Finite Element Method

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Module - 2 Lecture - 2

In the previous lecture, we had concentrated on the Rayleigh - Ritz Method and we had shown through two example problems that using the series representation of the solution which is what we are always going to do throughout this course of this type and choosing these functions phi_i which we had called as our global basis functions.

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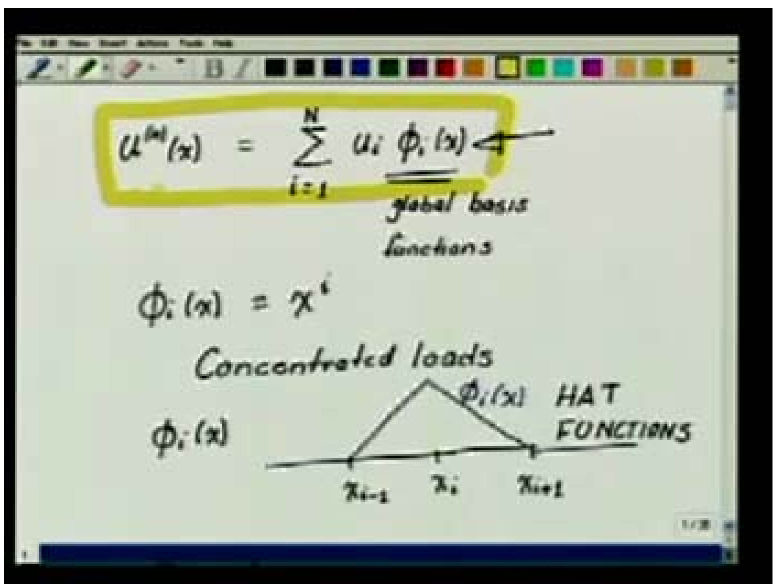

We choose our phi_i's differently. There is no restriction on choosing the phi_i to be of a particular type or of different type. It depends on what is the particular application for which we are using this representation that is going to decide what kind of phi_i we are going to use.

In the Rayleigh - Ritz Method, we have shown that one choice of phi_i's that we had taken was the polynomial type. That is the phi_i was a polynomial defined over the whole domain. We saw that for the case of smooth distributed body forces for the actual bar problem, these phi_i's did a very good job of approximating the solution to the problem. However they failed when we had concentrated loads applied at points. In fact, they did a very bad job of approximating the derivatives which are, for an engineer, of prime interest. So we said that fine these phi_i's fail for these kinds of the problems and these set of problems where concentrated loads are applied or when we have material interfaces are of practical interest to an engineer. To tackle that issue we introduced a new set of phi_i's which are again one should remember that these are globally defined functions. So we said that, let us take these points in the vicinity of the point of interest x_i and we will make phi_i as this (Refer Slide Time: 02:56). This phi_i is defined in such a way that it had value 1 at the point x_i and at the two neighboring points, these are specified points, that we have to define at these two neighboring points this phi_i became 0 and everywhere else in the domain, it was extended by 0 value.

So using these phi_i's which were the so called piecewise linear hat functions, we could again form a series representation similar to what we had done earlier. This series representation one has to keep in mind. From this series representation we found that, taking two term solution using appropriately placed points and defining the phi_i's with respect to these points we got the exact solution to the problem with the concentrated load. So this we said was the motivation for using the finite element method.

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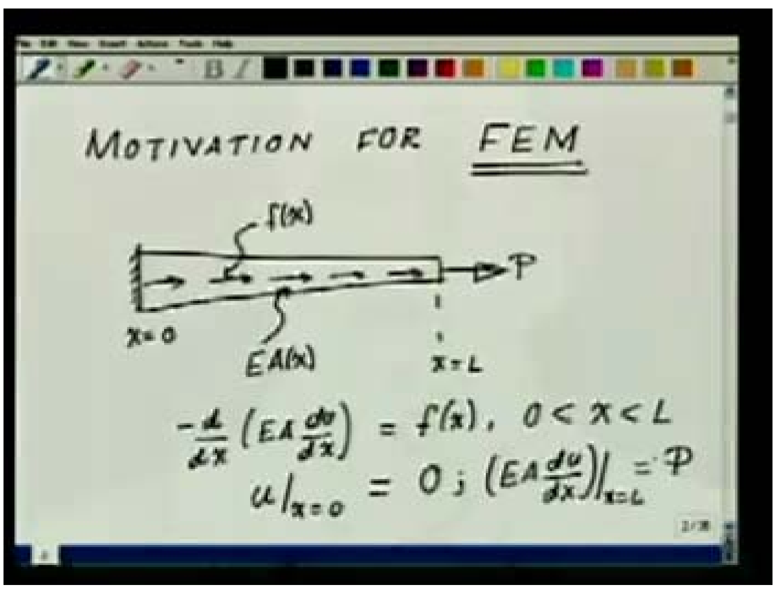

I am going to write the abbreviation as FEM. Now one thing that we had consistently followed was we had followed the regime of the Rayleigh - Ritz approach. That is, we had minimized energy functional to obtain the coefficients of the series solution. Here we are going to change track a little bit and we will always try to derive the sets of equations from minimization of an energy functional because in many of the engineering problems energy functional or a corresponding functional may not exist. So there is no question of minimizing of the functional.

We would like to follow a formulation which will always lead to the requisite set of equations for any problem of interest. Where does that come from? That comes from looking at the differential equation of the problem. So let us take again a model problem of this type.

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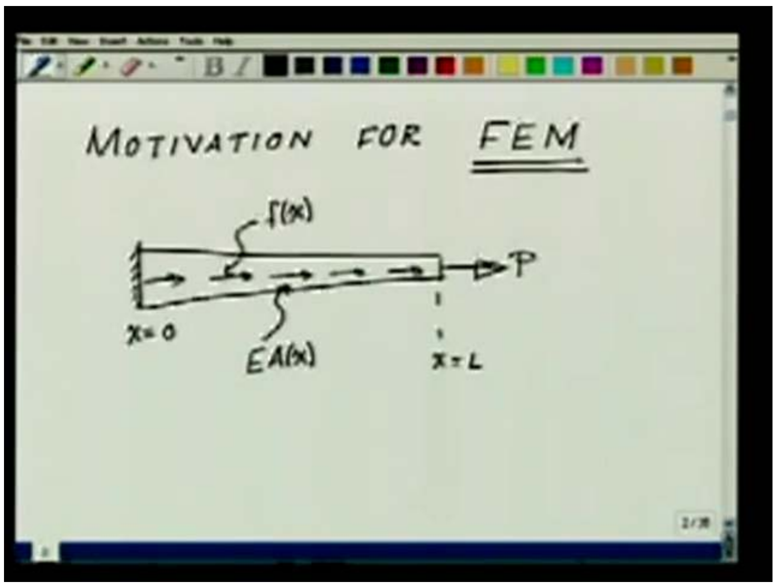

Here, we have some distributed load of fx in the body and end load P and again at x equal to0 we have constrained the body at x equal to L we have applied the end force. The material of the body varies because this is no longer an uniform beam I have taken; it varies as the function of x as EA(x). So if I write the differential equation corresponding to this bar, we have done this enough number of times already; it will be –d/dx of EA du/dx equal to f (x) for all x's lying between 0 and L. Obviously, for any boundary value problem, I would like to reiterate that we have to specify the boundary conditions also. We will have at the end x equal to 0. Remember that, we are only taking this as the specific example; this does not apply to all problems. We could take different boundary conditions at the two ends and similarly EA du/dx at x equal to L equal to P. So these are the boundary conditions for the model problem that we have taken here. Now what we had done earlier, we had talked about the Weighted Residual approach.

