Kinematics of Machines

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Module - 5 Lecture - 3

In today's lecture, we continue our discussion on velocity and acceleration analysis of the planar mechanisms. To start with we take up two examples: the first example will illustrate the use of the concept of velocity and acceleration image; the second example will illustrate the use of the concept of coincident points - that means instantaneously coincident points - belonging to various links. Let me start with the first example.

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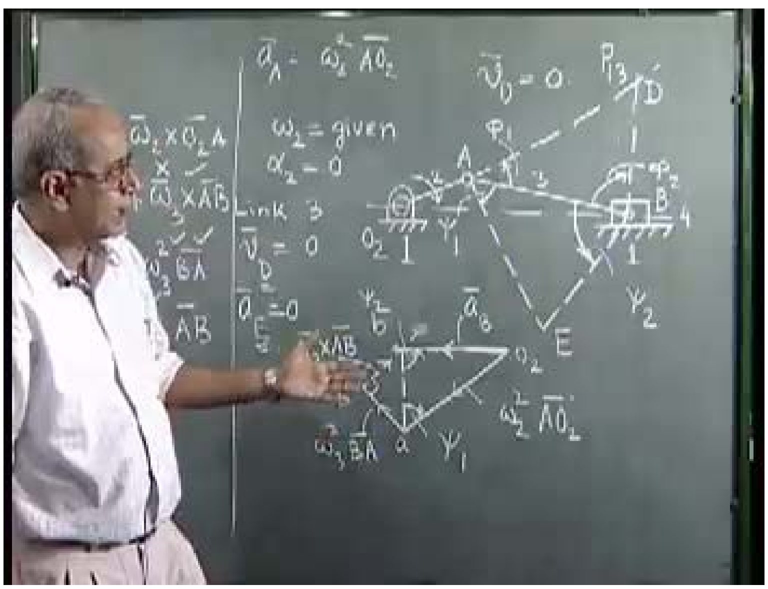

(Refer Slide Time: 01:03) We take the example of a slider-crank mechanism with O_2A as the crank, AB as the connecting rod which is connected to the slide. Suppose, the crank angular velocity is given as omega$_2$. Omega$_2$ is given. This is the fixed link number 1 (Refer Slide Time: 02:04 min); the crank is link number 2, connecting rod is link number 3 and the slider is link number 4. We also assume that this omega$_2$ is constant, that means angular acceleration alpha$_2$ is assumed to be 0. The question that is posed is - that at this configuration to find the point on the connecting rod that has 0 velocity. So the question is - find a point on the connecting rod, that is link 3, which has 0 velocity. Let me call that point D. So, V_D is 0 and D is a point on link 3.

To determine this, let me start with a velocity diagram which we have drawn earlier. Let the velocity of the point A which is perpendicular to O_2A is drawn to a scale and I represent it by small o_2a. This is the velocity of the point A which is completely known, V_A is omega$_2$ cross vector O_2A. Now, we consider two points A and B belonging to the same rigid body namely 3. So, we can write, V_B is V_A plus omega$_3$ cross AB. As we said earlier, before trying to draw the velocity of B, let us verify that this vector equation does not have more than 2 unknowns.

So far as velocity of B is concerned, we know the direction, that is, horizontal, but we do not as yet know the magnitude. Velocity of A has been completely determined and has also been drawn. So, this is completely known (Refer Slide Time: 04:43 min). Omega$_3$ cross AB is perpendicular to AB; so I know the direction of this vector, but I as yet do not know the magnitude. So, this vector equation has 2 unknowns as represented by these 2 crosses. This equation now can be drawn graphically. To satisfy this equation, because V_B is horizontal, I draw a line horizontal and passing through O_2 (Refer Slide Time: 05:15 min).

This vector omega$_3$ cross AB is perpendicular to AB. So through A, I draw a line which is perpendicular to capital AB, and these two lines intersect at the point b. So, this vector o_2b represents the velocity of the point B, and this vector ab represent omega$_3$ cross AB. This completes the velocity diagram, but the question is if we remember, we have to find a point on link 3, which has 0 velocity at this instant. That means we should find a point in the space diagram whose image is at O_2. As we see link 3, there are 2 points A and B which are corresponding points in the velocity diagram, that is, small a and small b.

