta$_{ri}$ is the total cam rotation during the rise period. This expression is valid so long theta lies between 0 and theta$_{ri}$. In this discussion and all subsequent discussions, we will only discuss the rise phase of the follower. Let us see for this type of simple harmonic motion, what are the velocities and acceleration of the follower.

As we know, velocity which is given by dy by dt can be written as dy by d theta into d theta by dt and d theta by dt is nothing but the angular velocity of the cam which is constant we assume, so this is the constant angular velocity of the cam so, dy by dt that is the velocity of the follower, can be written as omega, the constant angular velocity of the cam into dy by d theta. For this expression of y, if we take dy by d theta we can easily see this transfer to be omega into L by 2 into pi by theta_{ri} into sine pi theta by theta_{ri}. This is the expression for the velocity of the follower and as we see, because of the sine term it can never be beyond the range minus 1 to plus 1, so velocity is never going to be infinity.

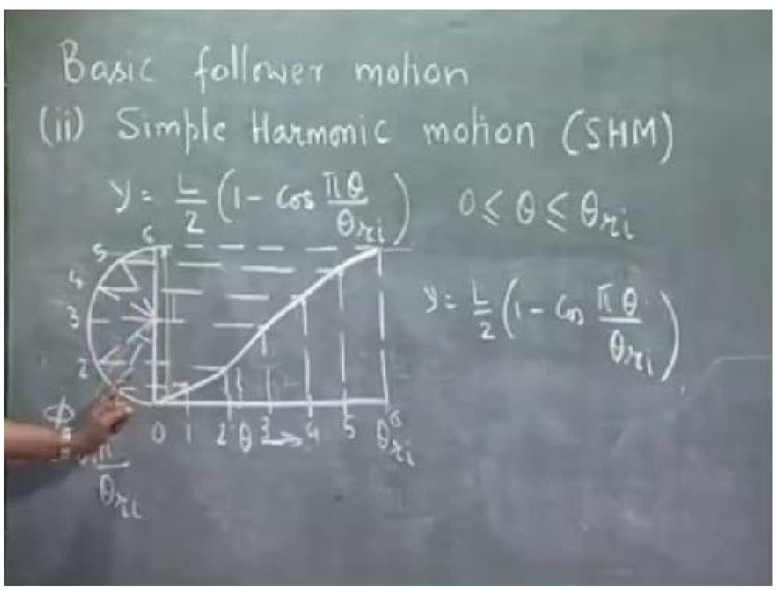
Similarly, if we look for the acceleration of the follower, that is, the second derivative of y, d two y by d t two, that we can get as omega square into d two y by d theta square. If this is the expression for y, we differentiate it twice what we get is, L by 2 into pi square theta_{ri} square into cosine pi theta by theta_{ri}. Again as we see, that the acceleration because of this cosine term can vary only between minus 1 to plus 1, the acceleration always remains finite. This avoids the difficulty of the uniform motion, where we just found that the acceleration goes to infinity at the beginning of the rise and again goes to minus infinity at the end of the rise here, it always remains finite.

If we look for the third derivative that is the jerk, let me talk about the acceleration a little bit more we can see, at theta equal to 0 that is at the beginning of the rise, acceleration let me call it ‘a’ and this is the velocity of the follower, this is the acceleration of the follower and at theta equal to 0 we are getting acceleration is omega square L by 2 into pi square by theta_{ri} square. It starts with acceleration at theta equal to 0. That means, during the end of the cycle this is the starting point and here, every derivative is 0, acceleration is 0 during this dwell period but as the rise starts with a value of the acceleration. So the third derivative goes to infinity. The jerk which is called the third derivative, jerk of the follower, that is the rate of change of acceleration which is d three y by dt three.

Here d two y by dt two is 0 and it starts with a finite value of d two y by dt two at theta equal to 0 that the beginning of the rise jerk tends to infinity. At theta equal to 0, rate of change of acceleration is very high and at the same problem will happen when theta is theta$_{ri}$. When theta is theta$_{ri}$ this becomes cos pi which is minus 1 so again, it has some negative acceleration at theta equal to theta$_{ri}$ at the end of the rise but at the end of the rise, again all derivative goes to 0, so there is a huge amount of rate of change of acceleration and again the jerk goes to a very high value.

