Finite Element Method

Prof. C. S. Upadhyay

Indian Institute of Technology, Kanpur

Module No. # 03

Lecture No. # 01

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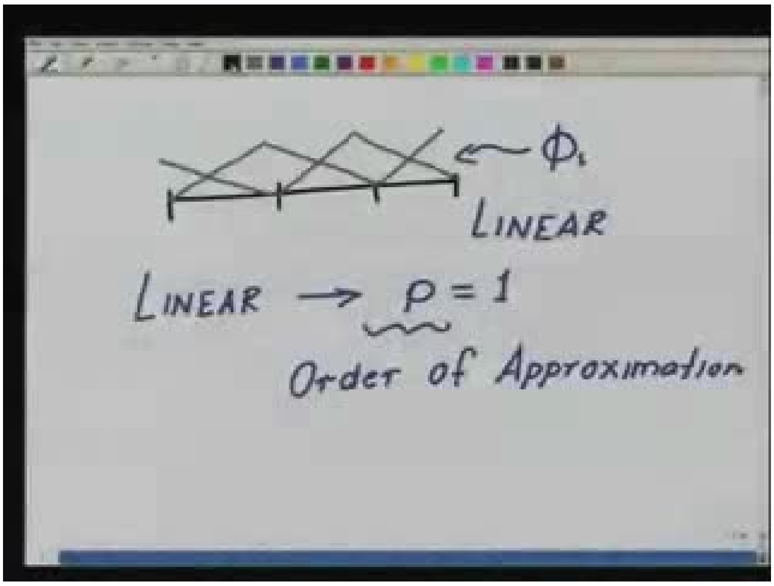

Till now, we have looked at the one-dimensional finite element problem using linear basis function. So, if you had this domain, what we have done is, we had partitioned the domain into elements, we had corresponding nodes and then called the element, we went ahead and constructed approximation using these functions phi i, which are linear. Also the question - is that all we can do or can we do more than checking such linear approximation? The answer is yes. So, what today we are going to look at, is higher order approximation for the finite element solution.

So, what is the notation? Let us first introduce when it is a linear approximation, then we will represent it by P equal to 1, where P we call has the order of approximation. Now, what we have implicitly assumed, I will assume it throughout this course, unless it is specified, specifically, that this order of approximation is the same in all the elements, that throughout the domain, we are going to use the same order of approximation; one can always use different order of approximation, in different regions of auto mode, that case we will not handle now.

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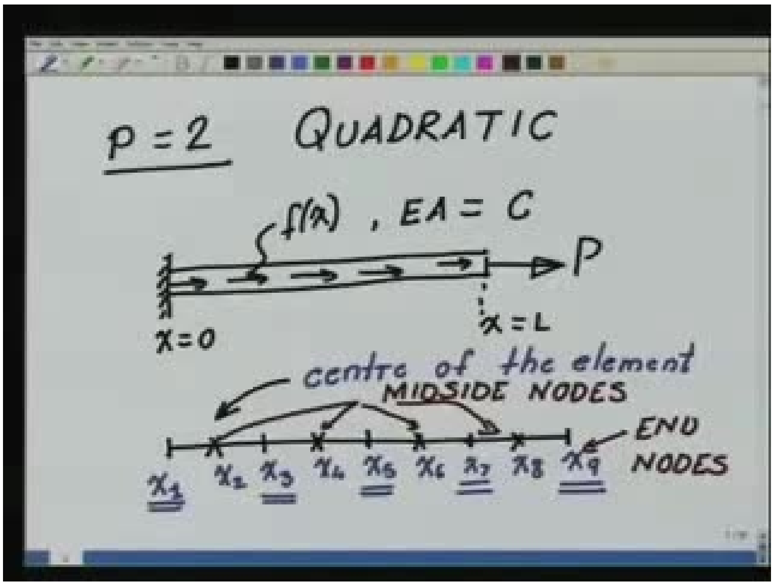

So, let us now see, how this order of approximation can be increased. So, the next example that would we have to do is, P equal 2; that is the order of approximation is quadratic. For this, let us see, what is it that we have to do, as far as the approximation is concerned. So, for this, you take the same domain that we had taken earlier, we send n load p, this is x equal to l, this is x equal to 0 and some distributor forces f x. For f x, let us also takes EA is equal to constant in the domain, that need not be true. So, using this, given this data for the problem, let us now go ahead and do the approximation. So, how are we going to do this? What we need to do now, is to first break the domain into element as we have doing till now.

May be I will take 4 elements here. So, these partitions that I have drawn, using these end vertical lines, these are our elements. So, this is element 1, this is element 2, this is element 3 and this element 4. So, in this element, in each of this element, what I am going to do is add an extra point. So, in each of the element, I am going to add an extra point. This point is located at the center of the element. So, now we are going to do this, the locations of this point. So, now, instead of having 1, 2, 3, 4, 5 points for the 4 elements that we have drawn, now we have 9 points. So, we will give them name. So, this is point x 1, this is x 2, this is point x 3, x 4, x 5, x 6, x 7, x 8, x 9.

What you should note, is that these extremities of the element are given by the point x 1, x 3; so, these are the extremities of element 1; extremities of the element 2 are the point x 3 and x 5; extremities of the elements 3 are the points x 5 and x 7; extremities of the element 4 are the point x 7 and x 9. These points x 2, x 4, x 6, x 8 are generally called mid time nodes or mid edge nodes. So, we will give a name to them; these are mid side nodes and the points x 1, x 3, x 5, x 7, x 9 are called these n vertices or simply the extremities of the elements. So, this point becomes end nodes or elements nodes or extremities of the elements. Now, what do we do next?

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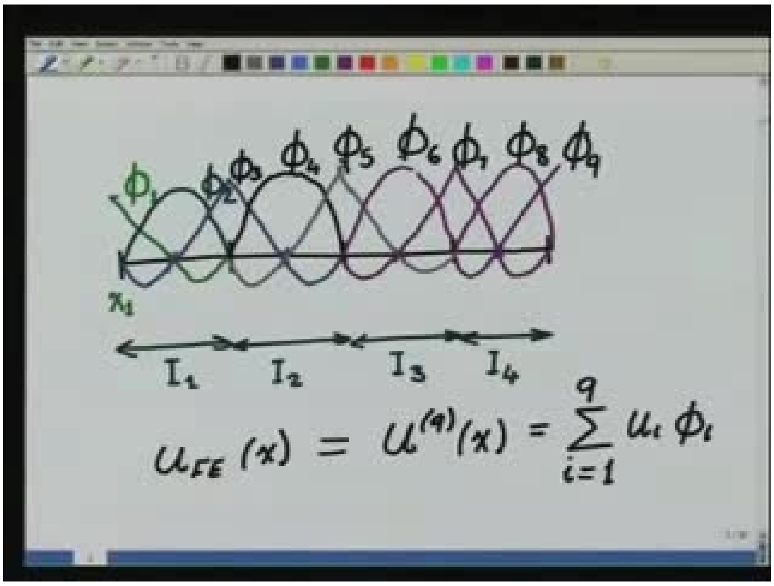

So, what will do is - over this point checked that we have created, we are going to use it somehow to create our basic function, which are quadratic in nature. So, I have this point set, I am going to construct the basis function, which now should be quadratic in the elements. So, first, I am going to draw; then, we will talk about how to construct mathematically.