I have to find the point D, whose image should be O_2, such that velocity of D is equal to 0. So, what we do? As we see, this line is perpendicular to capital AB, this line perpendicular to capital O_2A. Let me call this angle as phi$_1$ and this as angle phi$_2$. Note the direction that the phi$_1$ is counterclockwise at a and phi$_2$ is counterclockwise at b. Here in this space diagram at capital AB, I draw a line at an angle phi$_1$ counterclockwise at A and at B is clockwise, so I draw a line clockwise at an angle phi$_2$ and these two line intersect at the point D in triangle BAD. So, what we see that is this figure capital DAB is similar to this figure o_2 ab. The image of the point A is small a, image of the capital B is small b and image of the capital point D is small o_2.

This is the point D whose velocity is 0. At this instant, this point on the connecting rod has the 0 velocity. As we see, the angle between small ab and ao_2 is phi_1 and ab is perpendicular to capital AB and ao_2 is perpendicular to O_2A. That is why, if this angle is phi_1 the angle between these two lines, if I extend it also becomes phi_1. Now, ab is perpendicular to AB and BD is perpendicular to o_2b. Angle between them is $phi_{2;}$ so, the angle between them is also phi_2, because, these two lines are respectively perpendicular to these two lines. At this stage, it is very easy to note that this point D is nothing but what we earlier called P_{13} that is the relative instantaneous center of body 3 with respect to body 1, that is the fixed link; consequently, velocity of this point at this instant is 0. That this point is P_{13}, I will ask the students to do it themselves using the Arnold-Kennedy theorem of 3 centers which we have discussed in detail in one of our earlier lectures.

The next problem is to determine the point say E, such that E belonging to link 3, whose acceleration is 0 at this instant, we have to determine in the point E. To determine the point E on link 3 which has 0 acceleration at this configuration, let me first carry out the acceleration analysis. As we see the acceleration of the point A can be written as $omega_2$ squared AO_2 (vector AO_2). This is completely known because $omega_2$ is given and $alpha_2$ we have been assumed to be 0. (Refer Slide Time: 10:48) So, I can draw acceleration of the point a parallel to AO_2 to some scale and mark the pole as o_2 and this point as a, this is $omega_2$ squared AO_2.

We write the acceleration of the point B is acceleration of the point A plus $omega_3$ squared BA plus $alpha_3$ cross AB. So far acceleration of B is concerned, I do not know the magnitude, but I know the velocity which is horizontal. Acceleration of A is completely known and we have already drawn it here. From the velocity analysis, we have already determined $omega_3$, that is, angular velocity of link 3. $Omega_3$ is known, BA vector is known and so this vector $omega_3$ squared BA is completely known. I know the direction of $alpha_3$ cross AB, because it is perpendicular to the vector AB, but I do not know the magnitude.

In this vector equation again, we have two crosses. So, we can complete this vector equation in the diagram. First, we draw $omega_3$ squared BA that is parallel to vector BA from a. Let it be so much to the same scale. This is omega$_3$ squared BA. Alpha$_3$ cross AB will be perpendicular to this vector. (Refer Slide Time: 12:40 min) So, I draw a line which is perpendicular to this point and let me call this point b prime, but AB is horizontal. So through o_2, I draw a line horizontal. Where these two lines intersect gives me the point b such that the vector o_2b represents acceleration of the point b, O_2a represents the acceleration of the point a, b prime b this vector represents alpha$_3$ cross AB. There is no need to repeat up to this point, because this we have done earlier while analyzing velocity and acceleration of a slider-crank mechanism.

Let me try to answer this question E, such that acceleration of point E is 0 at this configuration and the point E belongs to link 3 (Refer Slide Time: 13:46 min). We have already got the image of the point A here as small a, image of the point B here as small b. If I draw this line then it is abo_2. If I draw a figure similar to this abo_2 with capital A and capital B representing the point small a and small b, wherever o_2 goes that point will be E, because, then the acceleration of the point E will be 0. To locate that point E, we note that this angle is psi$_1$ in the clockwise direction at a; this angle at b is psi$_2$.

