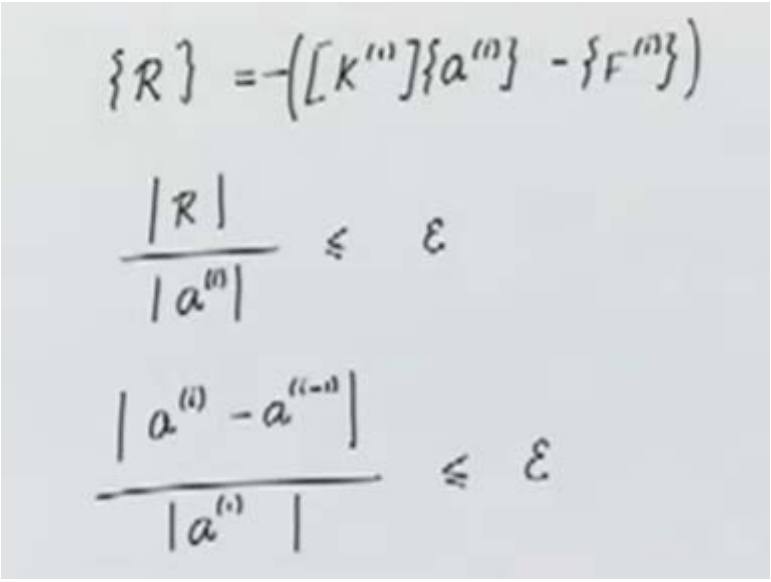
Finite Element Method
Prof C. S. Upadhyay
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Module -14 Lecture - 2

In the last lecture, we had stopped at the point where we discussed the stopping criteria for the non linear iterations and we had proposed one stopping criteria which was based on computing the residual at the residual vector R at the end of the current step. Which means this will be equal to I would say ([K i] {a i} - {F i}) - of this. This is what we have said our residual is going to be. Here by putting the computed solution at the end of the step back in the calculation of the stiffness matrix multiplying with the, taking the action of it on the computer displacement vector - F i this is what we will call as our residual to be used in the computation of the error.

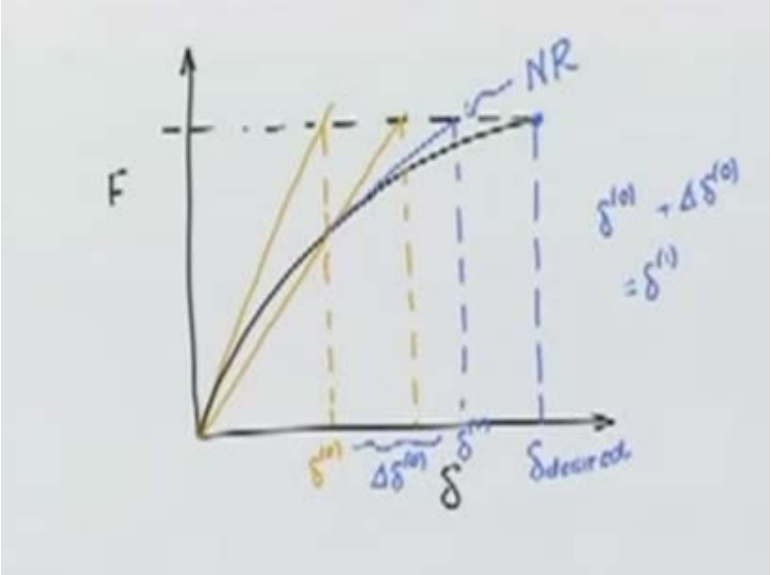
(Refer Slide Time: 00:37)



This computation and we have said that essentially if the length of the residual vector divided by the length of the solution vector is less than or equal to some tolerance then I stop. There are various stopping criteria's which are in use and the basic problem with all the stopping criteria is that we should avoid the termination of the solution process in a very very preliminary stage. That is if the stopping criteria tells me that the tolerance has

been reached and the solution has not converged then we will have a problem. There are various versions of stopping criteria, which are available another one which is available is by taking the length of the difference between the 2 solutions of the current step and the previous step divided by the length of the solution, and if this is the (2:40) then we stop.

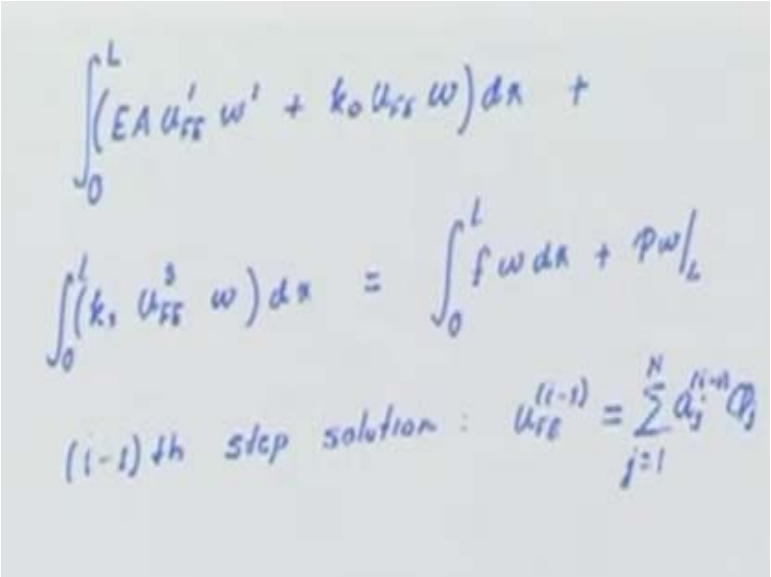
(Refer Slide Time: 02:54)