Now let me see, how we can construct the displacement diagram for this particular type of motion which we call simple harmonic motion. Because this displacement diagram drawing will be necessary if we follow a graphical method of cam profile synthesis, whereas this analytical expression is good enough for analytical method of cam synthesis.

(Refer Slide Time: 07:36)



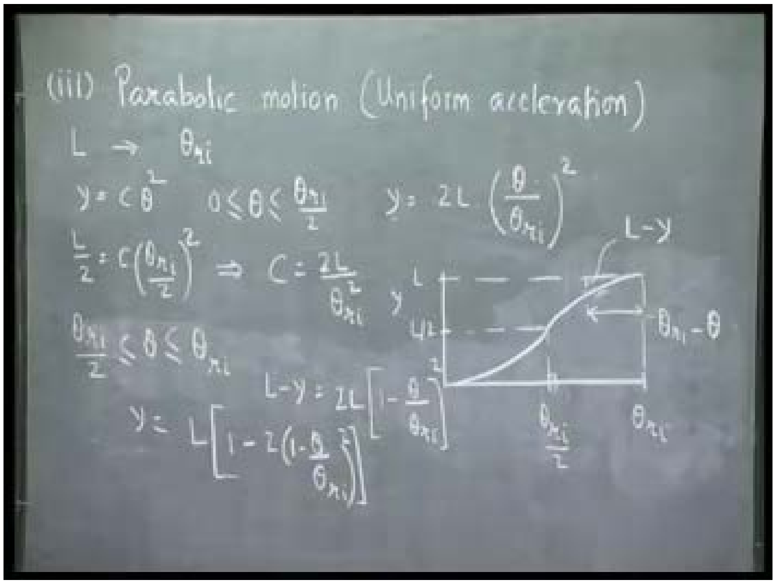
As before we draw the displacement diagram, this is a theta equal to 0 and this is theta equal to theta$_{ri}$, this is the lift of the cam L. To express this equation geometrically what we do, we draw a semi-circle with L as diameter. We draw a semi-circle with this lift of the cam as diameter. Our objective is, to draw the displacement diagram for this value of theta between 0 to theta$_{ri}$. To do that, we divide this theta$_{ri}$ into some equal number of divisions. I will explain it with six number of equal divisions, for more accuracy we can take higher number of divisions, but the procedure remains the same. Let me divide it into six equal parts, I have divided this interval 0 to $theta_{ri}$ in six equal parts. Then I also divide this semi-circle into six equal parts that means this is 30 degree, 60 degree, 90 degree, 120 degree, 150 degree and 180 degree. If I call these as station points, I name them 1, 2, 3, 4, 5 and 6. Similarly, here also I name them 1, 2, 3, 4, 5 and 6. Then I project these points horizontally and these points vertically, the corresponding points on the semi-circle and on the theta axis, these points of intersection if I join with a smooth curve that gives me the displacement diagram corresponding to this equation.

This can be very easily verified as we see, this distance is L by 2 and if this is the value of theta say, corresponding to the second station point, this is the value of the $theta_2$ then this distance is L by two cosine theta. Theta has been converted to pi, this angle is theta into pi by $theta_{ri}$. As theta goes to $theta_{ri,}$ this angle let me call it phi goes to theta into pi by $theta_{ri}$. This distance is L by two minus this distance which is L by two into cosine phi, so y is becoming L by two into (1 minus cosine pi theta by $theta_{ri}$) as theta goes to $theta_{ri,}$ phi goes to pi and both starts at 0 as theta goes from 0 to $theta_{ri}$, phi goes from 0 to pi. When theta takes of the value theta, phi takes of the value theta into pi by $theta_{ri}$

This geometrical construction reproduces this equation. This is the displacement diagram and this is the displacement equation for simple harmonic motion, which has one difficulty as I showed that, the acceleration is finite here at the beginning of the rise and at the end of the rise also but after that the acceleration is 0, before that this acceleration is 0, there is a sudden change in the acceleration.

The third type of basic follower motion that we will discuss is known as uniform accelerated motion or parabolic motion.