At A, I draw a line at an angle psi$_1$ with ab. With ab, the point o_2 is at an angle psi$_1$ at a. With ab, I draw an angle in the clockwise direction and psi$_1$. So, E must lie on this line. Similarly at B, I drawn a line which is almost 90 degree, I draw a line which is at an angle psi$_2$; this angle is psi$_2$ at b. o_2 lies on a line which is drawn on angle psi$_2$ in the counterclockwise direction to ab. Here, I draw at capital B, a line which is oriented at an angle psi$_2$ in the clockwise direction from capital AB. The corresponding between the small letter and capital letter should be noted.

Where these two lines intersect that gives me the point E (Refer Slide Time: 15:58 min), such that image of the point E is o_2, which means that the acceleration of the point E belonging to link 3 at this instant will be 0. It is needless to say that this point D with 0 velocity or this point E with 0 acceleration belonging to link 3 may lie outside the physical boundary of link 3; they belong to link 3 in the plane of motion link 3, but they may exist outside the physical boundary of this link 3. This is the point with 0 acceleration on link 3; this is the point (Refer Slide Time: 16:39 min) with 0 velocity on link 3 at this particular configuration. The use of the velocity and acceleration image of a particular link is shown to this example.

In our next example, we consider a 4-link RPRP mechanism and show the use of the instantaneously coincident points belonging to different links and how this could be used for velocity and acceleration analysis.

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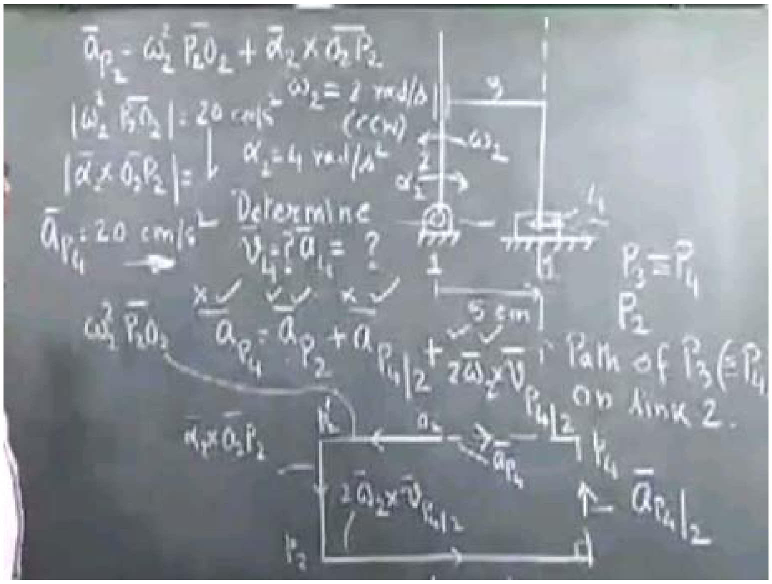

(Refer Slide Time: 17:17) This is an RPRP mechanism. Link number 1 is connected to link number 2 through a revolute pair, link number 2 is connected to link number 3 through a prismatic pair, link number 3 is connected to the slider which is link number 4 through a revolute pair and link number 1 and 4 are connected by a prismatic pair.

The problem is as follows. Suppose we are given the input motion omega$_2$, I take some numerical values, say omega$_2$ is 2 radian per second in the counterclockwise direction and angular acceleration of link 2 at this instant, say alpha$_2$; this is omega$_2$; alpha$_2$ is also given as 4 radian per second squared. The statement of the problem is determined with this input velocity and acceleration, at this configuration, what is the velocity of the slider 4 and acceleration of the slider 4?

Let me first carry out velocity analysis. Suppose, we are given this distance, at this configuration is, I take another numerical value, say 5 centimeter. To carry out the velocity analysis, I consider instantaneously coincident points at this revolute access of the block center, and call it P$_3$, which is always coincident with P$_{4.}$ There is no relative movement between these two coincident points

P_3 and P_4. P_3 belonging to link 3 and P_4 belonging to link 4. I also consider a point P_2 which belongs to link 2, but at this instant coincident with P_3 and P_4.