So several other stopping criteria are available, one can use any one of them. What we had said if you remember reinitiate the discussion here. That here is my F, here is my delta, here is my non linear solution corresponding to a particular level of the force of the load, here is my first solution or using the direct iteration where we essentially solve for the linear part so this becomes the delta zero, then I come down as we had said, and from here again we said that we will re compute and so on. This is for the direct iteration, what we had said is what if I come down here at the end of the first iteration and instead of doing this, I do this. Then I go to this part, I have the earlier solution to that essentially I am finding a correction. I had delta zero, to this I will add a del delta zero to get delta one. I want to add a correction so that I get the new solution and this computation of this correction to the previous solution is what we are interested in. This will be del delta zero, this will be delta one, when you see what it is doing? It is giving us faster

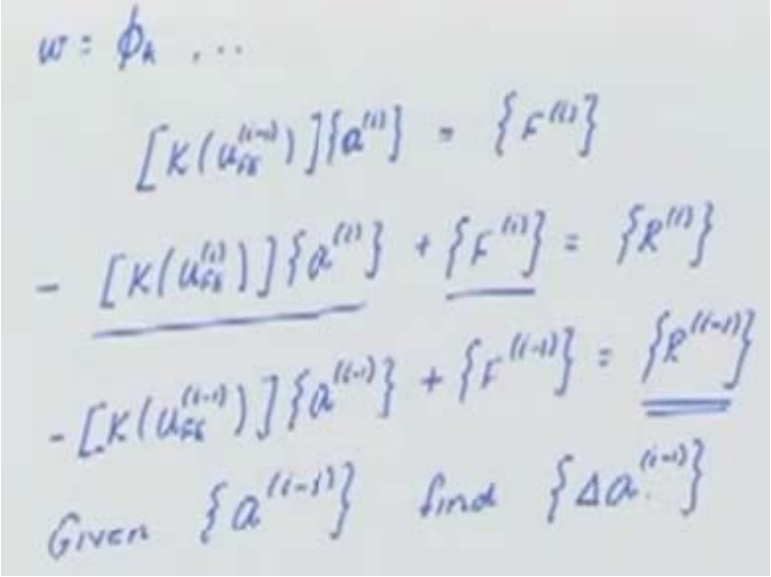
convergence to the desired solution this delta desire. This is the basic idea and what we had said this method is the Newton raphson method so today we are going to discuss this method in greater detail.

(Refer Slide Time: 05:15)



Let us go back to our problem and to refresh ideas let us have the weak form which is given integral zero to L EA u_{FE} prime w prime plus k_0 u_{FE} w, dx plus integral 0 to L k_1 u_{FE} cubed w dx, this is equal to integral of fw dx plus Pw at L. Let us say that I have obtained the solution, at the i-1th step, i-1th step solution. This will be given as u_{FE} at the step i-1, which will be equal to sigma j equal to 1 to number of nodes, a_j due to i-1 phi_j here phi_j is the bases function globally.

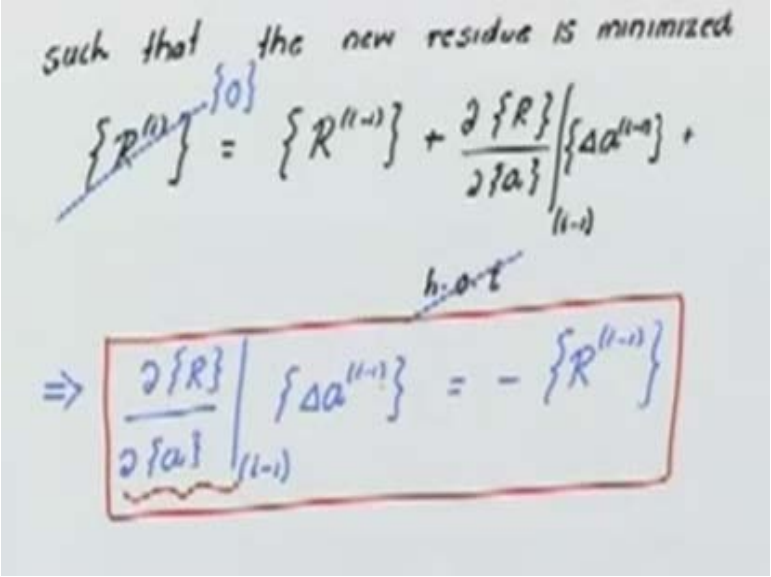
(Refer Slide Time: 07:14)



Given the solution now I can compute the residual, what will be the residual be I come back here I say that, here by putting w is equal to phi_k and so on. We get the form K which depends on u_{FE} at the step i-1 into a at the step i is equal to F i, this is what we had done earlier in order to solve for the a i. What we will do is, I will call K u_{FE} at the step i as we have done already in order to discuss the residual, minus this plus F i is equal to the vector R i. What does it mean, because the current solution is not the exact one then this matrix system will not be equal to the load vector. If it was then R i would be 0, if it is not R i will be non 0. This way I can compute R i at each step. Similarly I will have minus of K u_{FE} at the step i-1 into a at the step i-1, plus F at the step i-1 this is equal to R i-1.