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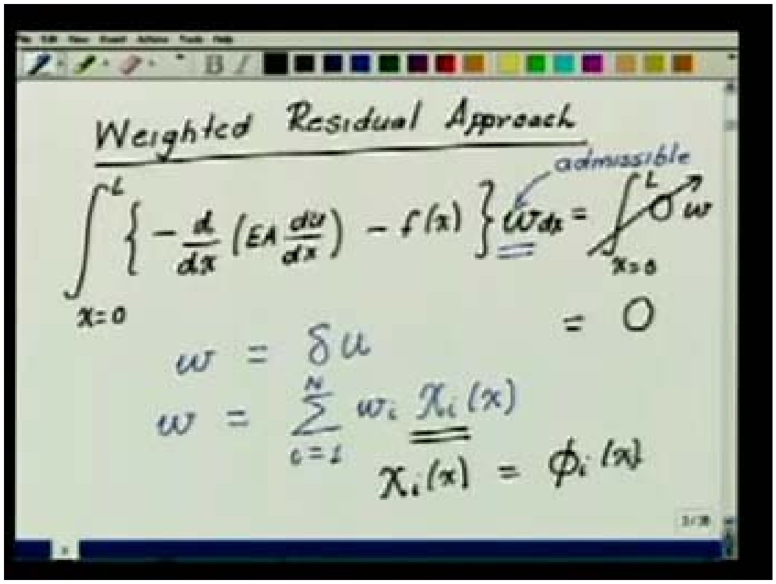

What was the idea of the weighted residual approach? The idea was, that I take the differential equation of, in this case, corresponding to the equilibrium of the bar and move the right hand side to the left hand side to get –d/dx (EA du/dx) minus f(x) equal to 0. Then, we take this whole expression on the left hand side which is called the residue. This expression is called the residue and multiply it by some weighting function w. This weighting function has to be an admissible function. Just to remember, there is something called Admissibility. So both sides I am going to multiply it with w and then I am going to integrate this expression from 0 to L. That is over the domain or over the length of the member; x equal to 0 to L, here also dx. So what we get is that this side essentially is equal to 0. So this is what we obtain. Then we had said that this is weighted residual formulation that we started off with. Now look at this w here; this function w. Till now whatever we had done, we have taken w to be specifically equal to variation of u; not any more. We are simply going to take any w which is admissible and admissibility we will have to define in a strict way later on.

Now if I take w to have representation of this type. Let us say I am using the series solution that we had found earlier; w to have representation of this type; such that this functions X_i (09:51) are not equal to the functions phi$_i$ that were used represents the solution to the problem. Then, this formulation is called a Petrov-Galerkin formulation. So when w does not have the same representation as u, it is called a Petrov-Galerkin approach.

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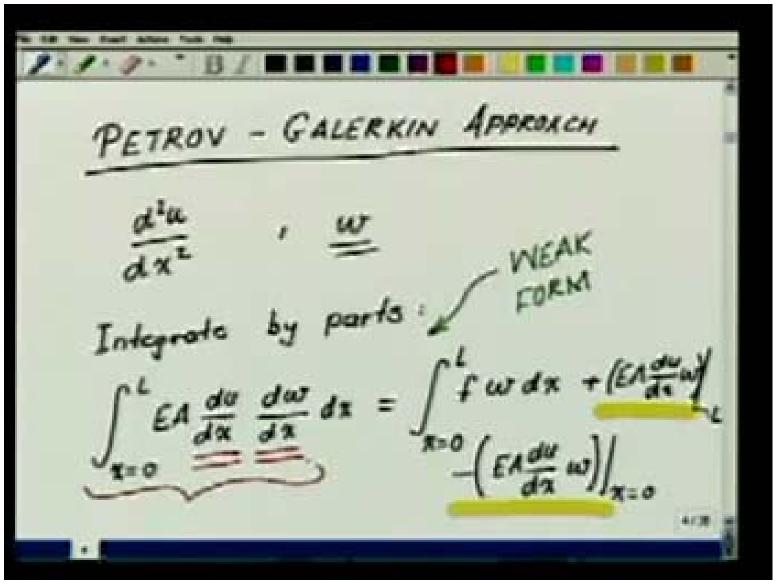

We see here again, we had already highlighted these issues in the previous lectures. Note that, in this problem the second derivative of u is involved in the integral while only w is involved in the integral. So in defining these functions for u and w there is a disparity. All we need for this integrals to be finite is that the functions phi_i should be smooth enough to ensure that the second derivative of u is finite and similarly, for w we need the function X_i to be smooth enough such that w itself is defined; that is the integral is finite. Now what we will do is, we can go ahead and integrate by parts the weighted residual form that we have obtained. We will integrate by parts once and we will see why we do that? So if you integrate by parts once you will end up getting integral x is equal to 0 to L EA du/dx dw/dx is equal to integral 0 to L f w dx plus I will have EA du/dx into w whole thing evaluated at x equal to L minus EA du/dx into w whole thing evaluated at x equal to 0.

Here we have two terms which correspond to the boundary terms. At the boundary x equal to L, this one corresponds to the boundary term at the boundary x equal to 0. So you see that naturally in this formulation by just doing integration by parts once, the boundary conditions have come into our formulation and if you remember what we called this formulation earlier this was called the Weak Form. Why was it called the weak form? Because you see that now, we have weak end, the derivative requirement for u that is only the first derivative of u is in our integral on the left hand side and we have passed that one extra derivative of u to w. So both u and w have the first derivatives on the expression on the left hand side. Another nice thing of this formulation is that, here if you look at these boundary terms;

what do we have from the boundary terms? In the boundary terms you see that if I look at this part EA du/dx and I will put this w a little apart, let us take at the point x equal to 0 or L.