So, while drawing, what had we said about the linear function, what we wanted these functions to do, what should have a value 1 at one of the nodes and 0 at all other nodes; the same principle we are going to use as far as construction of the quadratic functions. So, let us say the function phi 1 that is corresponding to the nodes x 1 or the point x 1. The phi 1 will be 1 at the point x 1 and 0 at all other points; so, if I want to make that phi 1 as this function. Similarly, I would like to make phi 2; phi 2 will be a function, which is 1 at this point and the point x 2, and 0 at all other points. So, what we see is that we have defined it in a local region; that is, phi 2 is only defined in the first element.

So, if I can draw that element here, this is element I 1, this is element I 2, this is element I 3 and this is element I 4, then I can draw phi 3. According, phi 3 should be 1 at this point and 0 everywhere else. So, by the same token that I have done phi 3 like this, then phi 4. phi 4 should be 1 here, 0 at all other points phi 4. Similarly, phi phi phi phi will be 1 here; 0 will be where are… Similarly, if I want to use, phi 6 will be this, phi 7 will be this, phi 8 will be this and phi 9 will be this.

So, what we got is phi 2 is here, so naming wise, this is phi 3, this is my phi 4, this my phi 5 phi; sorry, I made a mistake here. So, I have to go back and change my naming convention. So, before phi 2, this is going to be phi 3, this column is going to be phi 4, this one is phi 5, this one is phi 6, phi 7, phi 8 and then phi 9.

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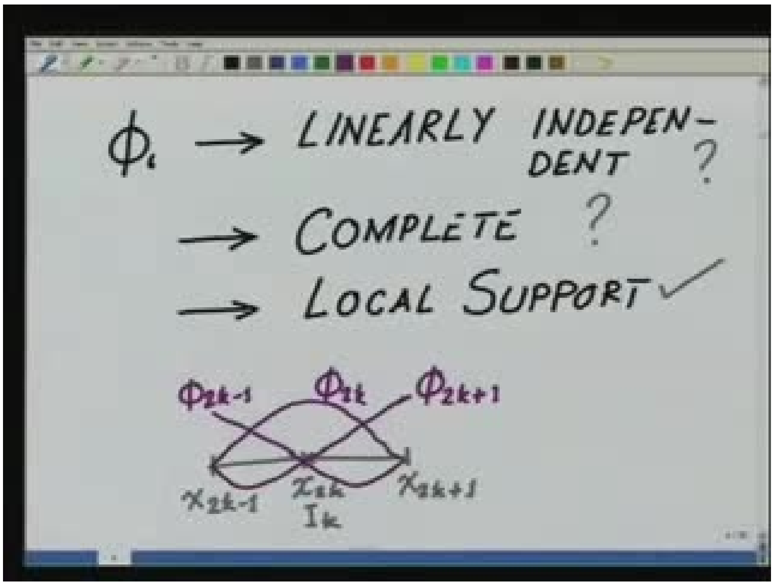

So, here, I have the 9 phi, which I have constructed; they look quadratic, we have to also give an expression to them mathematically; why are they quadratic? Because they have a value 1 at one point, and 0 at two other points, and then we extend it as 0 everywhere. So, they should be quadratic; that means they look like that. So, in some will this function, how am I going to write my 500 ml solution. So, u F E of x, we will use our earlier notation, is it is going to be, u is the 9-term solution u 9 of x. This is equal to sigma i is equal to 1 to 9 u i phi i.

So, this is a pixel optical. So, now, what is this, that the function has to satisfy in order to be admissible basis function? The first thing that we have done till now is that, this function phi i should be, it should be linearly independent; second thing that they have to satisfy is that they should be complete; and third thing that we have done by construction is they should local support, that is, they are only defined in one or two elements and in the rest of the element there, 0.

So, by construction, we see that this property has satisfied linear independence; we do not know this, we do not know. So, let us first make this function in such a way that they are quadratic, linearly independent and complete. So, how are we going to do that? So to do that, let us now consult this function element by element. So, if I look at element - a generic element I k - if you look at the generic elements I k, what are the nodes that this elements has or the point it will be? x 2 k minus 1 x 2 k x 2 k plus 1. Similarly, which is the phi, which is non-zero here? So, phi which are non-zero, here are phi 2 k minus 1 phi 2 k plus 1.

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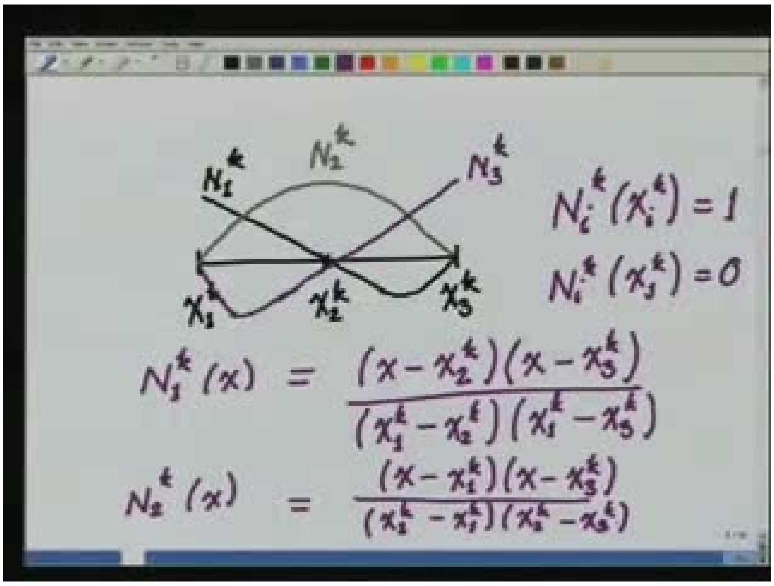

So, let us now go and transform this to our element notation that we had introduced earlier. So, from the element notation point of view, in the element, I will call the first node as x 1 of an element k; the second one, x 2 of an element k; and the third one is x 3 of element k.