This is what we have? I have the generic definition of the residual vector in terms of the generic solution vector A. Then we say that I want to find, given a i, given the vector a at the step i-1 find delta a.

(Refer Slide Time: 10:02)



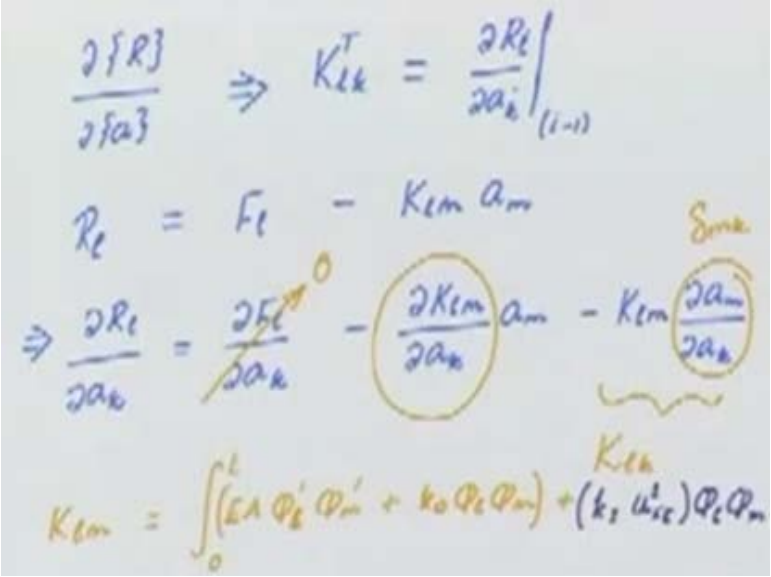
Such that the new residue is minimized, this is what we want? We will say that the residue. Given the solution at the step i-1, I want to go the step I, using this change in the a i, a i-1 in such a way that the new residue is minimized. What we will do is we will take the residue at the step i and expand it in a taylor series expansion in terms of the delta a i-1 about the residue in the step i-1. I want to expand since the residue becomes a function of a i, I want to expand the a i in terms of a i-1 around a i.

This expression in terms of a i we want to expand it in the taylor series expansion about a i-1. This will become the residue at the step i-1, plus we will have, I will write what this means. The partial of the residue with respect to the vector a evaluated at the step i-1 into delta a into the vector delta a i-1 plus I will have higher order terms. You see when I take the partial of a vector with respect to a vector I will actually get a matrix or a second order tensor. We will come to that.

We said that in the Newton rhapson method, it should be minimized rather we say it should be 0. And we are ignoring the higher order terms. And what we do? You say that well I take linear expansion about the solution obtained in the previous step, and I will

use that in order to obtain the new correction to the solution. This will give me, this implies that del R del a evaluated at the step i-1 into delta a at the step i-1 is equal to minus R at the step i-1. This is what drives our solution. If you see this, this is a matrix which is now, note given a solution at the step i-1. This known matrix into this change of the perturbation in the coefficients part is equal to the minus R and so from here we can find this.

(Refer Slide Time: 14:01)

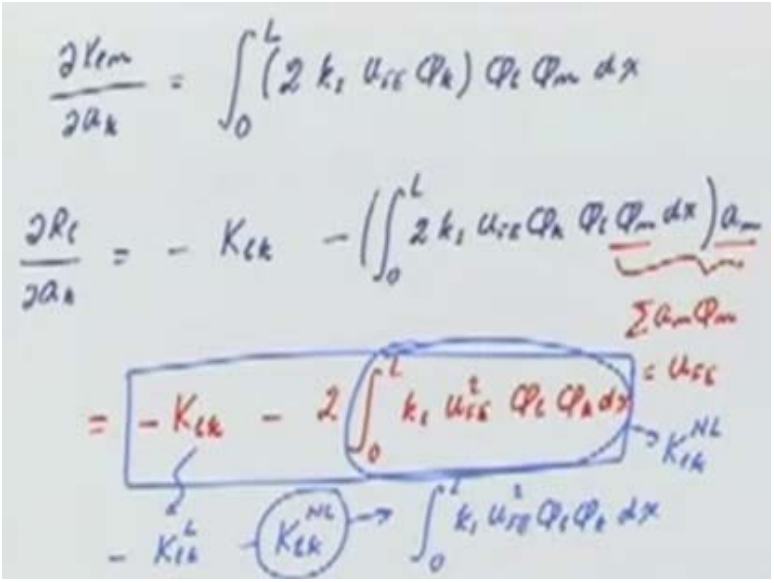


How do we write this del R? del R del a, this will be equivalent to saying that I want to find some matrix, which I will call by we will see the reason K_{ij} T entry of it is equal to del R_i del a_j, evaluated, because let me now change my naming system, that we don't have a problem. I will say lk, del R_l del a_k evaluated at i-1. How will I do it? What is R_l? The l th component of the residue vector. This will be from what we have done earlier in the definition of the residue. If you remember in the first place we had.