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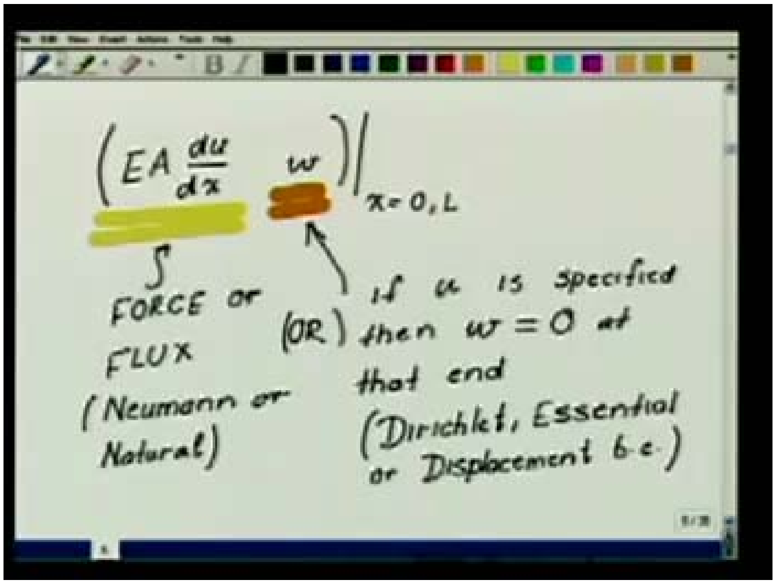

Then let us look at this part of the boundary term. What does this part specify? This specifies an axial force or FLUX, which is specified at the boundary. This is sometimes given another name; it is called Neumann or Natural. At a boundary, either this force or the flux term is specified and that is called the Neumann or a natural boundary condition. Why is it called natural? Because this term naturally appears in the weak formulation that we have obtained. Now let us look at the second part; what about this part w? This part w is corresponding to u; because what happens at an end where u is specified? The displacement u is specified. Then at that end, remember we had talked about admissibility of the w. If u is specified at that end, then w has to be equal to 0 at that end. So this set of conditions are called Dirichlet, Essential, or Displacement boundary conditions. Note that, at a particular end both these things cannot be specified. It is either or; it is either the force is given at the end or the displacement is specified at the end which means, if the displacement is specified w has to be 0 at that end.

You see that out of our formulation, we have obtained the admissible set of boundary conditions that can be applied at an end of the member. We cannot have any other type of boundary conditions that we wish. These are the only types of the boundary conditions that we can apply. For our model problem if you go back then, we end up getting an expression of this form

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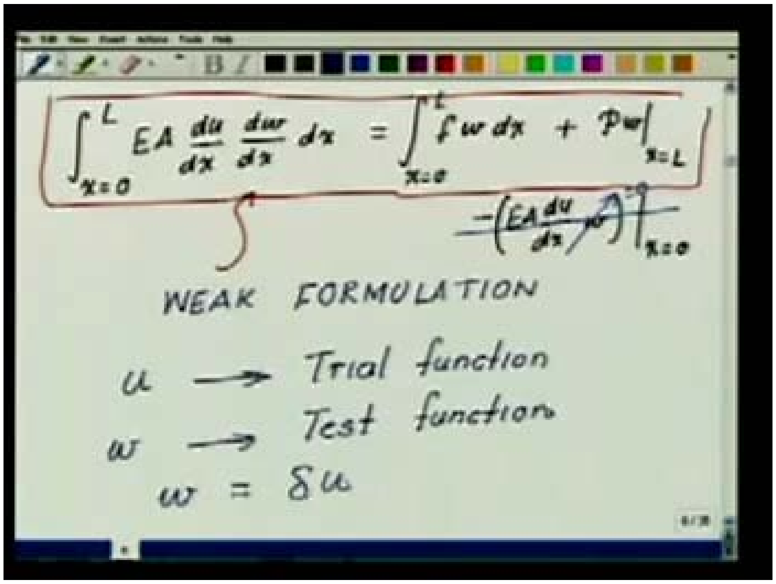

EA du/dx dw/dx dx is equal to integral x is equal to 0 to L fw dx plus the part at the end x equal to L, at the end x equal to L da/du/dx is equal to P. We put that P into w evaluated at the end x equal to L, minus da du dx into w at the point x equal to 0. But at x equal to 0, our u is specified. So in that case, at x equal to 0, I will put it, and then EA du/dx, w at x equal to 0. But you note that the end since u specified w is equal to 0. So this whole term goes. We are left with essentially this part (Refer Slide Time: 18:18). For our model problem, this is the weak formulation. Now let us again introduce some more finite element parlance. u in general in the finite element connotation is called the Trial function and w is called the Test function.

Note further, if you took w is equal to variation of u, we will get exactly the variation formulation for this model problem that we would have obtained using the minimization of the function. Note however that this is a very special problem where this is true; that is the weak form where w equal to delta u is the same as the variation formulation. It is not true in many problems of interest. So now what do we do? We said that, we have to obtain the series representation for u.

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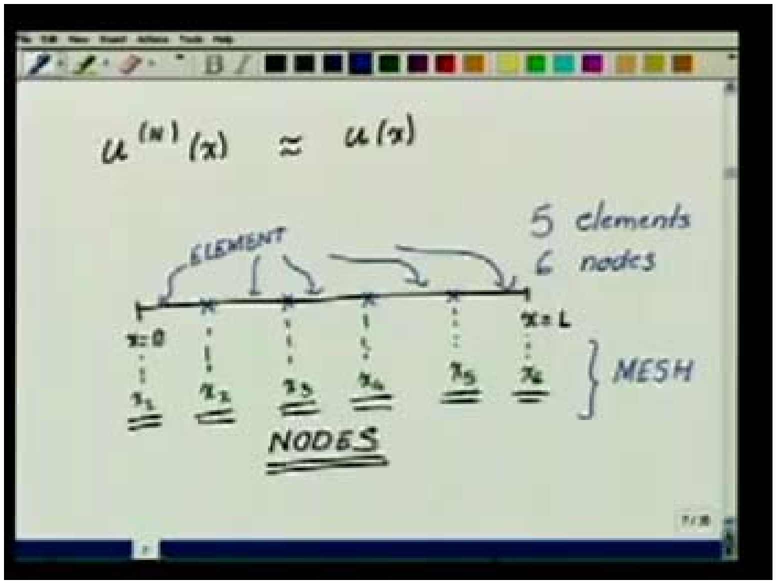

So we will say this is an approximation for u(x). To do that let us go and introduce some more terminologies. So what we will do first is, let us take the domain at point x equal to 0 as one extremity x equal to L as another extremity. In this domain, we are going to add additional points let us say we have added in this case four more points. We are going to give these points some names. This point will be the point x_1, this point will be the point x_2, this point will be x_3, x_4, x_5 and here this will be called x_6. In the same way, we can add n points. Here we have added six; we could add n points in the domain. Now these points are given certain names in the finite element terminology. These points are called Nodes and the interval in between two consecutive points is called an Element; so this will be an element. What we have in this is an element, this is an element, and this is an element. So here if you see we have 5 elements, 1, 2, 3, 4, 5 elements and six nodes. This set of nodes and elements together form the Mesh. This is the geometrical exercise that one has to do even before we start doing any calculations. Meshing is a part which is integral part of any finite element computations and this is a precursor to any solution process that we take. We have formed a mesh for our bar that we have taken. Now for this mesh, what we will do is again I will draw the mesh.