Similarly, the phi 2 k minus 1, in this element could become, would be called N 1 of element k and phi 2 k will be called N 2 of k. Similarly, phi 2 k plus 1 will be called n 3 of k. And like in the case of linear approximation, in this we want this function N 1 of k to be to have a value 1 at the point x 1 of k and 0 at all other points. N 2 of k should have a value 1 at the point x 2 of k 0; at all other points, N 3 of k should have a value 1 at the point x 3 k and 0 at all other points. So, what we can write in short is, N i k as the point x i k is equal to 1 and at all other points j it is equal to 0. Set property is what we want to have for this function. So, if we want to construct N 1 k with this property such that N 1 k is 0 at the point x 2 k and x 3 k.

So, how will I form the function N 1 k? This should minus at the point x 2 k, certainly, this representation should have this form, something into is there, then at x 2 k this expression goes to 0, it should also vanish at the point x 3 k. So, it should also have this part sitting there. You see that this expression vanishes at the point 2 and the point 3 of the element; now what else is required at the point 1? This expression should have a value 1. So, what I will do is simply divide by, what I have done in the bottom part, I have replaced x with x 1 k; you see what has happened, this is a quadratic expression, it vanishes at the point where you wanted to vanish and it has the value 1 at the point where we want it to have a value 1.

So, by the same token, can we define N 2 k. We are defining this thing only in the element phi k. This will be equal to what? Very simple; it will be x minus, it should vanish at the point 1, and 2 is going to be x minus x 1 k x minus not 1 and 2, but 1 and 3 x 3 k, and it should have a value 1 at the point x 2 k. So, it will be x 2 k minus x 1 k x 2 k minus x 3 k.

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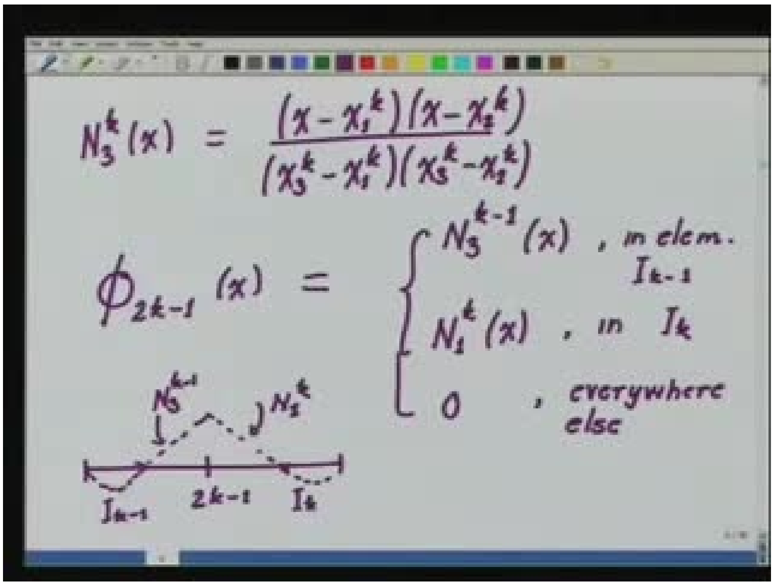

Similarly, N 3 k by the same token, can be written as x minus x 1 k into x minus x 2 k divided by x 3 k minus x 1 k into x 3 k minus x 2 k. So, we have constructed the three functions at the element level; this satisfy the three requirements we have imposed as far as the values are concerned. Now, what do we note? That if I look at the function phi 2 k minus 1 of x, where is this functions, and this is for we have to differentiate between the two things, this function 2 k minus 1 corresponds to an edge vertex, that is a node which is shared by two elements; that is, this corresponds to if k is 1, it is 1. So, it is 1, 3, 5, 7, 9 this function has to be non-zero in the two elements sharing that node. So, this will be equal to N 3 in the element k minus 1 for it will be N 1 in element k; hence, it will be 0 everywhere else.

So, what you said is this function, if I talk of this node 2 k minus 1; this is element I k minus 1; this is element I k. So this, the function is given as N 3 of k minus 1, which we now know how to construct and here N 1 in k.

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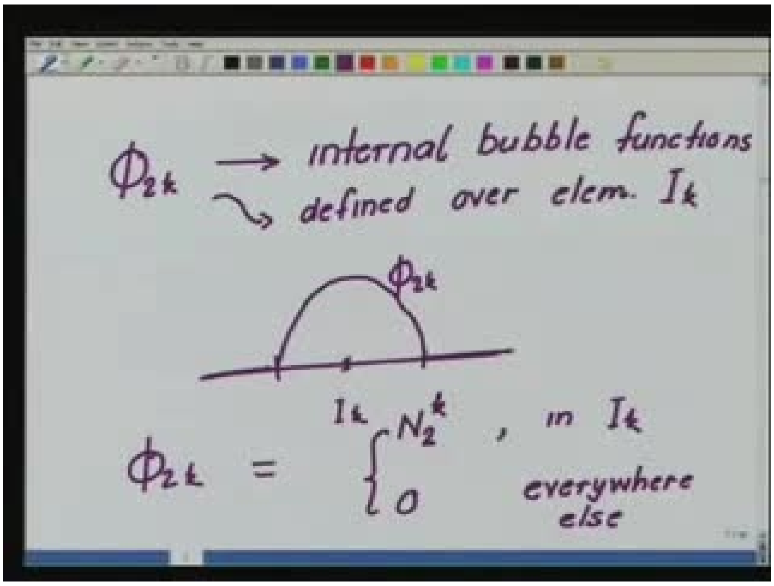

So, this we can construct, all the global of basis function, odd ones using these N I k(s) defining the element; what about the phi 2 k that is the even one? So, the even one which you see are defined only in the element. So, the even one - the phi 2 k k is equal to 1 2, the number of elements, these are sometimes called internal bubble functions and they are only defined over element I k, that is, this is my element I k. So, this is by the 2 k node and phi 2 k is non-zero in this element and 0 everywhere else. So, what we know now that phi 2 k is equal to by definition N 2 of element k in element I k and 0 everywhere else.