This is the definition of the residue. Let us go to what we want to do and we will have this as F_l - K_{lk} lm I will say, not K_{lm} has to be generic lm into a_m. If I say implies del R_l del a_k is equal to del F_l del a_k minus K_{lm} is itself a function of a_k, del K_{lm} by del a_k into

a_m minus again here we will have K_{lm} del a_m del a_k. What we have taken, our F is independent of the displacement; in the case that I have a follower load where the load is changing with the configuration F will also be a function of the displacement. Nevertheless, here it is 0. This quantity if I look at it is equal to $delta_{mk}$ that is it is equal to 1 only when m equal to k. This $delta_{mk}$, this whole quantity will become K_{lk}. This is 0 this is 1, then this quantity from the expression that we have K_{lm} is equal to integral 0 to L EA phi_l prime phi_m prime plus k_0 phi_l phi_m plus k_1u squared into phi_l phi_m.

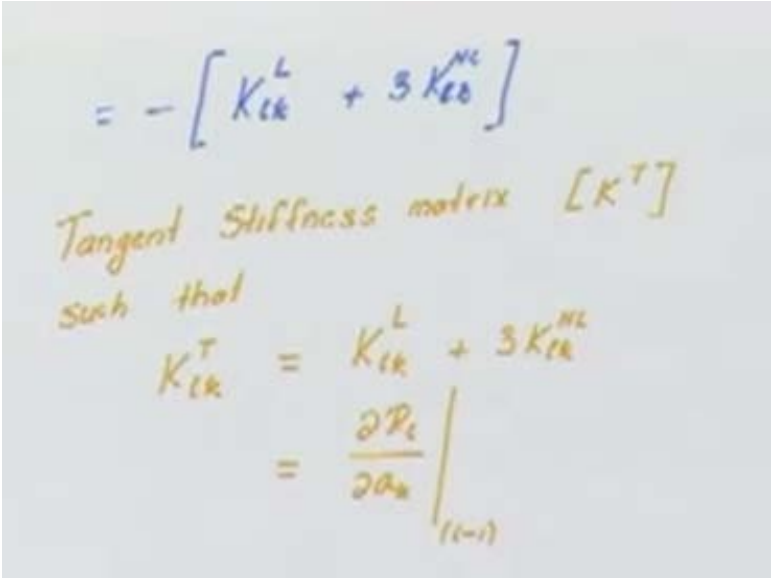
(Refer Slide Time: 18:43)



This is the definition of K_{lm}, u_{FE} squared is in terms of this coefficients themselves, this will become then I will get delta K_{lm} by delta a_k from the first part. If I come here first part the way the derivative will give me 0, because this part does not have the a_k sitting there. This part will get knocked off, the linear part will not contribute to this. Second part I will find the derivative of it with respect to…, it will become integral from 0 to L from what we have done here k_1 twice u_{FE} into derivative of u_{FE} with respect to a_k. What is derivative of u_{FE} with respect to a_k. It will be phi_k. It will be twice k_1 u_{FE} into phi_k into phi_l phi_m dx. What do we have to do? Here you see that delta R_l delta a_k is equal to minus K_{lk} plus or minus this part.

We will do that. This quantity will now become delta R_l delta a_k is equal to minus K_{lk} this K_{lk} contains both the linear and the non linear part, minus I will get here integral 0 to L $2k_1$ u_{FE} phi_k phi_l phi_m dx into if I go back this will be into a_m. Essentially since m is occurring twice, I get a_m into phi_m is nothing but a summation over this combined would be sum of a_m phi_m. And this would be equal to nothing but u_{FE}. This will be equal to minus K_{lk} -2 integral 0 to L k_1 u_{FE} squared into phi_l phi_k dx and this is my delta Rl delta a_k. You see here that this will have by itself minus of K_{lk} linear minus K_{lk} non linear, K_{lk} nonlinear part for what we have marked in the previous slide will be equal to integral k_1 u_{FE} squared phi_l phi_k. See from our definition what we have here this is nothing but K_{lk} non linear.