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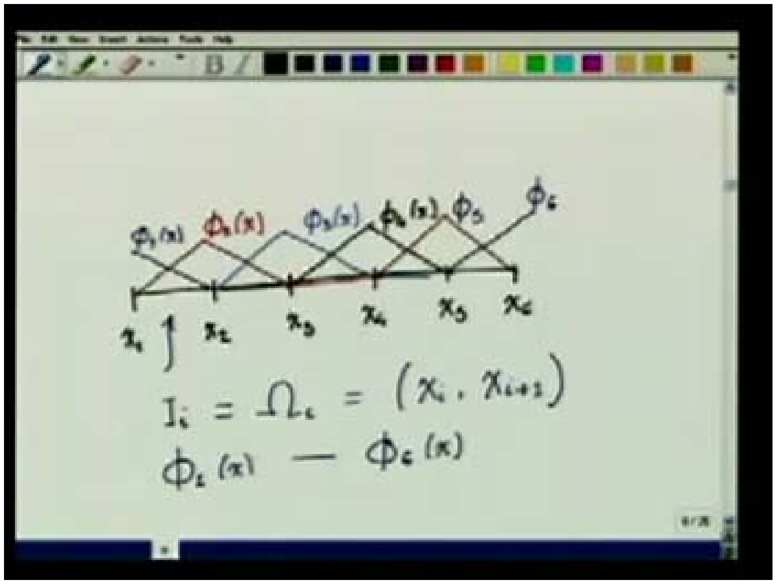

Now for this mesh, if you remember, in the last class we had introduced these functions phi_i which were somehow defined with respect to points. Here again we are going to define this function phi_i. How are we going to define these functions phi_i? This will be my function phi_1 of x such that it takes the value 1 at the point x_1 and all other points x_2, x_3, x_4, x_5, x_6 it is equal to 0. So if you see this function it is linear in the region x_1 to x_2 and it is 0 everywhere else. One more thing I would like to introduce is that these elements should be given a name. So these elements are going to be given names like this: element I_i is given by this interval or I could call it by $omega_i$ is set of all points lying between x_i and x_{i+1}. So in other words, this function phi_1 if you see it is non zero in element 1 and 0 in all other elements.

Next is, we are going to introduce the function phi_2. So phi_2 will be 1 at the point x equal to x_2 and 0 at the point x_1 and all other points. This is how we define phi_2. Similarly, we will go back and define phi3. phi_3 will be again a function which is 1 at the point x_3 and 0 at all other points. Again we can define phi_4 and then we can define phi_5 and finally we can define phi_6. This is phi_5, this is phi_6. So if you see that we had six nodes in our mesh and corresponding to these six nodes we have defined these six functions phi_i. So we get functions phi_1 of x to phi_6 of x.

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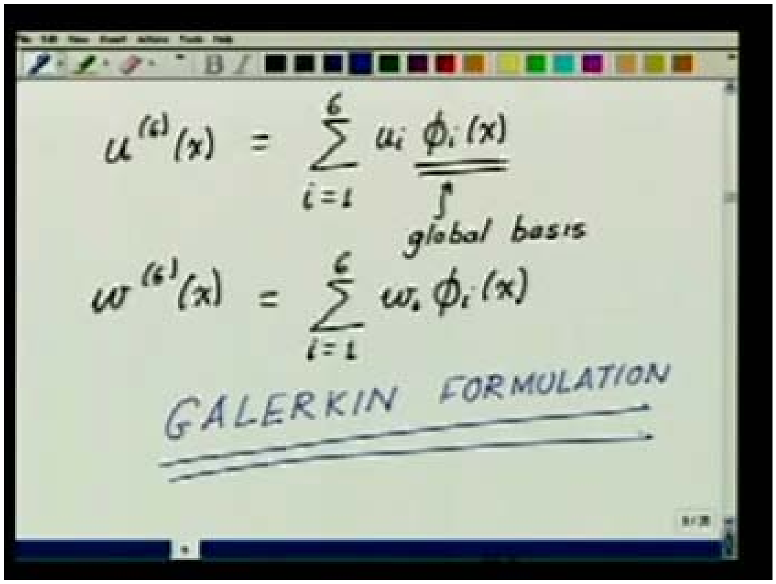

Next let us represent our u^6 of x is equal to sum of i is equal to 1 to 6 u_i phi_i of x. Again let me reiterate these phi_i's are called the global basis functions. Now what we are going to do further is that we are going to take w to be of the same form as the u. That is w that we use in our weak formulation will also be of type that is represented in terms of the same global basis functions. In fact this is what we will do throughout this course. When in the weak formulation or in our weighted residual formulation we use w to have the same representation as the u that it is represented in terms of the same basis functions, then the formulation is called the Galerkin formulation. So what we are going to do throughout this course is use the Galerkin formulation. Now we have the representation for the u and for the w. We need to find these coefficients u_i just like we did earlier; the same question arises again. How do we find coefficients u_i?

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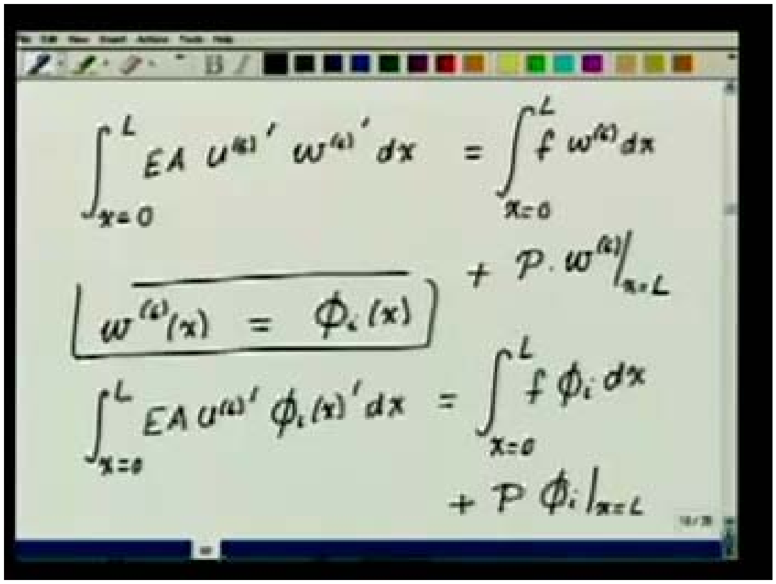

So we put this representation back in our weak formulation. So we will have x is equal to 0 to L, EA u^6 prime will call it w^6 prime dx is equal to integral x is equal to 0 to L f w^6 dx plus P into w^6 evaluated at x equal to L. So this formulation should give a sufficient number of equations to find the coefficients of u^6 that is u^1 to u^6. Now what do we do?

Since this w is the weight function, it is an admissible function that we taken it could be anything provided it satisfies the geometric conditions at the ends. That is these w_i's that we had used in the representation of w^6 is they can be chosen according to our wish. What will do is we take a choice w^6 x to be equal to phi_i of x. So if we take a choice of w^6 x to be phi_i of x we put it back in our expression that is in these series representation for w^6, we take all the w's equal to 0 excepting w_i which is set to a value 1. So here in this representation replacing it we get, EA u^6 prime phi_i x prime this is equal to integral of 0 to L f phi_i dx plus P phi_i evaluated at the point x equal to L. So this way, we can choose w^6equal to phi_1 put it in this expression will get one equation. Put w^6 equal to phi_2 put it in the expression we get the second equation and so on we can get six equations by putting phi_i in the expression for i equal to 1, 2, up to 6.