Let us see whether we know the path of the point P_4 in link 2. It is needless to say, because, P_3 and P_4 are always coincident, path of P_4 on 2 will be the same as the path of P_3 on link 2. To judge the path of P_3 on link 2, what do we do? We hold imaginarily link 2 fixed and try to move link 3. If we try to move link 3 it is obvious that because of the slider, 3 can only go in the vertical direction on link 2. The path of P_3 on 2 is this vertical line, given by this vertical line. This is the path of P_3 which is same as P_4 on link 2 (Refer Slide Time: 21:17 min).

We can write that velocity of P_4 is velocity of P_2 plus velocity of P_4 on link 2. This relation we have derived earlier within coincident points belonging to two different links - 2 and 4; V_{P4} is V_{P2} plus V_{P4} as seen by an observer on link 2. What we see that velocity of P_4 has to be horizontal, because of the prismatic pair in the horizontal direction between 4 and 1, all the points on link 4 must move horizontally. I know the direction of V_{P4} which is horizontal, but the magnitude is unknown. I can write velocity of P_2, because omega$_2$ is given, it is omega$_2$ cross O_2P_2. The vector O_2P_2 is known. It is 5 centimeter length along the horizontal direction, omega$_2$ is equal to 2 radian per second in the counterclockwise direction; so the velocity of P_2 is completely known.

I know the magnitude and direction, what is the magnitude? Omega$_2$ is 2, O_2P_2 is phi, so the magnitude is 10 centimeter per second and because of this counterclockwise omega$_2$, the velocity of P_2 is in the vertically upward direction. So, I can draw velocity of P_2 which is vertically up, and let me represent it as O_2P_2. What is the velocity of P_4 on 2, because the path is vertical that means the velocity which is tangential to the path must also be vertical. I know the direction, but not the magnitude. Again, in this vector equation, there are two crosses, so I can draw it. I have already drawn V_{P2} by O_2P_2 and $V_{P4/2}$ is also vertical. How about V_{P4}? It is horizontal? Two vertical vectors giving arise to a horizontal resultant vector implies that these vectors must be of 0 length.

If I draw this vertical I am drawing it intentionally a little to the right, actually it is lying on the same line, this is vertical (Refer Slide Time: 24:21 min). And these two vectors put together must give me P_4 and O_2P_4 which is of 0 length represents the velocity of the point P_4. Two vertical vectors resultant of that is horizontal implies that these vectors must be of 0 lengths. So, velocity of P_4 at this particular configuration is 0. That means this block is momentarily at rest and what is $V_{P4/2}$? As this diagram shows, this vector and this vector of equal length such that P_4 becomes coincident with $O_{2.}$ This I get, just as V_{P2} magnitude, which is 10 centimeter per second, but in the vertically downward direction.

We have completed the velocity analysis of this mechanism; let me start with the acceleration analysis. From the velocity analysis, we have already obtained that V_{P4} is 0; whereas, V_{P4} on 2 is 10 centimeter per second in the vertically downward direction.

To carry out the acceleration analysis, we write a_{p4} is a_{p2}, that is, the coincident point plus a_{p4} as seen by an observer on 2, which I write as a_{p4l2} plus Coriolis acceleration which is this case twice omega$_2$ cross $V_{P4/2}$ and P_2 and P_4 are coincident points belonging to link 4 and P_2 to link 2. From this acceleration equation, let me determine which vectors are known and which vectors are unknown. a_{p2} is completely known, because the input angular velocity omega$_2$ and input angular acceleration alpha$_2$ are prescribed. I can write a_{p2} is omega$_2$ squared P_2O_2 plus alpha$_2$ cross O_2P_2; everything is specified so a_{p2} is completely known. Magnitude of omega$_2$ squared P_2O_2 is omega$_2$ squared is 2 squared, that is 4; P_2O_2 is 5. So this is 20 centimeter per second squared. The magnitude of omega$_2$ squared into P_2 $O_{2,}$ this vector is 4 into 5, that is, 20 centimeter per second squared and the direction is along the vector P_2 O_2 that is horizontal to the left. Alpha$_2$ cross O_2 P_2 where O_2P_2 is 5 centimeter and alpha$_2$ is 4 radius per second squared, so the magnitude of this is also 4 into 5 is again 20 centimeter per second squared (same magnitude, 20 centimeter per second squared) and what is the direction? Alpha$_2$ is clockwise, so the transverse acceleration is vertically in the downward direction. So, this vector is completely known. How about $a_{p4/2}$? The path of P_4 on body 2 as we have seen is this vertical line (Refer Slide Time: 28:21 min). So the direction of this vector is completely known, this is along a vertical line, but the magnitude is unknown. We have already determined $V_{p4/2}$ as 10 centimeter per second in the downward direction and omega$_2$ is given to be 2 radians per second in the counter clockwise direction. Both the magnitude and the direction are known in these vectors. For a_{p4}, I know the direction that is horizontal and magnitude is unknown. In this vector equation, again we have only two crosses left, so this vector equation can be represented by a vector diagram.