(Refer Slide Time: 12:37)



In this parabolic motion, from the beginning of the rise up to the half of the lift, the follower moves with constant acceleration and then for the rest half to reach the end of the rise it moves with constant deceleration of the same magnitude, this is also known as uniform acceleration and deceleration. Uniform acceleration is the constant acceleration for the first half of the rise and the second half of the rise with the same value of deceleration. If we remember the school mechanics, because the follower is starting with 0 velocity and also finishing the end of the rise with 0 velocity, then if the lift has to be covered in a given time that is, for a given value of theta_{ri}. If the given lift L has to be covered for a given value of theta_{ri} which means for a given amount of time because theta_{ri} is nothing but omega into t and omega is constant then, the maximum acceleration is minimum if it moves with constant acceleration. Any other variation of acceleration will make the maximum value of the acceleration during the cycle more. The maximum value of the acceleration will be minimum, if it moves first half with constant acceleration and the second half with constant deceleration.

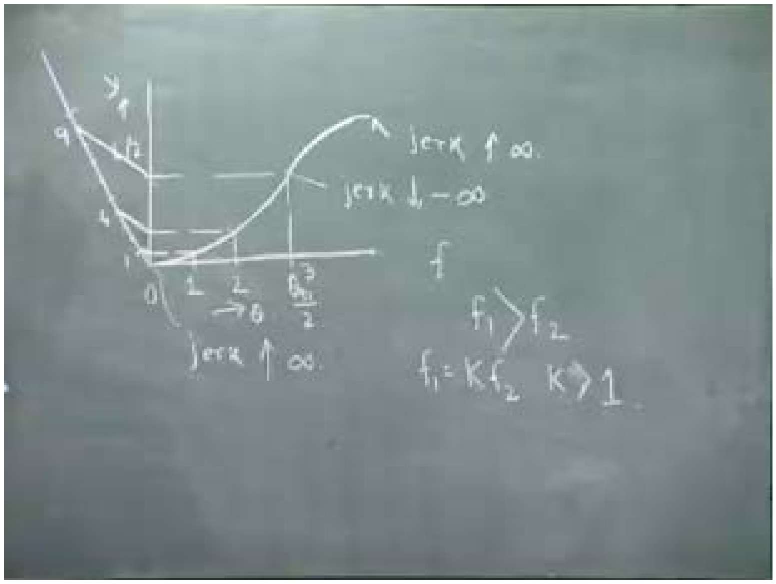
It is called parabolic motion because the displacement diagram is parabolic, actually there are two parabolas let us see how. As we remember, with constant acceleration the motion is starting with 0 velocity will be proportional to square of the time. S equal to ut plus half a t square, if ‘a’ is the acceleration, ‘u’ is the initial velocity is 0 at beginning of the rise, so the follower movement will be half a t^2 and t is nothing but theta, so I can write, y as c theta square and this takes place so long theta is $theta_{ri}$ divided by 2.

The value of the constant C can be easily found out if we remember that at the half of the value of $theta_{ri}$ that is at the half of the rise y is L by 2, it attains half of the lift when theta is $theta_{ri}$ by 2, that gives me the value of C as 2L divided by $theta_{ri}$ square, this is 4 so 2 cancels this 2 goes there 2L by $theta_{ri}$ square. The expression for this uniform parabolic motion during this range of theta from 0 to $theta_{ri}$ by 2, y is given by 2L into theta by $theta_{ri}$ by 2 whole squared. This expression is up to this.

For the second half that is, when theta lies between $theta_{ri}$ by 2 to $theta_{ri}$. For the second half when it goes with constant deceleration we can easily write the expression of y as follows. If we draw the displacement diagram y we have seen, $theta_{ri}$ by 2 this is $theta_{ri}$, this is L by 2 and this is L. This first half of the rise, we have got this parabola, for the second half when it goes with constant deceleration and again gives the velocity 0 here. We draw the parabola from here at the other end exactly the same parabola up to this point. We got the equation for this and this, we have to write the equation for this only and this parabola is same as this parabola. This distance is nothing but L minus y and what is this, which is equivalent to theta here, but here, as I see this is nothing but $theta_{ri}$ minus theta. We can write L minus y instead of y then, the constant remains the same and for theta I write, $theta_{ri}$ minus theta so that gives me one minus theta by $theta_{ri}$ whole squared. That gives me y equal to L into one minus two into (one minus theta by $theta_{ri}$) whole squared. This is the expression for the second half of the displacement diagram and this is the expression for the first half of the displacement diagram.