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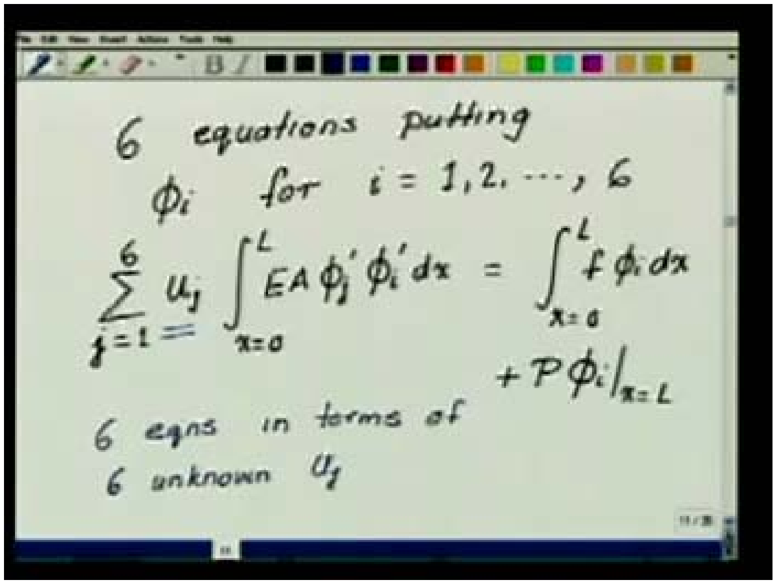

Now let us further go and put the expression for u^6 in our expressions for the equation corresponding to phi_i. So if I put the expression for u^6, I can write it like this. This will be j let us put it as j is equal to 1 to 6 u_j integral of x is equal to 0 to L EA phi_j prime phi_i prime. This is equal to integral x is equal to 0 to L f phi_i dx plus phi_i evaluated at x equal to L. You see by substituting this expression for u^6 in our weak formulation and using w^6 is equal to phi_i we get the ith equation in terms of these j unknown coefficients u_j. So we get six equations in terms of six unknown u_j and these are a set of simultaneous equation algebraic equations. Next game that we are going to play is, you see that these phi_i's are defined over the whole domain. So what we can do is, this domain by our construction we have partitioned the domain into elements given by these nodes x_1 to x_6 and elements I_1, I_2, I_3, I_4, and I_5.

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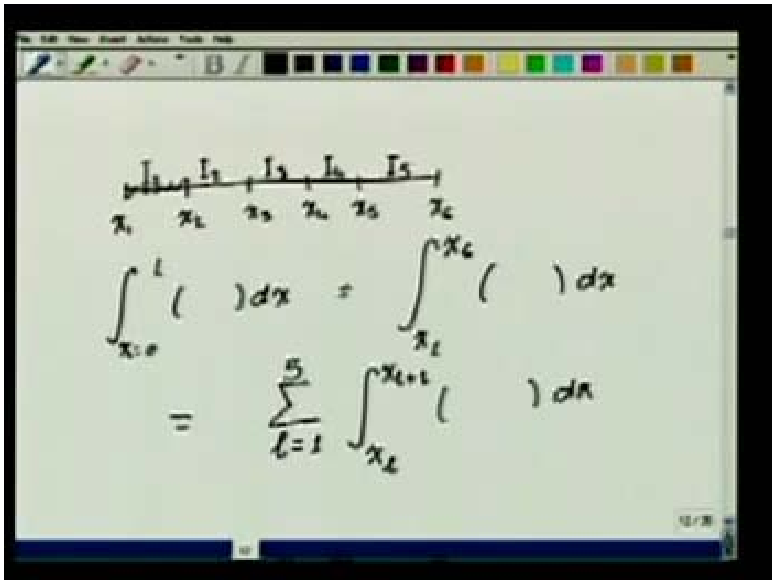

What we can do is that these integrals that we have from the weak formulation we can break them in to sum of integral over each of these elements, because integral from x equal to 0 to L of any expression dx is equal to integral in our case x_1 to x_6 of that expression dx which we can write as sum over l going from 1 to 5 so x_l to x_{l+1} that is the two extremities of the lth element and the same expression dx. This is basic integral integration that we have learnt in our earlier classes. This kind of a partitioning of the integral we are going to apply to our expression that we have derived. What do we get?

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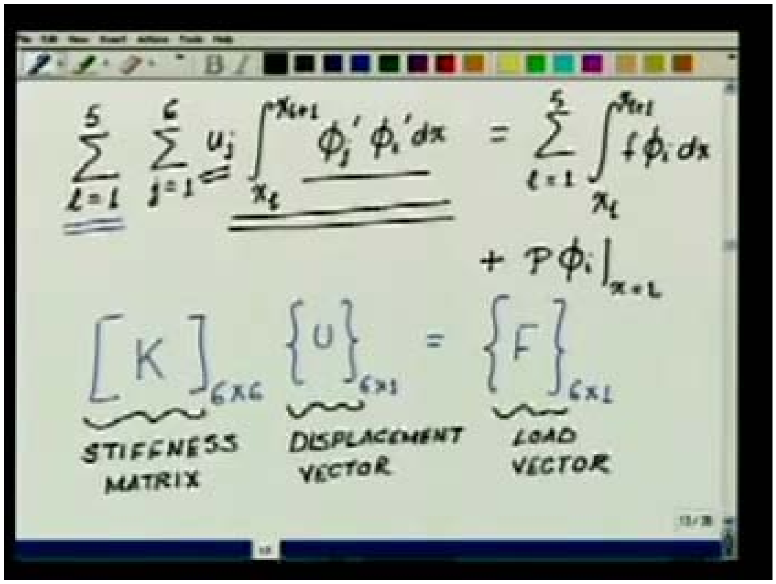

We will get sum over L is equal to 1 to 5 summation over j is equal to 1 to 6 u_j integral x_l to $x_{l \text{ plus } 1}$ phi_j prime phi_i prime dx is equal to summation over l going from 1 to 5 integral x_l to $x_{l \text{ plus } 1}$ f phi_i dx plus P phi_i evaluated at the point x is equal to L or at the point x_6, so now what do we have?

This integral we have broken into these partitions over these elements. Now if we look at it as such we can write these set of six equations as a matrix K which is of size 6 by 6 into a vector u which of size 6 by 1 is equal to a vector F. Now where does this come from? This matrix comes from this expression under the integral obtained over the whole domain. What are its uses? The element vector u are nothing but this coefficients u_j. This matrix as we had introduced earlier is given a name; it is called the global stiffness matrix. This vector is called the displacement vector and this vector is called the load vector. If I now want to go and find out what are the entries of the stiffness matrix, it is very simple. Looking at this expression here, what do we have? We have i rows which is 6 rows and 6 columns; where do the rows come from? Rows come from substituting w with the phi_i w equal to phi_i gives as the ith row and the jth column of the ith row comes out of this phi_j. So its very simple to look at K_{ij}. What is K_{ij} equal to?