Let me try to draw this vector diagram. For a_{p2}, I write $omega_2$ squared O_2P_2 to some scale; let me say this is O_2. This point I call P_2 prime, because this is only this component which is 20 centimeter per second squared to the left. Then this component is again 20 centimeter per second squared, but in the vertically downward direction. Again, I draw of the same length and this is $P_{2;}$ this is $alpha_2$ cross O_2P_2 and this vector is $omega_2$ squared into P_2 $O_{2.}$ This vector is completely known. Let me try to draw this vector at P_2. $Vp_{4/2}$ was 10, $omega_2$ is 2, so 2 into 2 is 4 into 10 gives 40 centimeter per second squared and it is in the horizontal direction, but double of this length; this is 40 centimeter per second squared and this vector represents twice $omega_2$ cross $V_{p4/2}$ and I know $a_{p4/2}$ is vertical. I draw a vertical line through this point. This point is horizontal and it is 90 degree and these four vectors (Refer Slide Time: 31:06 min) these two vectors represent to this, these vectors represents this and this vector is in the vertical direction. All these three vectors summed over must give me a_{p4} which is horizontal. So, I draw a line horizontal and wherever they intersect, that gives me the point P_4. According to this vector equation, this O_2P_4 represents acceleration of P_4 and this represents acceleration of P_4 as seen by an observer on body 2. This diagram represents the vector equation and it is easy to see because this is 40 which is horizontal line, this is 20 which is also horizontal line. So, this must be 20. For a_{p4}, I get as an answer 20 centimeter per second squared in the horizontal direction from left to right.

These examples clearly show the power of considering the instantaneously coincident point on various links when we have sliding joins on rotating links.

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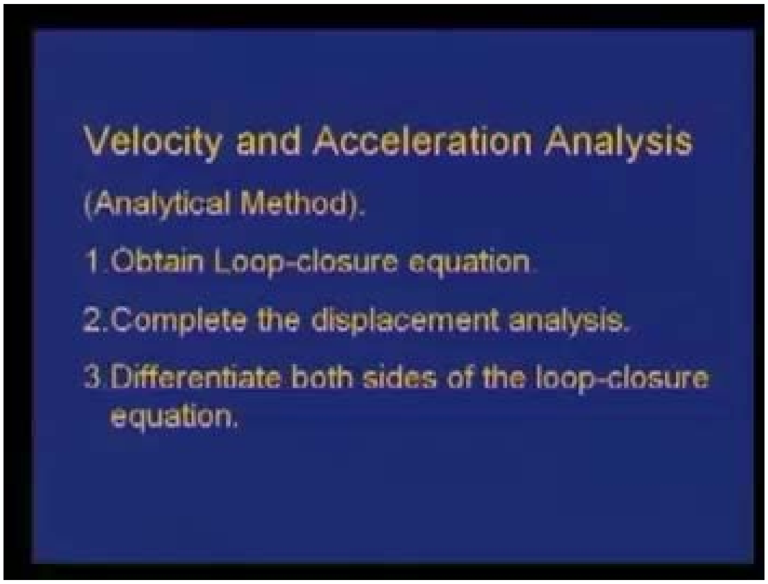

Now that we have completed velocity and acceleration analysis by graphical method, let me show you how to carry out such velocity and acceleration analysis through analytical method. In analytical method, just as in the displacement analysis, we start with the loop closure equation. That means, we represent the link lengths and sliding displacement as vector quantities and go through each loop that is present in the mechanism and write the corresponding loop closure equation in terms of these vectors. The first step for carrying out the velocity and acceleration analysis is obviously, to complete the displacement analysis. Now that this loop closure equation is valid for all instants of time, we can differentiate both sides of such loop closure equation to complete the velocity and acceleration analysis.