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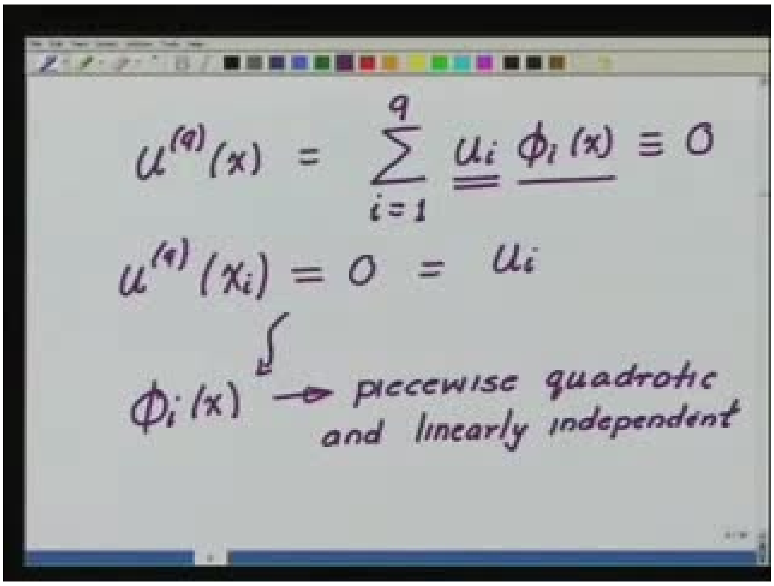

So, these, now you see that when we went, we came from the linear to the quadratic. The construction has been become a little involved; nevertheless, the steps that we are going to follow will be the same, irrespective whether it is linear, quadratic, and as we will see cubic forth order or fifth order, any order that we would like to have. Now, the question is that - are these functions linearly independent? So, once I have defined these functions phi i, then if you remember, we had written u 9 of x is equal to sigma i is equal to 1 to 9 u i phi i of x. How do we check linear independence? If you remember, linear independence means that if this function is identically equal to 0, then the coefficient u i should come out to be equal to 0, how do I show that the coefficient c y r 0? So, forgot, let us see a very important property of this function; what are these phi i? If you remember this phi i has a value 1 at the node with respect to all the point, with respect to they have been defined, and at all other points that we have put in the domain, these functions are 0.

So, if this function u 9 of x has to be 0 everywhere, it has to be 0 at the point x i also, is also equal to 0, which is equal to from the summation, what do we get? All other phi i(s) will vanish, only the u i will remain. This will be equal to u i. So, if I do this for all the 9 nodes, we are having the domain, what do I get that from this condition? That u 9 is equal to 0 everywhere, we get that all the coefficient u i are 0, which means, that the phi i, are… this will be called piecewise quadratic; that is, in every element they are quadratic and linearly independent.

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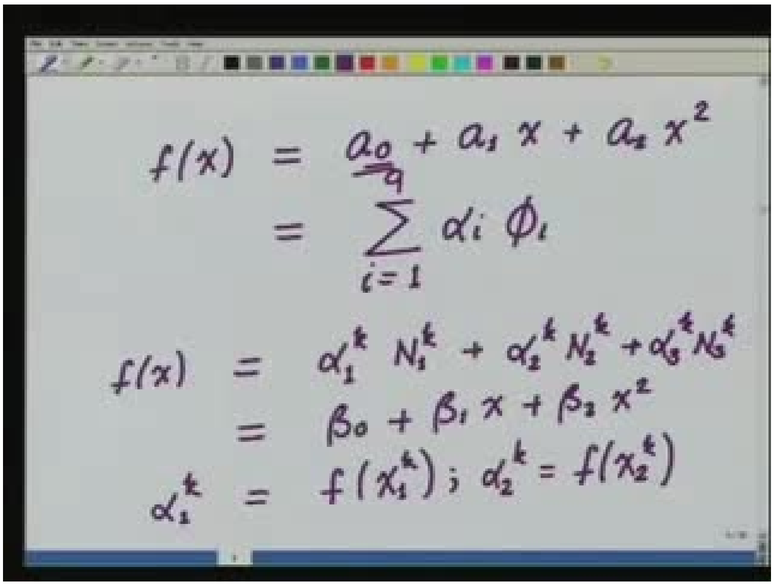

What about completeness? So, for the completeness - if we go, and what do we like, want to do? We would like to see, whether a function of this type, if the function f of x, this function is given as the generic quadratic a 0 plus a 1 x plus a 2 x square; if I have this function, then I should be able to represent it as the linear combination of a phi i; that is, I will call it as alpha i phi i. Now, what is this? Because, we are saying these approximating functions are piecewise quadratic for completeness, we want the series represented by these functions to exactly represent at least a quadratic polynomial, not more than that; so, either I have, if you see, this is, these are arbitrary coefficient.

So, this series has to exactly represent the constant represented by a 0 or a linear a 0 plus a 1 x or a quadratic, which is a 0 plus a 1 x plus a 2 x square. So, if it has to do it in the whole domain, then it should also be able to represent a quadratic in an element, so in the element if I write… So, instead of doing it in the whole domain, let us do it in the element. So, in the element, if I want to write f x it is equal to, I will call it alpha 1 of k N

1 of k plus alpha 2 of k N 2 of k plus alpha 3 of k N 3 of k, and this, let say that the quadratic polynomial that we have taken in the element, it is of this form beta 0 plus beta 1 x plus beta 2 x square.

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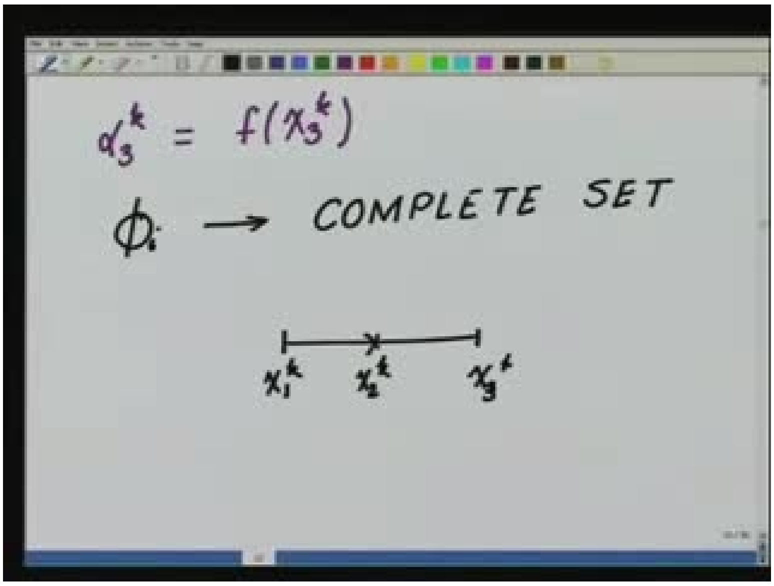

Now, by the various functions have been defined, what will be the alpha 1 of k be? Alpha 1 of k, where the equal to f evaluated at the point x 1 of k; alpha 2 of k will be equal to f evaluated at the point x 2 of k; alpha 3 of an element k will be equal to f, evaluated at the point x 3 of k. So, with this representation for the alpha, one should go and check that indeed, what we have done in the previous stage is really… I do not know, whether this is going to be like this. So, the equalities is question mark and what we can now check is that is not a question mark, but indeed this is an equality.