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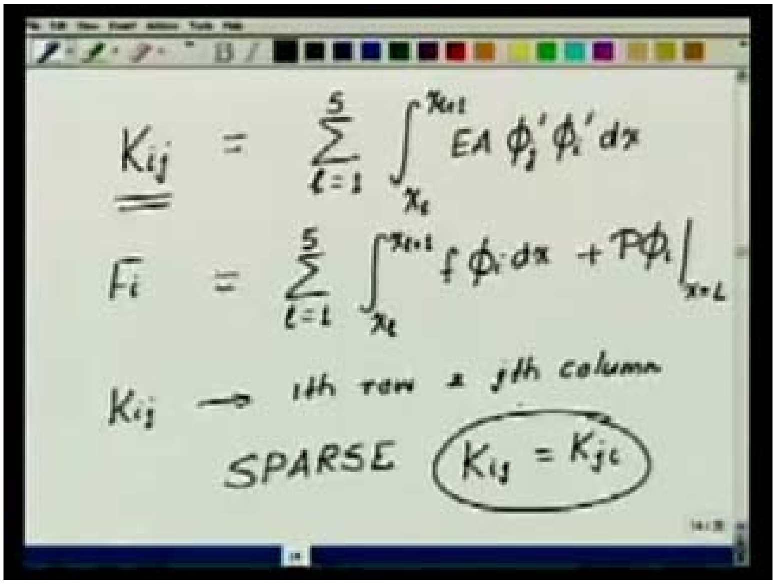

It is summation over l is equal to 1 to 5 that is over these five elements integral of x_l to $x_{l\ \text{plus}\ 1}$ EA, EA is the function of x we should remember, phi_j prime phi_i prime dx. What is the entry of the load vector f? So we will get F_i is equal to integral summation l equal to 1 to 5 integral x_l to $x_{l\ \text{plus}\ 1}$ f phi_i dx plus P phi_i evaluated at x equal to L. Lets now look at this ij th entry carefully. So K_{ij} corresponds to the ith row and j th column. Now one very important feature of finite element method is that, this matrix K is not fully populated. What does it mean not fully populated? It means that, many of the entries of the K matrix are going to be 0. That is K is sparse. Sparsity of a matrix has its own advantages as for as the numerical calculations are concerned. Another thing that you should note is that, if I go to the jth row and the ith column for the jth row and the ith column I will have K_{ji} is equal to summation integral of this expression of EA phi_i prime phi_j prime. You see that, in this case K_{ij} is equal to K_{ji}. This is another very important property of these matrices. This matrix K is said to be symmetric

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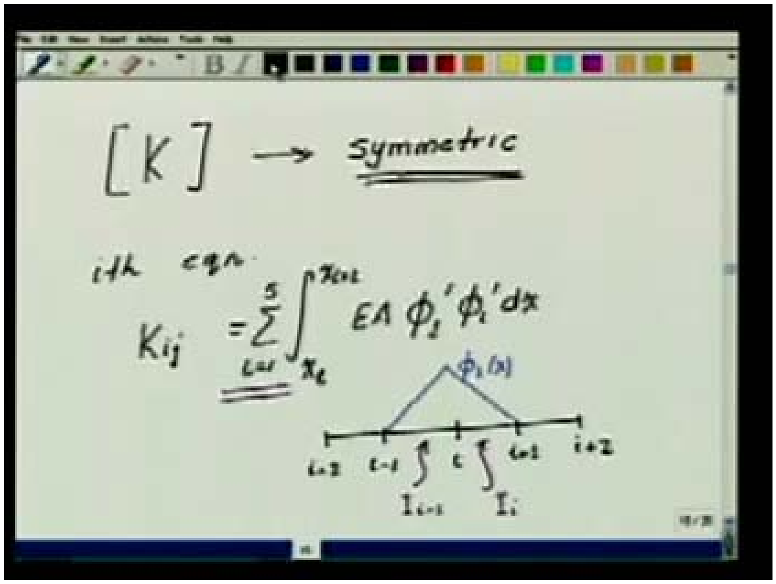

So let us now again go back to the ith equation. So we have, K_{ij} is equal to integral summation L is equal to 1 to 5 x_l to $x_{l\ \text{plus}\ 1}$ EA phi$_j$ prime phi$_i$ prime dx. Now one simple question tell me what are the elements in which phi$_i$ is non-zero. So phi$_i$ if I go and draw the mesh I will draw the full mesh if you remember, here is node I, here is, i minus 1 here is i minus 2. Okay, this is node i plus 1 this is node i plus 2. So if I look at phi$_i$ it is non zero, in only which elements in only elements i minus 1 and in element i. So phi is non zero in only in these 2 elements which is the element i minus 1 and the element i. So what does it tell me? Out of these summation over all the l's, there will be sum of these elements out of which contribution to this integral going to be 0.

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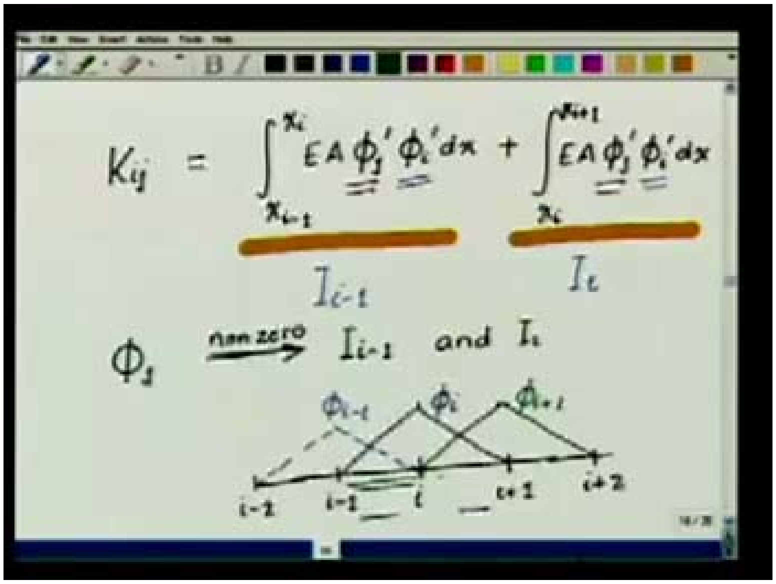

K_{ij} will be equal to integral as we have done, x $_{i\ minus\ 1}$ to x_i EA EA phi_j prime phi_i prime dx plus integral x_i to $x_{i\ plus\ 1}$ EA phi_j prime phi_i prime dx. If I highlight this, this is going to be elements contribution coming out of the element i minus 1. This will be second part will be contribution coming out of element i. You see that these are the only two elements which are going to contribute to the entries of the ith row. That is corresponding to weight function phi_i. This is one observation that we made. Now the other question is fine the integral over the whole domain is now restricted the integral over the two elements over which phi_i is non zero. Next the question is that we look at the j's. This phi_j's, now which of these phi_j's are non zero in the elements i minus 1 and i. So which of these phi_j's are non zero in the elements i minus 1and i.

Again let us draw the pictures that we had made earlier. So here is my node I, here is node i plus 1, here is node i plus 2, here is node i minus 1, and here is node i minus 2. So if you look, if you plot these figures for this phi_i's so this is going to be $phi_{i;}$ I am looking at only these two elements. This element and this element and I would like to know which of the phi_i's are non zero in these two elements. So one phi which is non zero is certainly phi_i, the other phi which is non zero here is this guy - this is $phi_{i\ minus\ 1}$ and another which is non zero here is $phi_{i\ plus\ 1}$. You see that, $phi_{i\ minus\ 1}$, $phi_{i,}$ and $phi_{i\ plus\ 1}$ are the only global basis functions which are non zero in this element which is element i minus 1 and element i. That is if I go and say that I want to chop this off here, the domain, then you see that this its quite clear that phi_i minus 1, $phi_{i,}$ and phi plus 1 are the only ones which are non zero in this chopped block.