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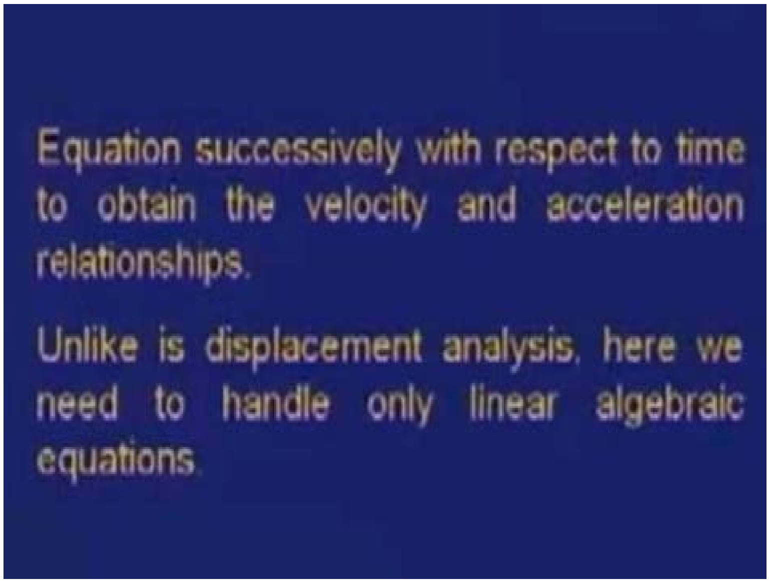

Successive differentiation of this loop closer equation with respect to time will give us required velocity and acceleration relationship. I would like to mention one point here, that is, in the displacement analysis to analytical method we always get non-linear algebraic equation; whereas, if the displacement analysis is complete which is always necessary to carry out the velocity and acceleration analysis. For velocity and acceleration analysis, we always get liner equations in the unknowns; that means, the problem is much simpler. As an example, let me start with a 4R linkage that means 4 links connected by 4 revolute joints.

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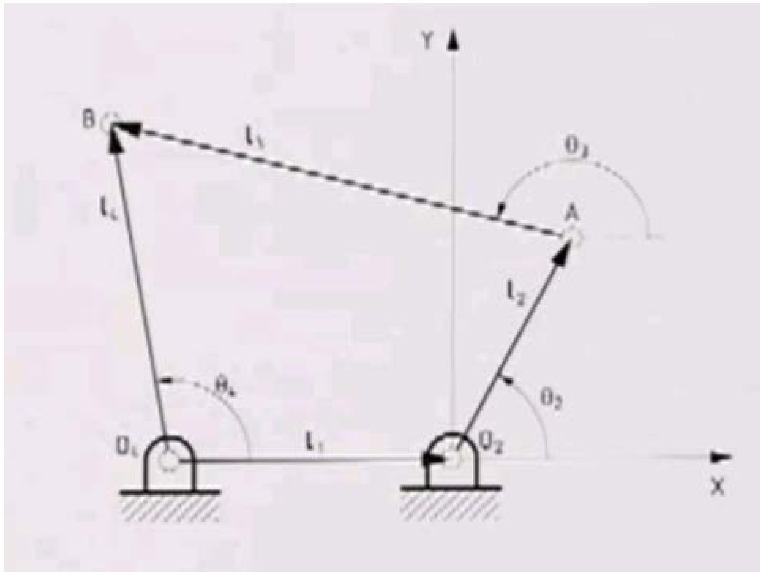

Let us look at the diagram of this 4R linkages namely O_2, A, B, O_4. As in the displacement analysis, we set up a coordinate system, Cartesian coordinate system xy with the origin at the revolute pair O_2. The fixed link is represented by O_4O_2, the input by O_2A, the coupler by AB and the follower or the output link by O_4 B. For this given configuration, that is, if theta$_2$ is given by displacement analysis we can obtain theta$_3$ and theta $_4$. After doing this displacement analysis, that is, knowing theta$_3$ and theta $_4$ we should be in a position to carry out the velocity analysis.

What do you mean by velocity analysis? Let us say, the input velocity theta$_2$ dot, the angular velocity of link 2 is prescribed and we have to find out the angular velocities of link 3 and link 4 which are given by theta$_3$ dot and theta$_4$ dot, respectively. Obviously, everything is measured positive in the counterclockwise direction.