Next we shall discuss, how I draw this parabola graphically because we may need to draw the displacement diagram graphically in case of graphical synthesis of the cam profile, these are the analytical expressions. Of course, we can take the values of theta and y and plot this graph but there is a simpler way of drawing a parabola, which we must have learnt in our drawing course but let me just recapitulate it.

(Refer Slide Time: 19:38)



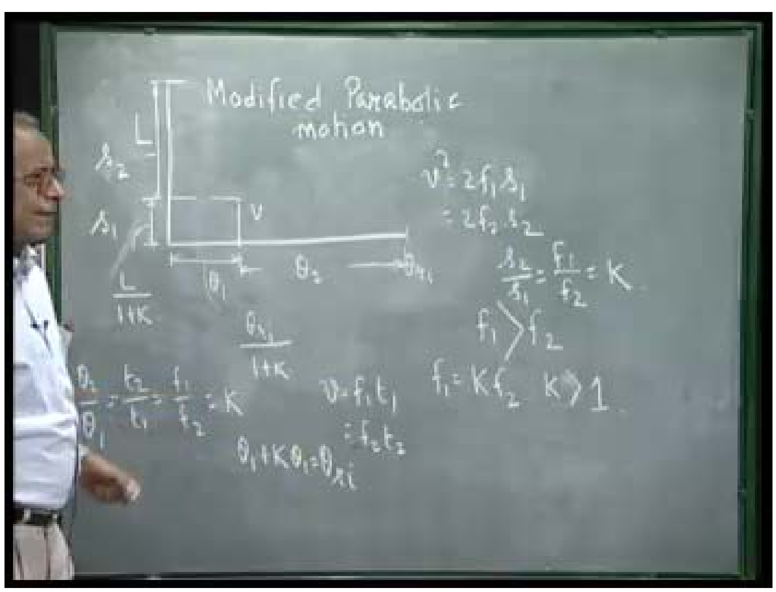
Let me now explain drawing of this parabola graphically for the parabolic motion. I shall explain it only for one of the parabola because the other parabola can be drawn from the other end as I explained for the analytical expression, this is 0, and this is theta$_{ri}$ by 2. I have to draw a curve that the theta$_{ri}$ by 2 it goes up to L by 2 and I have to draw this intervening parabola. To do this, we just divide this 0 to theta$_{ri}$ by 2 in some number of equal parts, lets say three parts, I am explaining with the help of three parts- one, two and three, and as we said y varies as square of the angle theta. We have to divide this interval as the square of one, two and three that is one, four and nine, so I draw a line through this point and make equal divisions one, two, three, four, five, six, seven, eight, nine. These are all equal divisions so I have to divide this in the ratio of 1:4:9. I divide this (Refer Slide Time: 21:45); I draw a line parallel to this line from here and this line from here. From here I project, from here I project and from here I project and I get the required parabola and exactly the same way I can draw the other parabola. But the thing to notice is that, here it is going with positive constant acceleration and now it goes with same value of negative constant acceleration. Here, the acceleration changes from plus f to minus f. There again the jerk goes to very high value from plus f to minus f and here from minus f to 0 at the end of the rise here again the jerk acceleration goes from minus f to 0 suddenly so the jerk goes to plus infinity and the same problem is here, the acceleration before the rise was 0 and suddenly it starts with constant acceleration so the jerk goes to infinity.

These are the problems with the parabolic motion but it has the advantage that to cover the same lift in a given time interval, that is given value of $theta_{ri}$ the maximum value of the acceleration is minimum for this. Here of course the maximum is constant, that is whatever f we have got, but for other motions the maximum value of the acceleration will be more. Some times we also use modified parabolic motion, for example in the case of automobile valve, the valve has to open very fast. Similarly, the valve has to close very fast at the end of the return. During these two portions, I will use the larger value of acceleration and then follow with a constant acceleration the rest of the path.

That means, the acceleration initially is f_1 and for the rest of the period the deceleration that is the magnitude of the acceleration is f_2 then f_1 is larger than f_2. That way the valve will open very fast, then rest of the opening will be gradual with a lesser value of acceleration. And we can say f_1 equal to some constant k and f_2 where k is more than 1. This is the starting acceleration here and the rest of the deceleration there.