What it tells us is that out of all the phi_j's remember that here we had six phi_j's, only the three phi_j's which are for j equal to i minus 1, j equal to i, and j equal to i plus 1 are non zero in this integration range.

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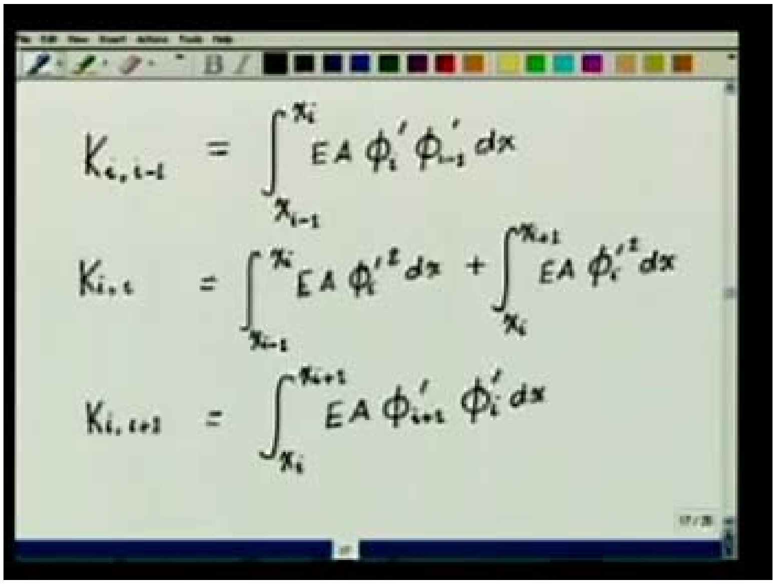

So what do we have? To make things clear we will say that, only for the ith equation $K_{i,\ i\ minus\ 1}$ that is for the ith row the ith minus 1 column is going to be equal to $x_{i\ minus\ 1}$ to x_i EA phi_i prime $phi_{i\ minus\ 1}$ prime dx. You see one thing that for the other part, that is for the element i that is from integral x_i to $x_{i\ plus\ 1}$, this function $phi_{i\ minus\ 1}$ is 0 so that part disappears that's why we are not written this. Similarly for the ith equation the ith diagonal entry will be given as since phi_i is non zero in both elements i minus 1 and I, so we will get EA phi_i prime square dx plus integral x_i 2 $x_{i\ plus\ 1}$ EA phi_i prime square dx. Similarly $K_{i\ i\ plus\ 1}$ is equal to integral. So if you see here $phi_{i\ plus\ 1}$ that is for j equal to i plus 1 $phi_{i\ plus\ 1}$ is 0 in the element i minus 1. So that part is not going to contribute to the integral even though phi_i is not so; because the product is 0. So what we will have is, integral x_i to $x_{i\ plus\ 1}$ EA $phi_{i\ plus\ 1}$ prime phi_i phi_i prime dx. These are the only three entries, column entries of the ith row which are going to be non zero. So if I write that matrix what will I get?

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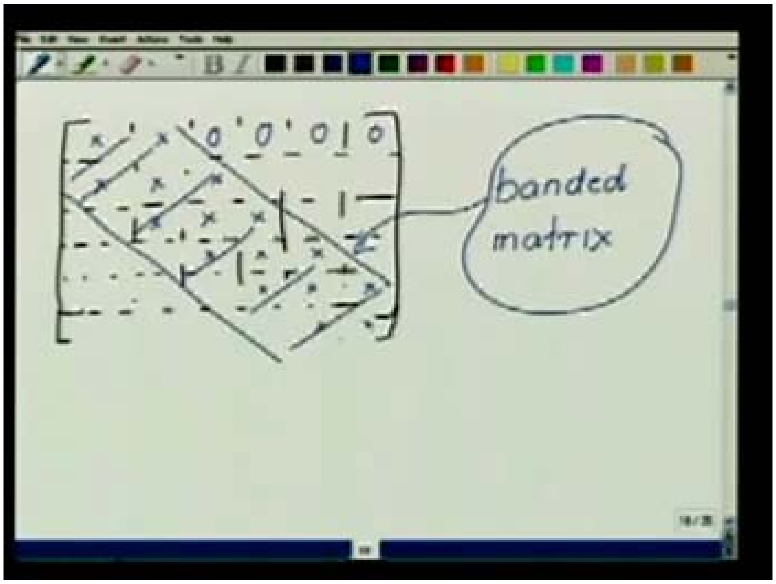

So here is the 6 by 6 matrix 1, 2, 3 here up to extended further, so in that matrix if I go and see what will happen is that for the first row only the first column and the second column are going to be non zero. For the second row, the first column that is the i minus 1th column, the ith column and the i plus 1th column are non zero everything else is going to be 0. Since it is the third row, the second column that is the i minus 1th column ith column and i plus 1th column are going to be non zero. For the fourth one, now you will have the third, fourth and fifth. For the fifth one, you will have the fourth, fifth and the sixth and for the sixth one, you will have the fifth and the sixth. So you see that this is the matrix if you look at it which has non zero entries in only these small bands. So this is call banded matrix.

This is another very important feature of the finite element method that, the stiffness matrix you get in the finite element method is generally of this form where you will have diagonal entries and the some elements in the neighborhood of the diagonal non zero and everything else is 0. These are all zero entries. This is another very important aspect of the finite element method that we should be aware of. So now where are we? If you see this is the ith equation, again rewrite it will have the i minus 1, i and the i plus 1th columns as non zero.

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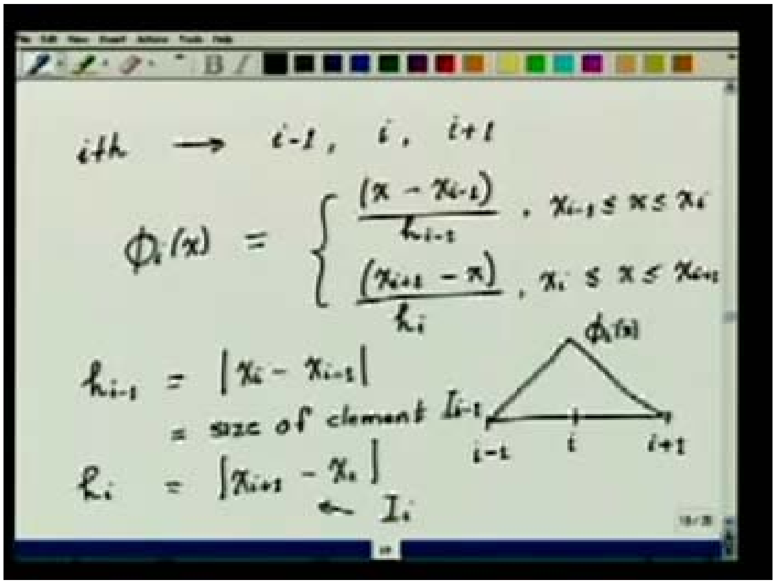