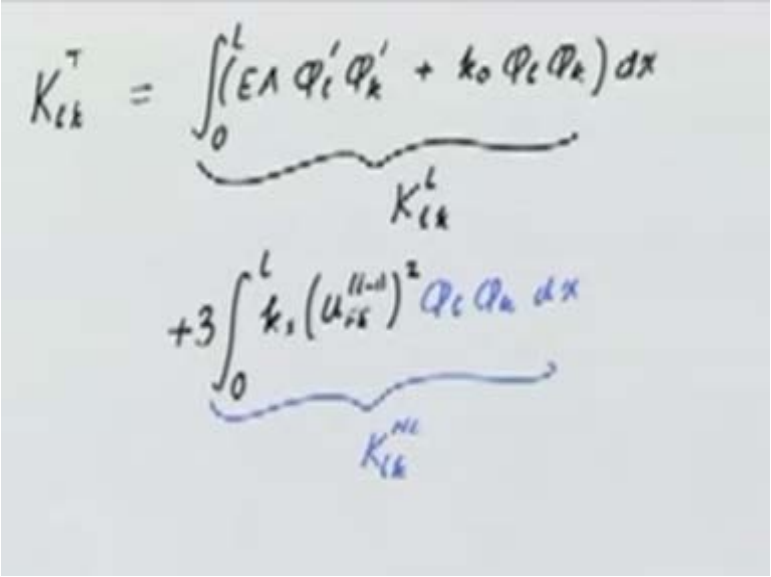
(Refer Slide Time: 22:35)



This will become equal to minus of K_{lk} linear plus 3 K_{lk}, K_{lk} non-linear, this is what we are going to get out of this computation. You have to be very careful here you see this part together they form u_{FE}. Once I have this then now I have the representation this matrix, because it came from the partial of the residue with respect to the solution vector at the current solution vector. It is called the tangent stiffness matrix.

We have tangent stiffness matrix that is the local slob in the drawing that we have made in the beginning, stiffness matrix is K T such that I will write what we mean by this K T $_{lk}$ is equal to for this problem K_{lk} plus 3 linear plus 3 K_{lk} non linear or this is also in the generic way del R_l del a_k at the step evaluated at the step i-1. The components of the tangent stiffness matrix are obtained by taking that partial of the R_l evaluated with this.

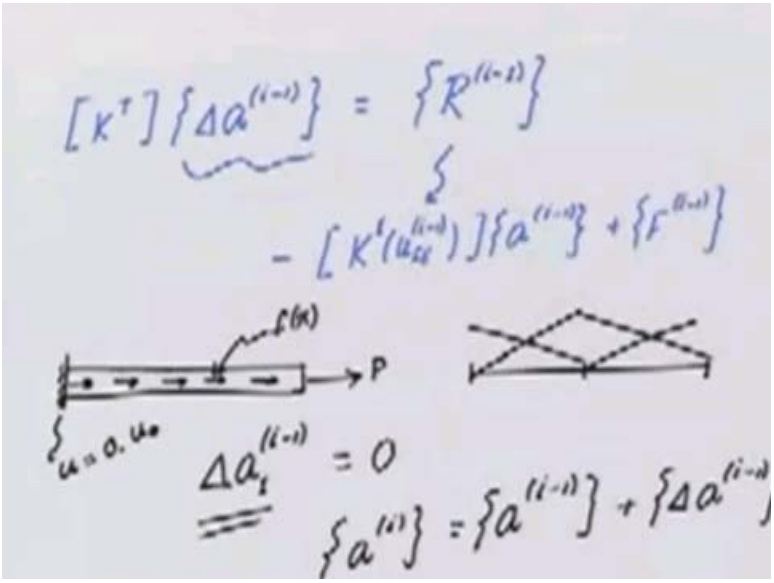
(Refer Slide Time: 24:49)



In this case what will the tangent stiffness matrix entry be because we have to evaluate it, we have not done the evaluation at the previous step. It will be for us this problem K_{lk} tangent and I have thrown out the minus part why because this minus on this side and if I go back the minus here and the minus here will cancel them out. We will be left with essentially the K T acting on delta a is equal to R and so on. We will have let's say K T $_{lk}$ will be equal to integral 0 to L EA phi_l prime phi_k prime plus k_0 phi_l phi_k dx.

This part remains fixed you don't have to update this all the time once, you have created this part which is your K linear $_{lk}$ plus integral 0 to L 3 times k_1 u_{FE} at the step i-1 whole squared into phi_l phi_k dx. This is nothing but K_{lk} non linear. For this problem this is what we will get as a tangents stiffness matrix at a step i-1.

(Refer Slide Time: 27:01)



Let me go back to my drawing here it is a local tangent or the local slope at the current solution level that we are talking about. This way we can construct the tangents representation each time and I will get the K T into delta a at the step i-1 this is equal to R at the step i-1. What is R at the step i-1, it is by putting u_{FE} at the step i-1 in the expression for the stiffness matrix and the (27:33).