This is a simple algebraic exercise, and one can show that, yes, indeed just linear combination of these three functions is able to exactly use a quadratic polynomial defined in the elements and thus in the whole domain, which means that this function phi i, the set of function phi i represented in the complete set. So, this in the construction of this phi i is… one has to be very careful, one cannot arbitrarily define this function phi i as we wish.

Now, if you see this point x, another point we have to note is that the point x 2 of k that we have taken, x point x 2 of k, we have said it is at the mid side, it is not been, it could be anywhere; but as the convention, it is taken at the middle of the element always, but one should remember it need not be the case, specially when we have solve element, which we will be dealing with later on.

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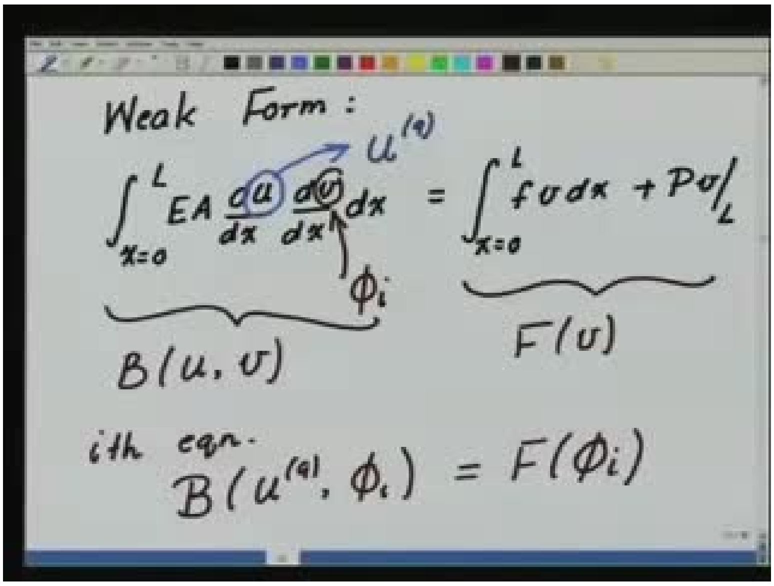

Now, that we have defined the phi i, what next? So, once we have defined the phi i, the next thing that we have to do is to go ahead and find the coefficient u i, in order to form the solution to the problem. So, to form the coefficient where do we start from? We start from the weak formulation, and for the problem that we have taken, the weak form is very easy, it is x is equal to 0 to L, it will be EA d u dx f v dx plus p into v evaluated at L.

Now, in weak form what do we do? We go and substitute, instead of u 9. So, this will be replaced by u 9 and to form the i th equation, what we do? We replace v with phi i. So, to make the notations a little bit compact, this part as we have done a few lectures ago, they are going to represent as the bi-linear form, as I said it is linear in u and v. So, it will be called as the bi-linear form given by this compact notation b u v; and this side, this we are going to call by F of u, which is the linear function and given by the name F of u, because here as the number of functions increase, the algebra get tedious. So, we would like to be compact only for that reason, we are introducing these things. So, the i th equation is given as B of u 9 phi i is equal to F of phi i.

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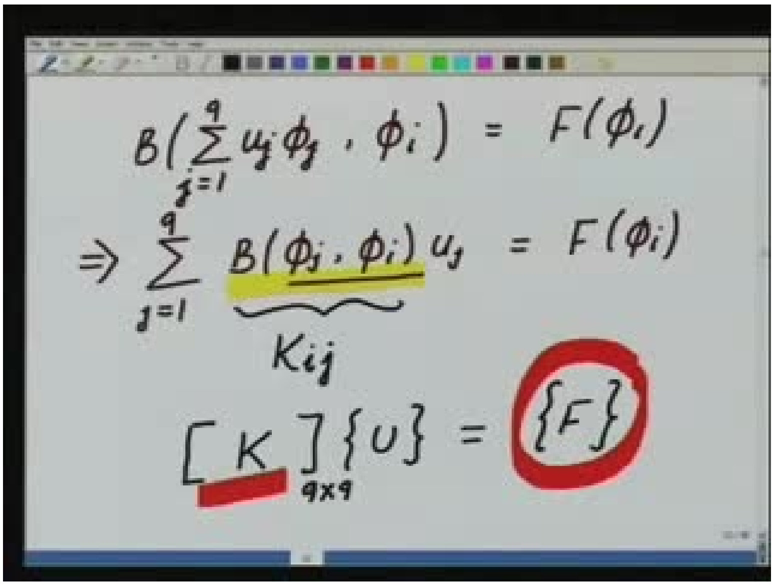

So, now if I have this, now I can expand the u 9. So, what will that be equal to? I can write it as B of sigma i is equal to 1 to 9 u i u not i - we will call it j - u j, phi i is equal to F of phi i. Now, since this is bi-linear, by the definition of bi-linear functions this summation over this coefficient I can take out. So, this is equivalent to writing form of j equal to 1 to 9 b of phi j phi i whole thing multiplied by u j is equal to F of phi i.

If you see for the i th equation, this becomes, this term becomes the coefficient of the j th unknown. So, what is it called? If you want to go back to what we have to done earlier, this term will be equal to k for the i th equation j th call. So, this is k i j. Now, our job is… So, if this is k i j, then we can write the whole problem in the form of the global stiffness matrix k into the unknown displacement vector u, where now this is k is 9 by 9, u is the vector of phi 9 this is equal to the vector F.

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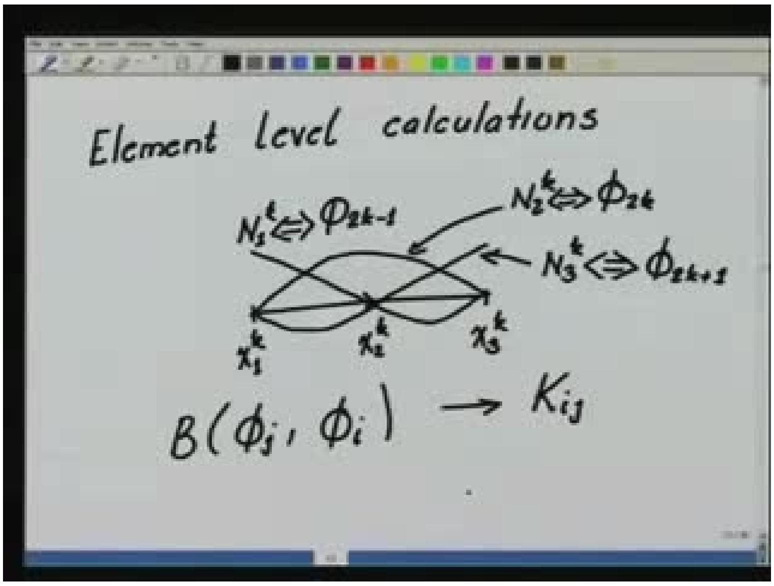

So, our job is to find the entries of this phase and of this vector. So, if we can find anything, then we have assembled to make this problem and all that we have to do is invert this matrix; first, we have to apply one-way condition, invert the matrix and solution is there. So, to do this, what do we do? We have to again go back to the element calculation; so, this is element level calculation.