In that case, it is very easy to show, if we are given the value of k then how to draw the displacement diagram or write the expressions that is very simple.

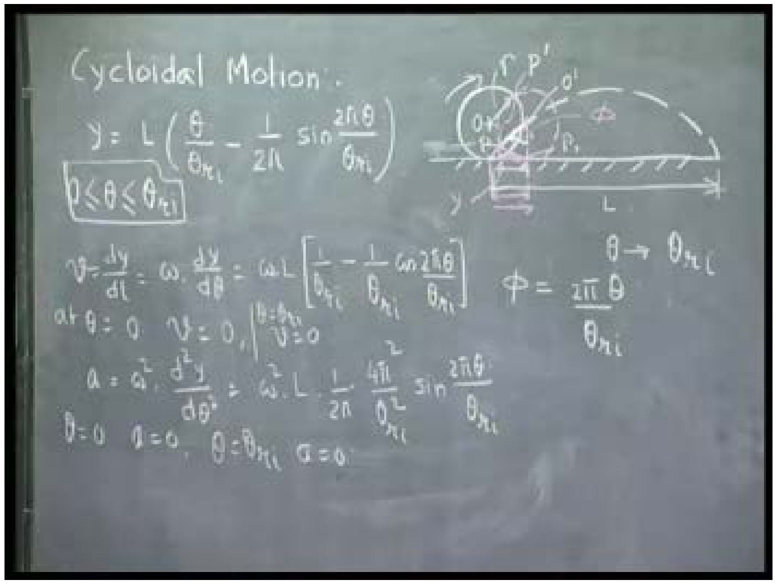
(Refer Slide Time: 24:51)



This is $theta_{ri}$, then up to what angle of theta it will move with constant acceleration f_1 that you can easily show that, this should be $theta_{ri}$ divided by one plus k and with this, how much of the lift will be covered? If this is the lift, then the rise with constant acceleration $f_{1,}$ this will be L divided by 1 plus k. This is very easy to see, it starts with 0 velocity, attain some velocity v here and then it decreases and again becomes 0 velocity here. We can easily show that as we know v equal to $f_1t_{1,}$ if this time interval is t_1 and also v equal to f_2t_2 but t_2 is this time interval if this is t_2; when we talk in terms of time but theta and time are proportional. We can see t_2 by t_1 is f_1 by f_2 is equal to k and t_1 plus t_2 is the total time. We can write if this I call, $theta_2$ and this I call $theta_1$ then $theta_2$ by $theta_1$ which is same as t_2 by t_1 is k. $Theta_1$ is total $theta_{ri}$ divided by one plus k, because $theta_2$ is k $theta_1$. $Theta_1$ plus k $theta_1$ is $theta_{ri}$. $Theta_1$ I can easily solve for $theta_{ri}$ by one plus k.

Similarly, here the distance covered if I call it, s_1 then v square is $2f_1s_1$, similarly, from the other end starting from 0 velocity it is covering a distance s_2 and reaching the velocity v so this is v square is $2f_2s_2$. In the same way I can get, s_2 by s_1 is f_1 by f_2 is equal to k and s_1 plus s_2 is the total lift. This I call s_1 and the rest of the lift I call s_2 and as we see s_2 by s_1 is k and s_1 plus s_2 is L, so s_1 turns out to be L by 1 plus k. There are very simple things, you can do it yourself. This is called modified parabolic motion, there are two parabolas but not identical, for a part of the rise and the rest of the rise.

(Refer Slide Time: 28:32)



Next, we discuss another type of basic follower motion, known as cycloidal motion. The word cycloid we may know is the curve which is generated when a circular wheel rolls without slip on a flat surface. Suppose we talk of a circular wheel of say radius r is rolling over this flat surface and if I look at the curve which is generated by any point of this wheel, for example say right now we concentrate this point which is on contact with the horizontal surface. As this wheel rolls, this point will go up to 2r height and then again come down in one cycle of rotation. This is 2pi rotation of the wheel and this curve generated by any point for example this point p, this is called a cycloid.