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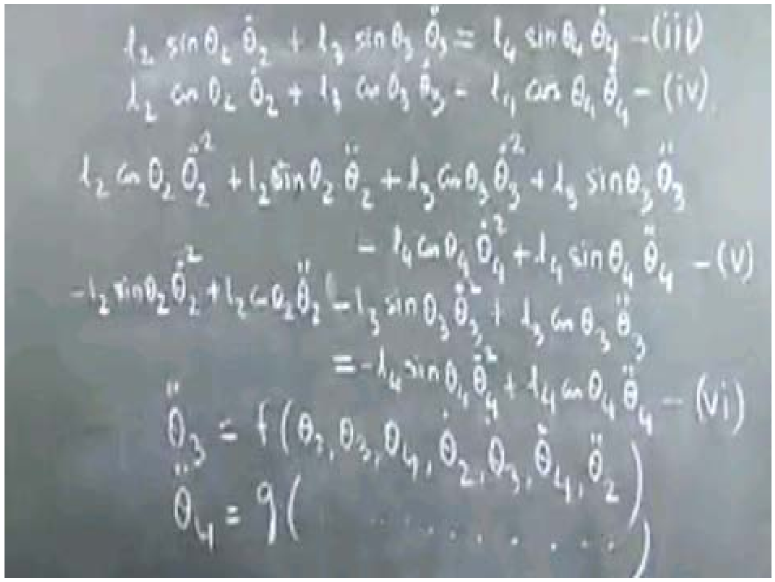

Let me write the loop closure equation. Vector l_1 plus vector l_2 plus vector l_3 is vector l_4. This is exactly the same as we did in the displacement analysis. Let me write now these two-dimensional vectors in terms of complex exponential notation that is l_1 plus l_2 e to the power I theta$_2$. The vector l_2 can be represented by this complex exponential notation, magnitude is l_2 and the orientation of this vector with the positive x-axis is given by theta$_2$. Similarly, l_3 vector is l_3e to the power I theta$_3$ is equal to l_4 e to the power I theta$_4$. As we know, e to the power I theta can always be written as cos theta plus I sine theta. We equate the real and imaginary parts of this particular complex equation, which is same as equating the x-component and y-component of two sides of this vector equation. We will get l_1 plus l_2 cosine theta$_2$ plus l_3 cosine theta$_3$ is l_4 cosine theta$_4$. Equating the imaginary parts, we get l_2 sine theta$_2$ plus l_3 sine theta$_3$ is l_4 sine theta$_4$.

This complex equation is equivalent to two real equations which I mark as equation number 1 and 2. These equations are valid for all instants of time where theta$_2$, theta$_3$ and theta$_4$ are functions of time whereas link length l_1, l_2, l_3 and l_4 are time independent constants. Because these equations are valid for all instant of time, I can differentiate both sides of these two equations with respect to time and I can write from first equation l_2 sine theta$_2$ theta$_2$ dot and differentiate with respect to time, plus l_3 sine theta$_3$, theta$_3$ dot is equal to l_4 sine theta$_4$ theta$_4$

dot. Similarly, from the second equation, I get l_2 cosine $theta_2$ $theta_2$ dot plus l_3 cosine $theta_3$ into $theta_3$ dot is equal to l_4 cosine $theta_4$ into $theta_4$ dot.

We have two equations where because the displacement analysis has been completed I know the values, the given values of $theta_2$, the value of $theta_3$, and $theta_4$. The input angular velocity $theta_2$ dot is given and our objective is to determine the two unknown angular velocities $theta_3$ dot for the coupler and $theta_4$ dot for the output link. The thing to note is that these two equations in these two unknown $theta_3$ dot and $theta_4$ dot are linear. We can easily solve for these two unknowns from these two linear algebraic equations. To eliminate, say, $theta_4$ dot I multiply this equation, let me call it, number 3 and this equation, if I call, number 4.