Now, let me redraw the whole figures again for the element; this is x 1 of an element k, x 2 of an element k, x 3 of an element k and if you remember, so this was N 1, N 2, N 3; so, this was N 1 of an element k, which is equivalent to phi 2 k minus 1. This is N 2 of element k, which is equivalent to phi 2 k, and this one is N 3 of element k, which is equivalent to phi 2 k plus 1 in the element k. Just like we have done for the linear problem for the phi equal to 1 approximation, here what do we know, that the phi is which are non-zero in the element are these three phi(s), phi 2 k minus 1, phi 2 k and phi 2 k plus 1 equal to 1, 2, 3, 4 that is a number of up to the number of element.

So, what element is going to contribute to, is the 2 k minus 1 th equation, 2 k th equation, and 2 k plus 1 th equation, and which rows of these equation are going to be non-zero, which column of these rows are going to be non-zero? The columns which are going to be non-zero for each of these rows are the 2 k minus 1, 2 k and 2 k plus 1. So, these columns only, the element is going to contribute, and none of the other column it is going to contribute. So, if I take that b b, that I had written phi j phi i, which is nothing but as we have said k i j.

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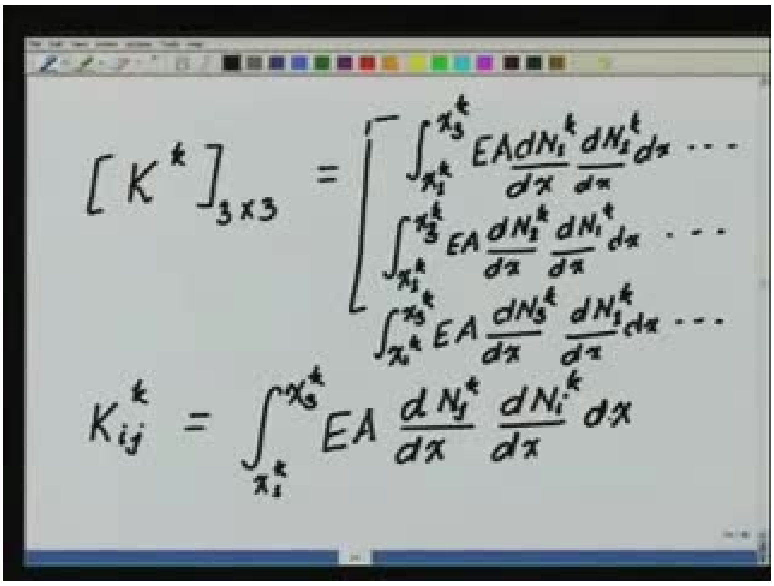

Which of the i and which of the j will get contribution from the element? As we have said, i equal to 2 k minus 1, i equal to 2 k and i equal to 2 k plus 1. Similarly, j is equal to 2 k minus 1, j equal to 2 k and j is equal to 2 k plus 1. So, let us find the contribution, how we had found these contributions earlier.

So, what we are going to define now, it is for this element k. The element stiffness matrix - now what will be the size of this element stiffness matrix? Because there are now 3 n(s) or 3 phi(s), which are non-zero in the element. So, it will be a 3 by 3 matrix. So, what will the entries be? I will write only of few entries of this and then it will be integral form x 1 k to x 3 k EA d N 1 k dx; this I will check. So, for the first row of this, I will EA d N 1 k dx into d N1 k dx will be the first column entry EA d N 1 k dx into d N 2 k dx, as the second column entry and the third will be d N 1 k dx into d N 3 k dx is the third column entry.

And similarly, if I come here to the second row, so it will be x 3 k EA d N 2 k dx d N 1 k dx and so on; and the third will have integer x 1 k to x 3 k; you see that limits of the integration are from one x exterminate of the element to next exterminate of the element. So, that is should be kept in mind d N 3 k dx d N 1 k dx d x. Can I write the k i j in the element k? This will be equal to integral x 1 of k to x 3 of k of EA d N j k dx d N i k dx d x.

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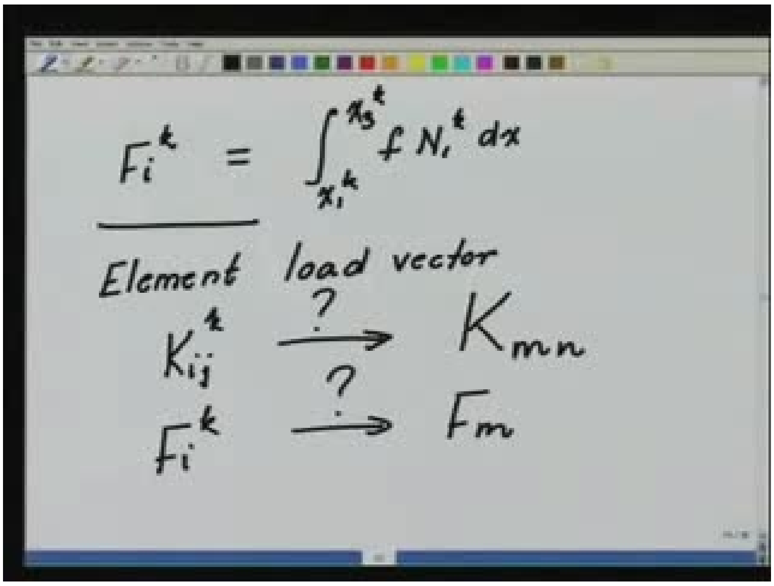

Similarly, F i in the element k will be equal to nothing but x 1 of k to x 3 of k f N i k dx; remember, that here we are not going to add the part due to the p that is going to be added, when we are applying the (( )) or the force found the condition; this is all that the F i k from the element will have. So, once we have this, then we have obtained the element level stiffness matrix and the element level load vector. So, this will go to the element load vector.