Let us see the expression for this cycloid and for the cycloidal motion the follower displacement y will be given by L into theta by theta_{ri} minus one by 2pi sine 2pi theta divided by theta_{ri}. This expression can be easily shown from this diagram. Suppose, this is the lift L and that corresponds to theta rotation of the cam. This is the lift L and this corresponds to theta going from 0 to theta_{ri} where theta is the cam rotation and the wheel rotates from here to there by 2pi. Wheel rotation if I denote by phi, then phi can be written as corresponding to theta it will be 2pi by theta_{ri} into theta. When theta is 0, phi is 0 and when theta is theta_{ri}, phi is 2pi. Due to this rotation, let us say it is rotated by phi and the wheel has taken of this position. The wheel centre has moved so much and this point P has come here and we have to find out this height. If I call this centre of the wheel as O which has gone to O one and the new point of contact is here which is, P_1 and this point P has moved to this point which I call P prime. What is the height of P prime and can I express it in terms of theta?

This is my y, theta is nothing but represented by phi and phi and theta related by this. Theta is the cam rotation phi is the rotation of this wheel. This vertical radius OP has now become O prime P prime and this rotation is phi. Because the wheel is rolling without slipping, I know this distance y and this is the lift (Refer Slide Time: 33:12), so the displacement of the follower is measured in this direction. This is the lift of the follower, so the movement of the follower is given by this distance. From P it has gone there and what is the corresponding movement in the direction of the follower movement is this, this is y. Because the wheel is rolling without slipping this arc length on the wheel is same as this distance, the distance covered by the centre of the wheel or the distance covered on the road.

This arc length is same as this and what is this arc length? Arc length is if radius of the wheel is r then r into phi and y is this minus this distance, which is r sine phi. So y is r into (phi minus sine phi). Let us see what is r, by one complete rotation of this wheel it has covered a distance L, so r is obviously L by 2pi and now let me put the value of phi in terms of theta which is 2pi theta by theta_{ri} minus sine 2pi theta by theta_{ri} and that gives with this expression for the cycloidal motion. 2pi cancels so it gives me L into theta by theta_{ri} and this 2pi I take inside, one by 2pi sine 2pi theta by theta_{ri}.

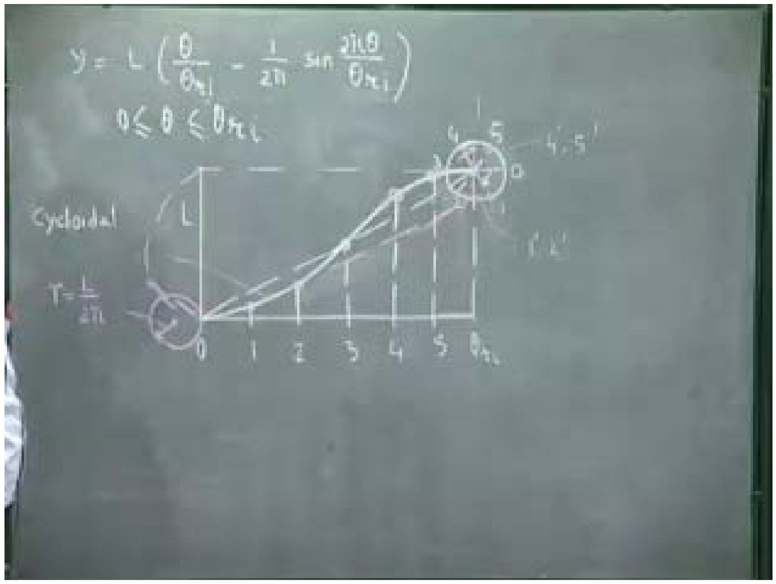
This is the expression for the cycloidal motion obviously it is valid in the range 0 to theta_{ri}. In this range, the displacement of the follower is expressed by this equation. We can study the velocity and acceleration of the follower, at the beginning and the end of the rise. For example, if we remember from the simple harmonic motion the one difficulty was that the acceleration from 0 took up to some finite value at the beginning of the rise, so the jerk was very large. Let me calculate the acceleration of the follower. This is the expression for the cycloidal motion which is valid for the entire rise period.