If I multiply equation 3 by cosine $theta_4$ and equation 4 by sine $theta_4$ and subtract then I get right-hand side 0. So what have we done? We multiplied by cosine $theta_4$ the topic version. I get l_2 $theta_2$ dot sine $theta_2$ cosine $theta_4$ minus cosine $theta_2$ sine $theta_4$ which gives me sine $theta_2$ minus $theta_4$. Here, I have multiplied by cos $theta_4$ and here I have multiplied by sine $theta_4$ and subtracted. I get l_3 $theta_3$ dot sine of $theta_3$ minus $theta_4$ and this is equal to 0. Let me check. sine $theta_2$ cos $theta_4$ minus cos $theta_2$ sine $theta_4$, sine $theta_3$ cos $theta_4$ minus cos $thera_3$ sine $theta_4$. So, I eliminated $theta_4$ dot from these three equations and I obtain the unknown $theta_3$ dot as minus l_2 by l_3 sine of $theta_2$ minus $theta_4$ divided by sine of $theta_3$ minus $theta_4$ into $theta_2$ dot.

If the input angular velocity $theta_2$ dot is given and we have carried out the displacement analysis such that the values of $theta_2$, $theta_3$ and $theta_4$ are known, I can easily find out the angular velocity of the coupler $theta_3$ dot. If it turns out to the positive then $theta_3$ dot is counterclockwise; if it is negative then $theta_3$ dot is clockwise.

Similarly, I could have eliminated $theta_3$ dot from these two equations and we obtain $theta_4$ dot. That I leave for the students to carry out as an exercise and complete the velocity analysis.

Next, let us see starting from these two equations 3 and 4 how can you carry out the acceleration analysis? To carry out the acceleration analysis, let me start with these two velocity equations which we have numbered as 3 and 4. These two velocity equations are also valid for all instant of time. So, I can again differentiate these two equations with respect to time and maintain the equality sign. Let me differentiate the first equation, then I get l_2 cosine $theta_2$ $theta_2$ dot squared plus l_2 sine $theta_2$ into $theta_2$ double dot. I get these two terms from the first term if I differentiate with respect to time.

Similarly, differentiating this term with respect to time, I get two more time namely, l_3 cosine $theta_3$ $theta_3$ dot squared plus l_3 sine $theta_3$ $theta_3$ double dot and that is equal to again differentiate right-hand side, I get l_4 cosine $theta_4$, $theta_4$ dot squared plus l_4 sine $theta_4$ $theta_4$ double dot. This is one acceleration relationship and let me number this equation as 5.

Similarly, from the second equation I get minus l_2 sine $theta_2$ $theta_2$ dot squared plus l_2 cosine $theta_2$ $theta_2$ double dot. These two terms, I get by differentiating this first term with respect to time. From the next time, I get two more terms namely minus l_3 sine $theta_3$ $theta_3$ dot squared plus l_3 cosine $theta_3$ $theta_3$ double dot and that is equal to differentiating the right-hand side with respect to time, I get minus l_4 sine $theta_4$, $theta_4$ dot squared plus l_4 cosine $theta_4$ $theta_4$ double dot.

Let me call this equation as 6. Since the velocity analysis has already been completed now $theta_2$ dot, $theta_3$ dot and $theta_4$ dot are known. The input angular acceleration $theta_2$ double dot has been prescribed. Our objective is to obtain the two unknown acceleration $theta_3$ double dot that is of the coupler and $theta_4$ double dot that is of the output link. These two equations 5 and 6 are again two linear simultaneous equations in two unknowns namely, $theta_3$ double dot and $theta_4$ double dot. It is very easy to solve these two unknowns and I leave the students to complete the algebra and find out $theta_3$ double dot and $theta_4$ double dot in terms of $theta_2$, $theta_3$, $theta_4$ which have been obtained from the displacement analysis, this is prescribed; these two have been obtained from the displacement analysis; $theta_2$ dot which has been prescribed and we have already obtained $theta_3$ dot and $theta_4$ dot from the velocity analysis and the input angular acceleration $theta_2$ double dot. Exactly the same way, we eliminate $theta_4$ double dot, we solve the $theta_3$ double dot and then we eliminate $theta_3$ double dot to obtain $theta_4$ double dot which will be again the functions of all these variables.

That completes the acceleration analysis by analytical method.

At the end of this lecture, we have completed our discussion on velocity on acceleration analysis of planar linkages, both by graphical method and analytical method.