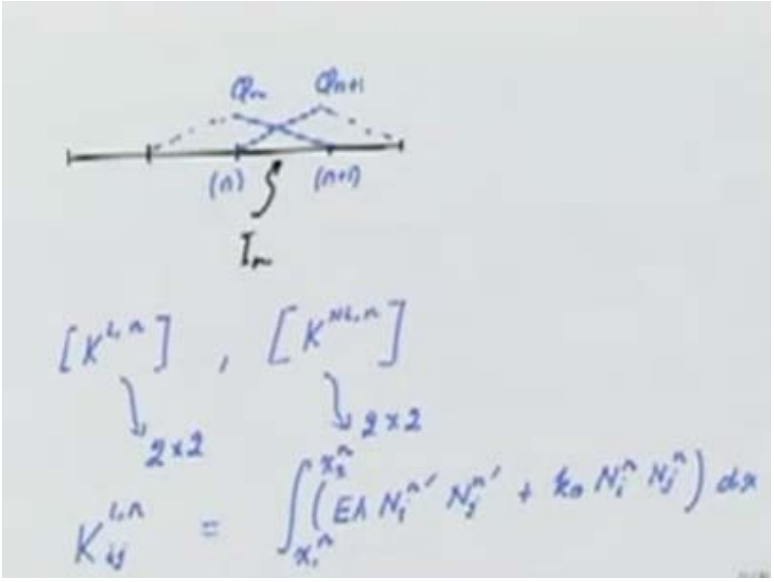
This will be K due to u_{FE} at the step i-1 into a i-1 minus with this plus F at the step i-1. This is how we are going to get the residue vector and once we solve this then we are ready to, you can compute this by inverting K T. You see the feature of K T, K T will also be symmetric for this problem.

It will be symmetric, it will be invertible what do you do with respect to the boundary conditions on delta a. If I have this bar you remember with f here, the u is given as 0 lets say or it could be given as some other value, but wherever they u is specified my delta a has to be 0 at that point.

I would get the delta a_1 for any step is equal to 0. If I take a let us say 3 element meshes. I could have the u given as u_0 bar u_0, which is non-zero, in that case I will get some contribution due to this basis function, but my delta a_1 here as to be 0. This condition has to be enforced, we enforce it using the standard penalty formulation or the way we have been doing it and we go ahead and solve for delta a i.

You get the solution use it, update a i will be equal to the vector a i-1 plus delta a i-1. This way we go ahead and construct the solution in iterative way and we stop using the stopping criteria's that we had mentioned earlier.

(Refer Slide Time: 30:18)

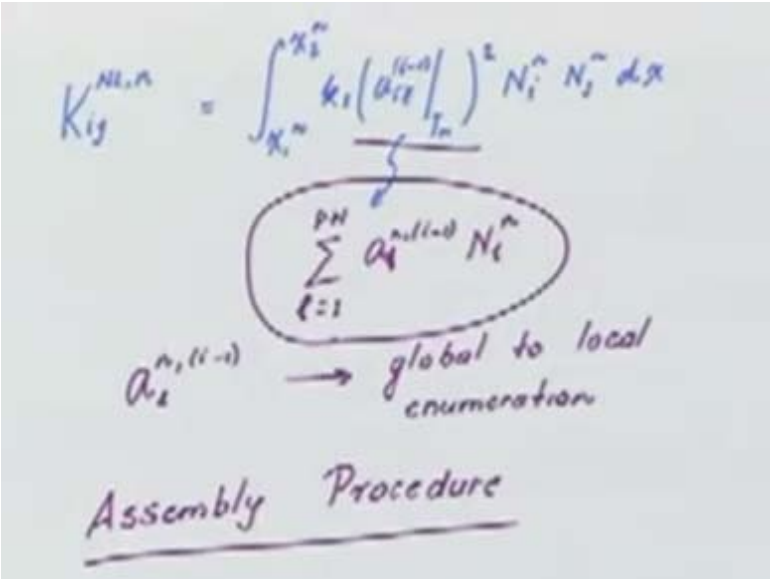


The question is how do we do it at the element level again we have let us say a mesh of some elements. Let me pick a generic element I, I_n in this element, let's say I have these basics functions which are non-zero. I am taking a linear approximation to bring the point out. I will be corresponding to the node n and n plus one phi_n phi_{n+1}.

This is what we will have at the element level. What we are going to do is at the element level we are going to compute the K linear for the element n, K non linear for the element

n. How do we compute, it will be essentially, if I have a piecewise linear approximation this will be a 2 by 2 matrix, this will be a 2 by 2 matrix. This we one can compute once and for all and keep it or we can redo every time depending upon the size of the problem, and the convince we have with respect to storage. Then this K_{ij} linear in the element N will be equal to integral x_1 of the element n to x_2 of the element n, EA N_i of the element n prime, N_j of the element n prime plus k_0 N_i of the element n, N_j of the element n this is where i and j goes from 1 to 2.

(Refer Slide Time: 32:27)