If we remember, for one of the basic follower motion like simple harmonic motion it started with a finite acceleration which created a very large value of the infinite jerk. Let me see what kind of initial values the cycloidal motion creates. If we remember the velocity I can write dy by dt which is omega times dy by d theta. What we get is, omega into L differentiating this we get, one by theta_{ri} minus if I differentiate this 2pi by theta_{ri} 2pi cancels with this, one by theta_{ri} into cosine 2pi theta. The velocity at the beginning and the end of the rise that is theta equal to 0, velocity v is 0, theta equal to $\text{theta}_{ri,}$ so 2pi theta by theta_{ri}. This is 2pi by theta_{ri} so this 2pi has been cancelled from here and cosine 2pi theta by theta_{ri}. At theta equal to 0 velocity is 0, if we put theta equal to theta_{ri} then again this is cosine 2pi which is one so again velocity is 0. It starts with 0 velocity, which is desirable and ends the rise, that it starts the dwell also with 0 velocity which is desirable.

Now, let me calculate the acceleration. Acceleration will be omega square into the second derivative of y with respect to theta. If I differentiate it twice, what we get, this does not give me any thing and this gives me 1 by 2pi into 4pi square by theta_{ri} square, we get it twice and minus minus makes it plus, sine 2pi theta by theta_{ri}. This is the value of the acceleration for any value of theta. At theta equal to 0, again acceleration is 0 and at the end of the rise, the theta equal to theta_{ri} again acceleration is 0. As this found here, at both theta equal to 0, v is 0 and theta equal to theta_{ri}, v is 0. Cycloidal motion has this advantage that at the beginning and end of the rise not only the velocity is 0, acceleration is also 0. That gives a little smoother motion then as compared to the harmonic motion.

Next, I will show how to draw this displacement diagram corresponding to this cycloidal motion.

(Refer Slide Time: 39:50)



We know that, the expression for the cycloidal motion the follower displacement expression we have just now shown is L into theta by theta$_{ri}$ minus one by 2pi into sine 2pi theta by theta$_{ri}$. Now, I will explain how to draw the displacement diagram corresponding to this expression. This is 0, this is theta$_{ri}$ and this is the lift L. As we have shown that, it is really rolling a circle of radius r equal to L upon 2pi such that one full rotation of this wheel will bring the wheel to the end of the lift, this is the lift. To draw this curve, what we do is, we draw the same wheel here of radius L by 2pi and then divide this rise period into equal number divisions, let us say six divisions and mark these station points 1, 2, 3, 4, 5 and 6 -these are the six equal divisions. This circle we also divide into six equal divisions: 0, 1, 2, 3, 4, and 5.

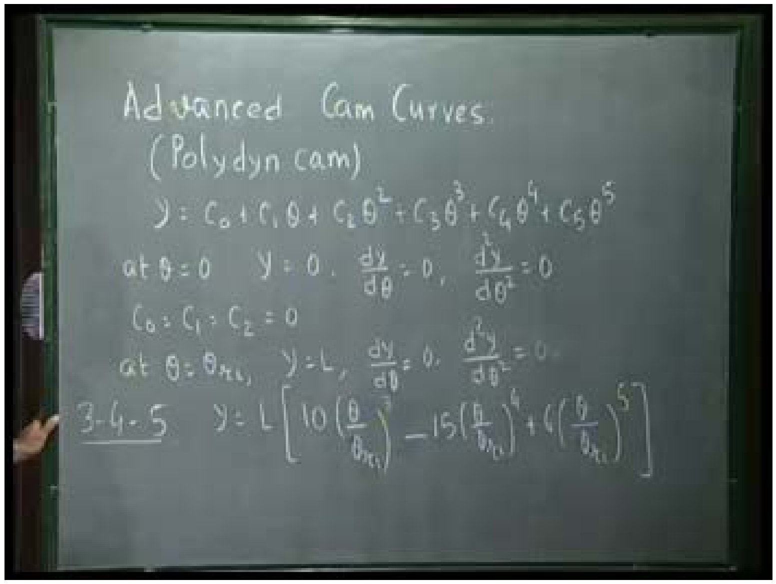
We draw this diagonal, then this straight line represents this part of the curve, L into theta by theta$_{ri}$. Because, as theta goes to 0 to ri, y goes to L and linearly, so this straight line represents this portion which is a straight line. Now I have to subtract this. What I do is, I project these points on to this vertical diameter, this I call 1 prime, 2 prime and this is 4 prime, 5 prime and the centre of the wheel which takes zero and three. From this, I draw a line which is parallel to this and from this point again I draw a line and these two lines are parallel to this diagonal.