So, once I have these, now what we do? We have to find the correspondence between these element level entries and the global entries; that is, where in the global stiffness matrix should be this k i j go globally? It is which m n it should go to, that is where I have to assemble these things. Similarly, where will be the F i at the element go to globally, which F this equation has this information, we have to help.

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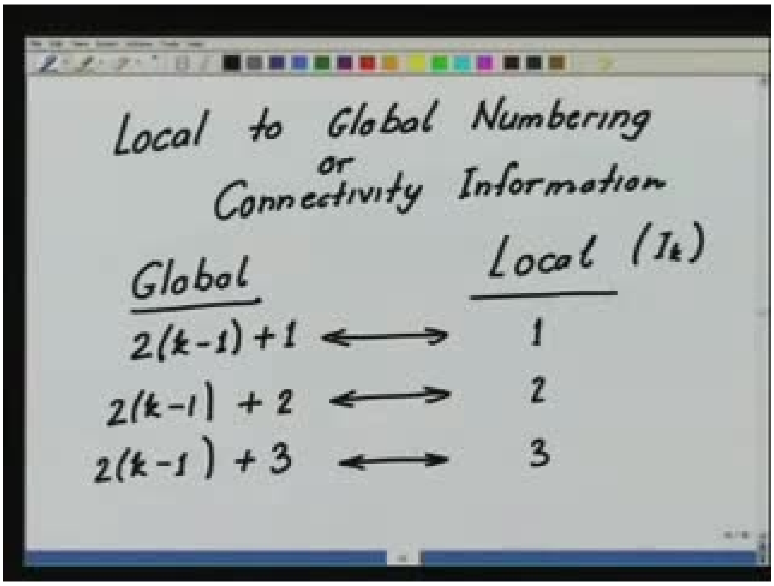

So, what has been done earlier, as far as this information is concerned, we had created a local to global numbering or 90 times it is called the connectivity information. So, this connectivity information has to be constructed. So, let us now do it for the elements. So, what I will do is, here I write the global number, and here, I will write the corresponding local number for the element k. So, this is the local number K in the element; I have how many? 1, 2, 3; this global number, does it correspond to 1? Corresponds to which global basis function? N 1 k corresponds to phi 2 k minus 1. So, it should correspond to globally. I will re-write it in another form, 2 k minus 1 plus 1 which is nothing but 2 k minus 1.

This one will be N 2 corresponds to phi 2 k globally; so, it will be as… rewrite as this plus 2; similarly, N 3 k will corresponds to phi 2 k plus 1. So, remember what we have done. So, before we go and do the element calculation, we will stretch the entries of the global stiffness matrix and the global load vector equal to 0.

The process of the assembly that we had talked about in great detail, in the previous lecture, essentially is adding the elemental contributions to this global stiffness matrix and the global load vector, in order to complete the global equation, because what we have said, is integration over the full domain can be partitioned into the integration over the elements, and when we add these elemental integrals together, we get the global representation.

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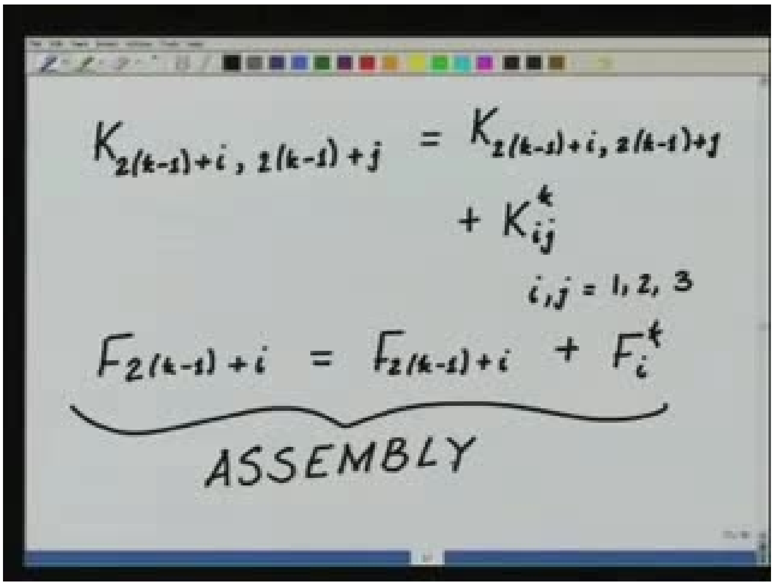

So, the global entry which is going to get contribution from element are going to be: k 2 k minus 1 plus i, 2 k minus 1 plus j is equal to… what we are going to do? It is to the original number, which was sitting there, I am going to add k i j. So, what has happened to the global stiffness entry k 2 k minus 1 plus i k 2 k minus 1 plus j? I am going to add the elemental i j th. So, this is the part, as the assembly part, as far as the stiffness is concerned and this for i, j is equal to 1, 2, 3. Similarly, the F 2 k minus 1 plus i will be equal to F 2 k minus plus i 1, to this I am going to add F k i. So, this is the assembly procedure.

So, we have obtained the element calculation, using the function which we have defined the elemental level, and assemble them to get the global equation, in terms of the basis function that we are after, the coefficient for the global basis function. One should remember, that in any finite element approximation, we are really looking for the coefficients of the global basis function, and the element calculation, is a simple way of doing the integration, relevant that have to be done, to obtain the entries of the global stiffness matrix and the global load vector by doing at the element level, adding it up to get the global equation.

One question that should come to mind now is - earlier, we have seen for P equal to 1, that the i th equation had only non-zero entries corresponding to the i minus 1 column, the i th column and the i plus 1 th column; in this case, what are the non-zero entries for the i th row?

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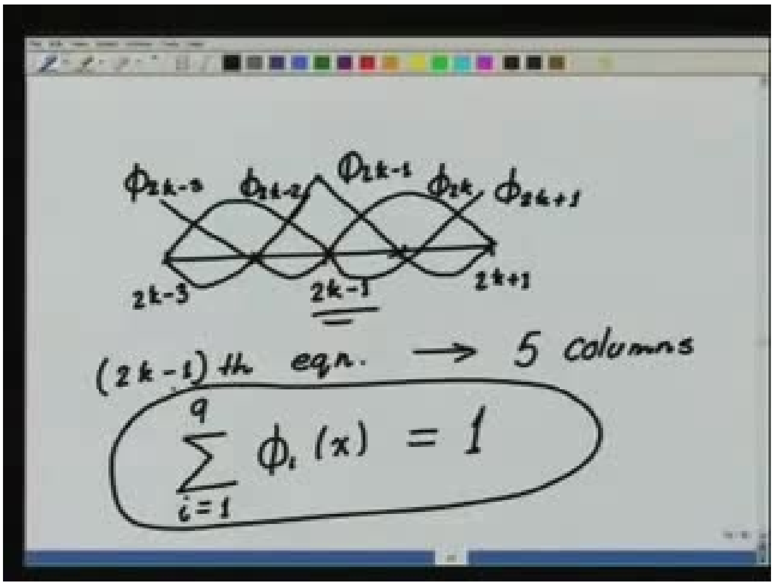

So, let us check; now take the i th row; I will take this one as the 2 k minus 1 th point. So, this is 2 k minus 3. Here we have to differentiate between this nodal function and the internal bubble function. So, let us take the nodal function that we have. So, if you see that I am talking on the equation corresponding to this one. So, I am corresponding to the equation corresponding to this function which is phi 2 k minus 1.