Remember that here we are talking in terms of the integral of this phi$_i$'s. Now this is all fine as for as working on paper is concerned. First of all let us see how we would like to bring it to a form which is computationally effective. That is the form which is the computer can understand. First thing is we would like to define in this functions phi$_i$ of x. If I draw it here, it is i here, it is i minus 1, and here it is I plus 1. This is what my phi$_i$ of x is such that it is 1 at the point xi. So what will get as definition of phi$_i$ of x it will be equal to x minus $x_{i\ \text{minus}\ 1}$ divided by h of element i minus 1 for $x_{i\ \text{minus}\ 1}$ less then equal to x is less then equal to x_i and it is equal to $x_{i\ \text{plus}\ 1}$ minus x divided by h_i for x_i less than equal to x less than equal $x_{i\ \text{plus}\ 1}$. What do we mean by h_i minus 1 and h_i? This h_i minus 1 is equal to x_i minus x_i minus 1. This is equal to size of element i minus 1. Similarly h_i is equal to x_i plus 1 minus x_i which is the size of the element i. Here comes another that will come across for finite element method that for all the elements in the domain.

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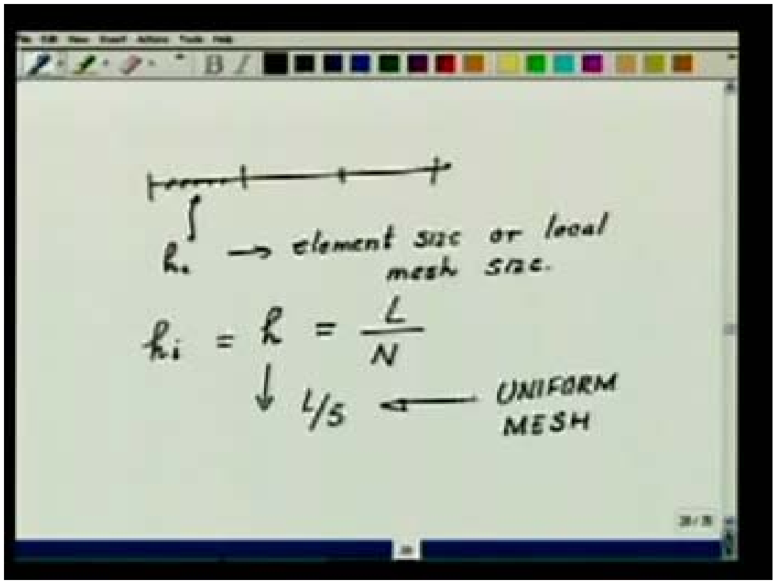

This is the element we will have corresponding to size h_i. This is call element size or local mesh size. We take the simplest possible where all these elements are of the same length, that is of the same size. Then what will happen? h_i will be equal to h which will be equal to the length of the whole domain divided by the number of elements. So in this case for example 5 element mesh in that case, h will be equal to L by 5. When the mesh consists of element of the same size then it is called a uniform mesh. So we have defined the mesh again and we have defined the the function phi_i over this mesh. So let us now stop here for the time being and recap as to what we have done in this lecture.

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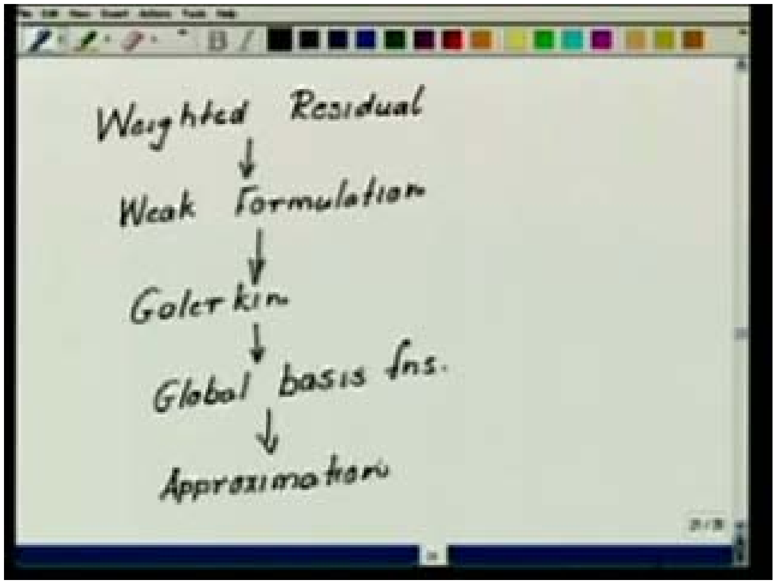

In this lecture, we have introduced Weighted residual formulation from which we obtain the weak formulation for the model problems of interest and from which we went to the specific is the Galerkin form and for the Galerkin form we introduced the global basis functions which are piece wise linear's and in terms of the global basis functions, we define the approximation. Then we said how to obtain the requisite number of equations to obtain the coefficients which are set in the approximation.

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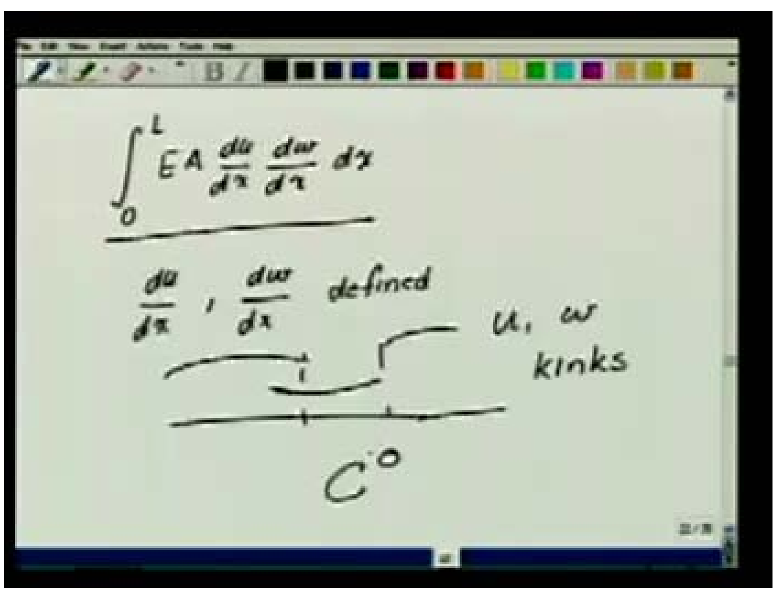

To close this part, I would like to reiterate another part and another very important part which we look at this formulation, just let us look at the right hand side only. Then you see that in this expression, in this integral the first derivative of u and first derivative of w are set. So for the integral for the finite, the first derivative of the u and the first derivative of w have to be defined. Define means that these derivatives can do this kind of the thing that is they can jump at the points. I have taken this inside they can jump at points, which means that these functions u and the w can have things. That is the reason why we have used this kind of a basis functions and this kind of basis functions are called c^0 functions. In the next lecture, what will do is we take this further and come down to equations defined at the elemental level and we talk how to find the equation, assemble them and add them together to form the global equations.