From this corresponding station points, this line pass from 1 prime, 2 prime which was the projection of this one and two and the vertical diameter, from here I draw the vertical lines to intersect this line corresponding to 1 prime 2 prime and from here, I draw the line to intersect this line which is drawn through this centre which we call 0 prime and 3 prime. This line which is drawn from 4 prime and 5 prime, I draw these vertical lines. If I join these six points, I get the desired cycloid. This curve represents this equation, it is very easy to see that this vertical height, which I am subtracting is nothing but x plus this, because this rotation is 2pi into theta by $theta_{ri,}$ radius is L by 2pi, this is the sine of this angle and the vertical height is nothing but the sine of that angle which I have subtracted from this vertical subtraction.

This is the cycloidal motion, if we remember the cycloid is really this curve, this is the curve generated by a point on this wheel. We have the expression, this is valid for 0 less than theta less than $theta_{ri}$ and this is the geometrical construction corresponding to this equation.

Now that we have discussed all types of basic follower motions like uniform motion, simple harmonic motion, parabolic motion or modified uniform motion or modified parabolic motion or cycloidal motion, we have seen there is one kind of difficulty is that, some derivatives of the follower displacement like velocity, acceleration, jerk etc. maybe very high at the beginning or end of the rise or at some intermediate point during the follower movement.

(Refer Slide Time: 47:02)



To avoid all these, what it is sometimes done, we cannot avoid it totally, all higher order derivatives cannot be made 0 but some derivatives can be made 0 at both beginning and end of the rise by using what is known as advanced or polynomial cam curves and the corresponding cams are called polydyn cams. These advanced cam curves what we do is, we write the displacement of the follower as a polynomial of theta, an algebraic expression using powers of theta, say I write up to fifth order. I write C_0 plus C_1 theta plus C_2 theta square plus C_3 theta cube and so on, up to the fifth order term. These I leave as constants which are so set such that, value or the derivative of y at theta equal to 0 and theta equal to theta$_{ri}$ can be controlled.

For example, what we need at theta equal to 0, y has definitely got to be 0 and we also want the velocity to be 0 which can be ensured by having dy by d theta 0 and suppose we also want the acceleration to be 0 which can be ensured by having the second derivative of y with respect to theta to 0, which immediately tells us that C_0, $C_{1,}$ $C_{2,}$ all have to be 0. Because if we put theta equal to 0 here, y at 0 gives me C_0 is 0. Similarly, if we take derivative C_1 stands at the first derivative at theta equal to 0 all other term goes to 0 so C_1 needs to go to 0. Similarly, from the acceleration I will get C_2 equal to 0.

At theta equal to $theta_{ri}$ at the end of the rise, I need y has got to be equal to the lift of the cam that is L, but I want it to have 0 velocity and 0 acceleration. C_3, C_4, C_5 should be so chosen that these three conditions are fulfilled at theta equal to $theta_{ri}$, y is L but derivatives are 0. We can solve for the remaining three unknowns namely C_3, C_4, $C_{5.}$ This is a matter of simple algebra, we can do it easily and we can solve, three linear equation we will get in C_3, C_4, C_5 and if we solve those, we will finally get the expression as, y equal to L into 10 theta by $theta_{ri}$ cubed minus 15 theta by $theta_{ri}$ to the power 4 plus 6 into theta by $theta_{ri}$ to the power 5. The coefficients 10 by $theta_{ri}$ cube then 15 by $theta_{ri}$ 4 and 6 by $theta_{ri}$ 5, we can solve for C_3, C_4, C_5 using these three conditions. As the third, forth and fifth order terms are remaining this is called a 3-4-5 term. This is what we call, advanced cam curve. There are other kinds of advanced cam curves which we are not going to discuss in this course.

Let me now summarize what we have done today. Today, we have discussed all types of basic follower motions namely, simple harmonic motion, uniform or parabolic motion or uniform acceleration motion and cycloidal motion. We have also tasked upon a little bit about the polynomial expression for the follower displacement. We have seen their analytical expressions and also how to construct the displacement diagram corresponding to these analytical expressions. Our next task will be that, if y as a function of theta is either prescribed or given in the form of a displacement diagram, how to design the cam profile or how to synthesize the curve of the cam profile? That we will start in our next lecture.