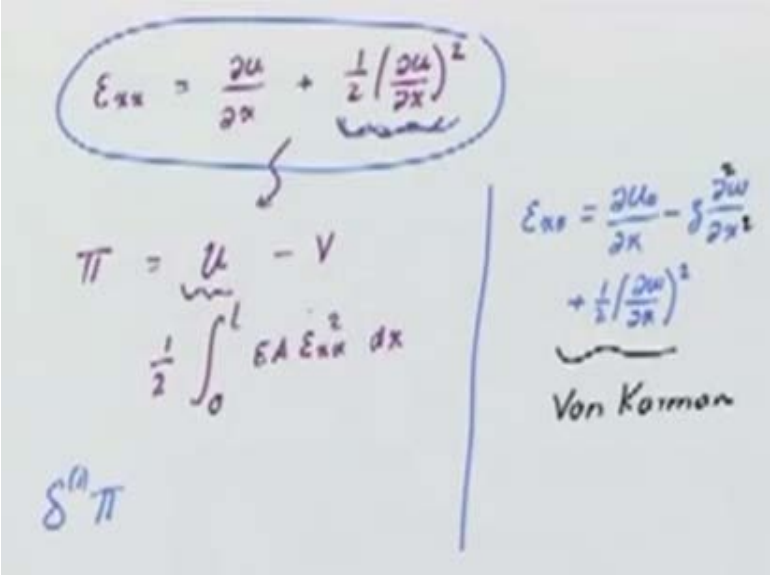


In this case in the generic case it will go from 1 to P +1. Similarly K_{ij} non linear in the element n will be equal to integral x_1 of the element n to x_2 of the element n, k_1 into u_{FE} from the step i-1 in the element n whole thing squared into N_i in the element n N_j in the element n dx. What is this? u_{FE} in the element n this will be sigma i equal to 1 to P + 1, a for the element n i or I would put it as a_l of the element for the step i-1 into N_l for the n.

This is what the u_{FE} in the element will be now what is a_l n i-1, this will come from the global to local enumeration. We have to keep in store the solution at the previous step that is at the i-1 step from where we extract the coefficients, use it to construct solution at

the function u_{FE} in the element n. We go to numerical integration, we will evaluate this function at each integration point find the square of it put it there multiply with the value of the shape functions N_i N_j at these integration points and we do the summation as usual.

(Refer Slide Time: 35:08)



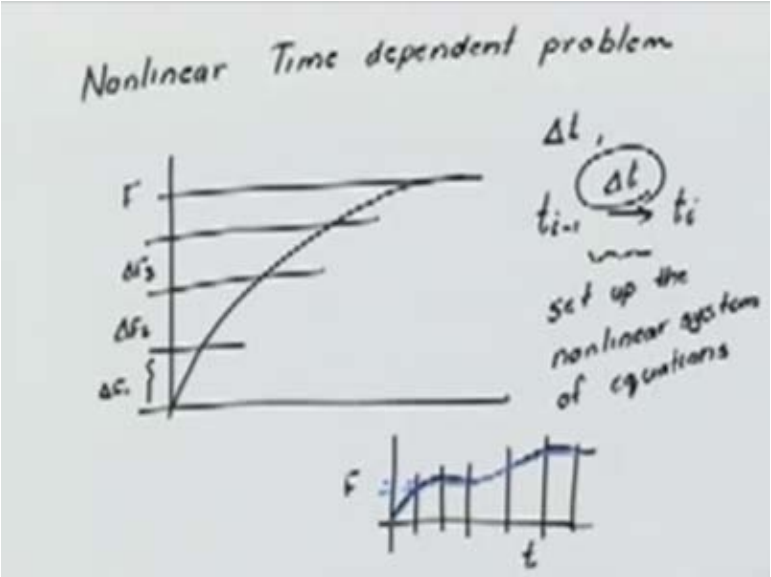
This will give me the element wise the non linear stiffness matrix. Then we follow the usual assembly procedure using the global local enumeration that we had talked about till now, we have outlined a procedure with which we can construct solutions to non linear problems. We could also do the same for the case of large deformation problems where the strain e_{xx} is given as del u del x plus let's say half of del u del x whole squared. Let's say we use the standard strain energy expression, the minimization of the total potential energy here this can be put in the expression for the total potential energy pi which will be in terms of u-v for the bar problem, this u will have half integral 0 to L EA e_{xx} whole squared dx.

We will have exactly the same scenario, as we got in the previous problem where this quantity is now going to contribute to the non-linearity of this quantity. When you take

the first variation of pi from there we will get the non linear expression in terms of del u del x, again we do the same procedure of finding if I am using the direct solution then I will go ahead and find a solution using this solution I am going to update the solution to the next solution or if I am using the Newton raphson method I will have to find the tangents stiffness matrix arising of this expression.

Similarly I can have the case of the non-linear beam deformation, where in the case of the beam e_{xx} would be equal to del u_0 del x minus z del w del x plus half del w del x whole squared this is from the von karmon theory. Again you see that in the strain itself I have this non linear term in terms of del w del x sitting and del u del w del x squared here.