This is the phi that we are going to contribute to the equation corresponding to this. Certainly, it is going to be this phi, it is going to be this phi and this phi. So, this is going to be phi 2 k minus 3, this is phi 2 k minus 2, this is phi 2 k and this is phi 2 k plus 1. What can you see? That for the 2 k minus 1 th equation will have non-zero entries corresponding to the 2 k minus 3 column 2 k minus 2 column 2 k minus 1 column 2 k column and 2 k plus 1 column. So, how many columns will be non-zero? 1, 2, 3, 4, 5; so five columns will have at least, not at least, at most five columns will have non-zero entries; for all other phi(s) the column entries are going to be 0, and because all other phi(s) are going to disappear or become 0 in the region, where this phi 2 k minus 1 is defined to be.

You see that if right global stiffness matrix, the number of non-zero entries has increased in a row. This is the another property of these approximation that we have going to see and how they reflect in terms of the part city of the global stiffness matrix and that also this part city, we are talking about, we have talk about solver. So, this we will tackle at a later date. See, there is the very curious thing about this function phi k; this function phi k, if you add them up anywhere, you see their value is equal to 1; why is it so?

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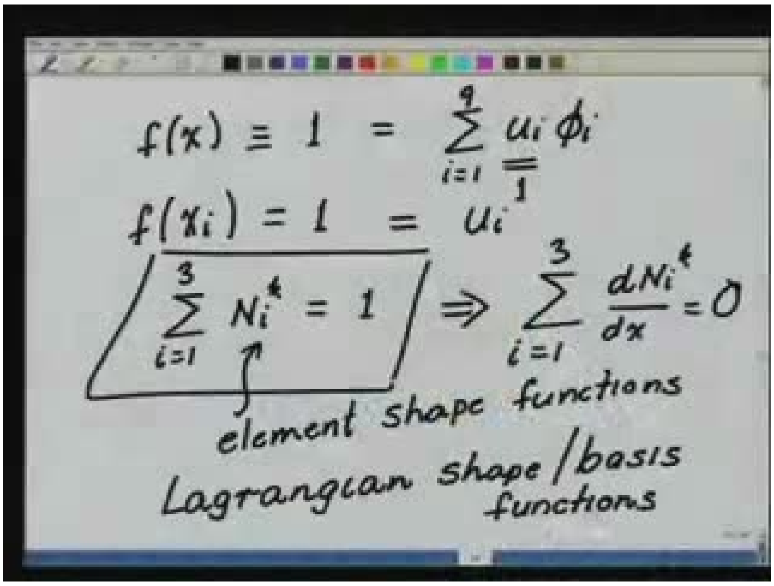

Remember, we had said that these functions phi k should be able to exactly represent any polynomial up to a quadratic. So, a constant is also global polynomial, which should be represented exactly, and in constant I will take my function f x, which I want to represent to be 1. This will be equal to sigma u i phi i. Now, is this function is equal to 1 everywhere in the domain? It has to be 1 at the points x 1, x 2, x 3, x 9. So, if I go to the point x i, this is still equal to 1, but then this will be equal to u i, which tells me that the u i here on the right hand side are all equal to 1 is the u i are all 1, then what we get is some of the phi i is equal to 1 and this is also true in the element; same thing applies in the element. So, in the element, some of the shape function will be i is equal to 1 to 3 is equal to 1; this is a property is that we should keep in mind, and what it also tells us that it implies, that if I do this summation, this has to be equal to 0, because this sum is equal to constant that delivered the constant is 0; these are certain things, which we should have in mind. So that when we go and write the computer program to do the finite element analysis, we can check all function, these basis functions of the element shape functions that we have called, which have N i is are nothing but the element shape function, we can check whether they have been correctly program or not.

The definition of this element shape function that we have used, this definition is called Lagrangian shape or basis function; we will see how to generalize these, to go for a keys order approximation in future.

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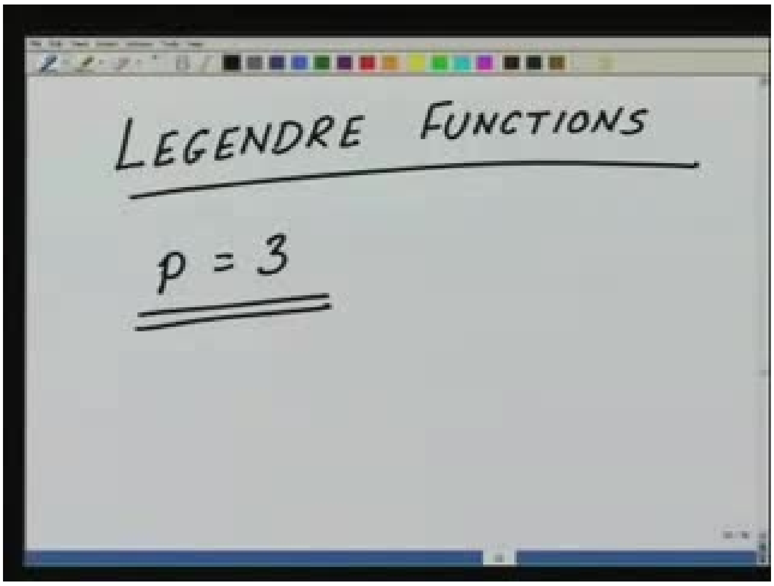

So, this Lagrangian basis functions that we have constructed, are note, the only one that we can have. So, those who are interested, there is also a very popular family of shape functions that are basis function, that can be used is Legendre function, but we are not going to discuss it, here in this analysis.

In the next class, what we are going to do is, we are going to look at the cubic approximation, the P equal to 3 approximation, also construct basis function, how to do the elemental calculation, how to go for the load vector and from there we will generalize to the P for the approximation. And once you have done the P for the approximation, then we can do a finite element computation using any order of approximation in the domain.

After that, we will see how they are going to benefit us; just doing them is not enough, why we should do them, how they are going to benefit us is going to be in the future lecture.

Thank you.