(Refer Slide Time: 37:35)



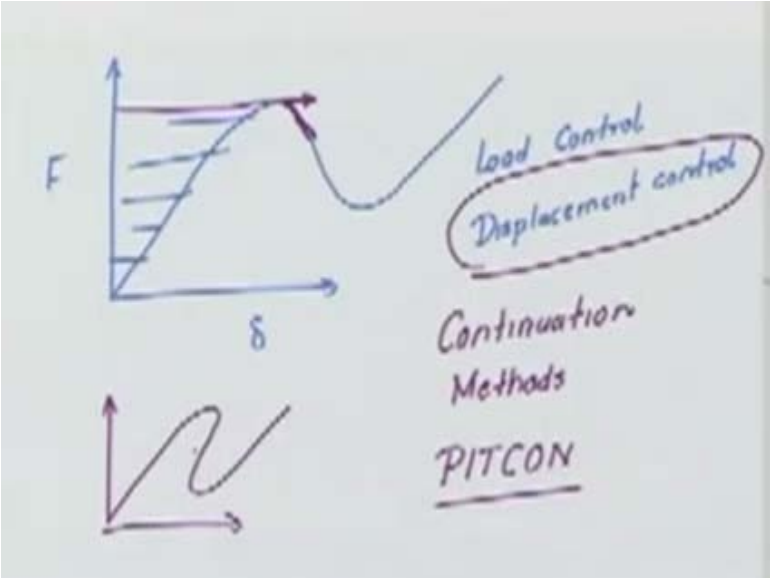
Sitting there and this again has to be taken care of in following the same procedure that you had followed till now; the other issue is that when I have a non linear time dependent problem. In this case that is a good idea that lets say I want to go to a particular load level. Let's say this is the F. Its better in fact I could have done it for the problem that we have been discussing till now the Newton raphson regime that I break the loads into this small steps delta f_1 delta f_2 delta f_3 and so on.

And I solve the problem in steps for these incremental loads. When I am in a time dependent regime then I will have to frame first the semi discreet formulation with the non linear stiffness matrices sitting there and the mass matrix if it is non linear sitting there. Then at every time station, I am at a time station delta t, I am at a time station i-1. I want to go to the time station i, using the delta t. Then here I set up the non linear system of equations and we can use the standard Newton raphson strategy that we have done and we can solve this. The good idea will be that break this into small steps the time station is broken into small steps for that time station, the load is broken into small steps.

That is the amount by which the load increments. This is my t, this is my f. I will have this. I will go to the increments in the load and when I will solve the problem accordingly. I will have to set up the non linear systems of equations at every time step and go ahead and solve it.

This procedure has to be handled in a very very careful way, specially the convergence or the stopping criteria has to be properly formulated boundary conditions have to be properly imposed in order to get a good solution. If there is something wrong in the code you will see that the number of iterations taken for convergence will increase significantly or you may not even converge.

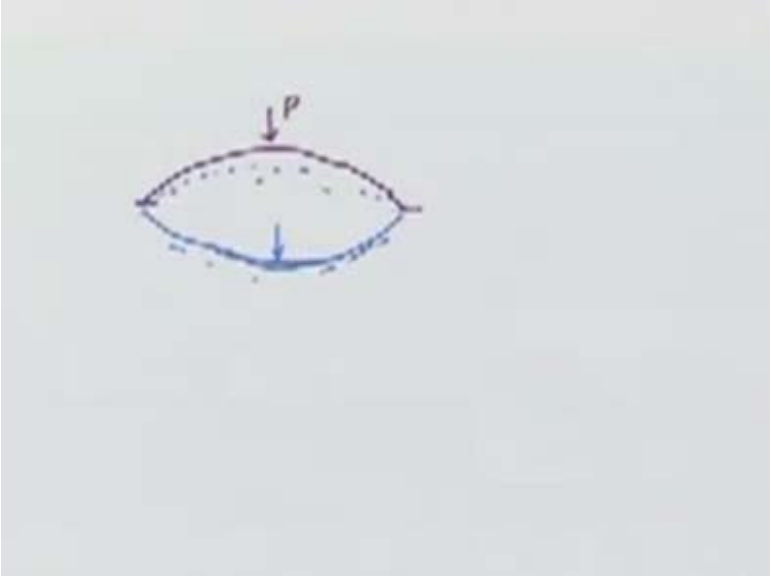
(Refer Slide Time: 40:37)



If you look at these problems where it does happen that the load response curve does something like this then we have to get into something called here. We are increasing the load clearly. The generic approach is called the load control approach. If I have this kind of situation I will have to go to a displacement control approach which means that at this point I cannot really go beyond this point with my load control approach, because here the tangents stiffness matrix will become singular. I have to do something in order to come to this part of the solution, where I go to this displacement control or we will use the so called continuation methods which are available in the literature.

There is a nice code which is downloadable from the pits bern site, called PITCON (41:55) called continuation method code. Which will solve all this exotic problems where the solution does this kind of a behavior or it does this (Refer Slide Time: 42:05) this is called a snap through, this is called a snap pack.

(Refer Slide Time: 42:16)



Especially this kind of a situation, the snap situation arises when I have an arched member and I am applying a middle load (42:20). What will happen, this member will initially deform like this, slowly as the load is increased and suddenly it will come to this configuration. Where it will become stable after that it will keep on deforming like this, it snaps to this configuration and then it becomes sufficiently stiff.

All these things can be done for that you need to go these advanced courses, the bottom line in this course has been to enforce the concept of the series representation of the approximate solution in terms of the special basis functions which we have created, which this creation of the basis functions is a essentially the heart of the finite element method, because they have local support, they have the continuity requirement that we have enforced and it is possible using this approach to tackle any problem, come up with the variation formulation or the weak form and construct suitable approximations to the problem, with that we would like to stop here.