Finite Element Method
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Module - 14 Lecture - 1

In this lecture, we are going to discuss how to solve non linear problems using the finite element method. Non linearities in physical problems can arise due to many reasons for example you may have material non-linearity as in the case of plastic deformation of materials, materials with damage in them, materials with cracks, growing cracks, so on. If you have a response curve like this, this is stress against strain (Refer Slide Time: 01:08), this is a plastic curve I could have non linear elastic behavior that is load up and down along the same curve, here I load up and come down along a different curve.

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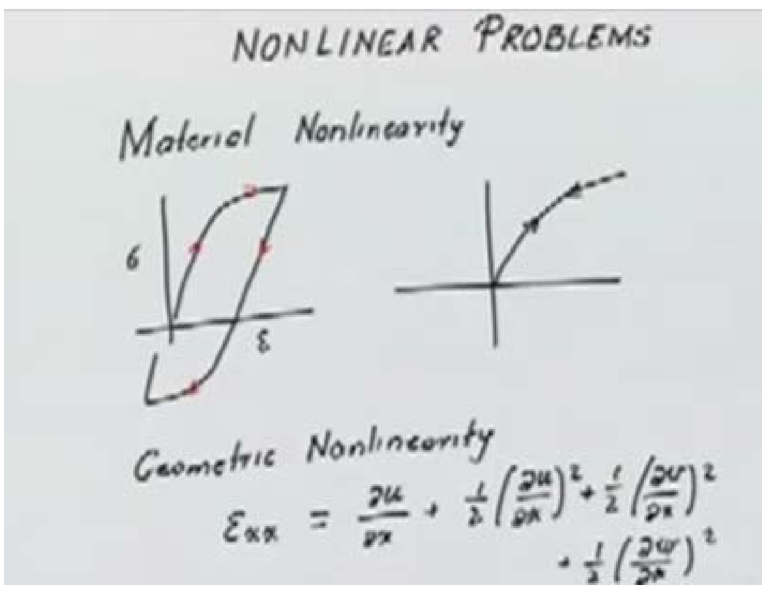

This is essentially a non linear elastic problem, this is plastic problem here there is dissipation. You could have also geometric non-linearities, geometrical non-linear problems, where the stress or the strain is the green strain that is it is a function of not only del u del x here for example but it will also have the other parts which is half of del u del x whole squared plus half of del v del x whole square plus half of del w del x whole squared. We may have various sources on non-linearity, we could have a lossy material in any material which has loss, dissipates energy in its response that will lead to a non linear phenomena to a non linear problem.

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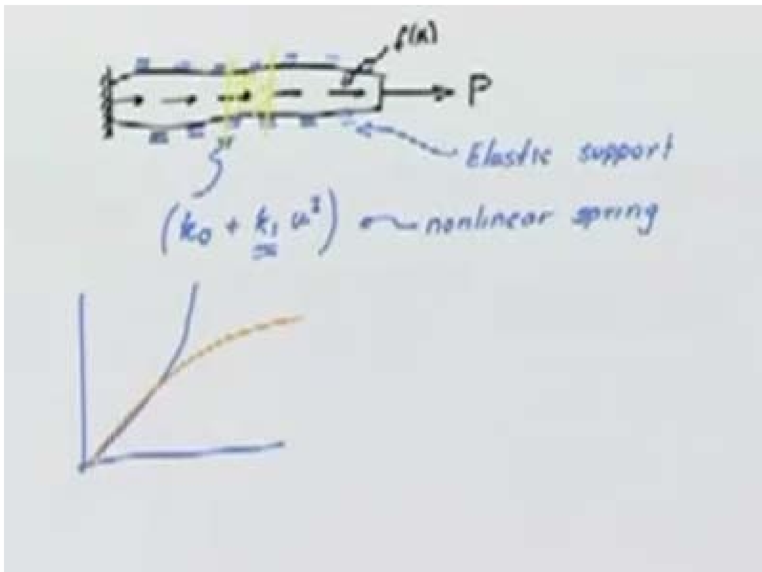

We are not going to look at these very complicated problems, but we will look at the essence of solving non linear problems through a very simple one dimensional problem. We go back to our bar problem here is a bar subjected to an end load P and in axial load effects as we have been doing till now we are not talking of the time dependent problem now we are only sticking to the static problem and this is resting on an elastic support but, so this is a continuous elastic support imagine that this is a rubber pad on which this is resting, so elastic support, but here the elastic support can be modeled as a distributed spring, its spring constant which changes with the position x.

We will have the spring constant in such a way that the elastic spring constant will be given through a constant part plus something which depends upon the displacement itself. Let's say I have this quadratic effect. This is a non linear spring. If you see that if the k_1 is positive then we have essentially as the displacement increases a hardening behavior, if the k_1 is negative as the displacement increases we get a softening behavior. Accordingly we have this kind of a representation of the spring. Let us take the k_1 to be positive.

The question is how do we go ahead and bring it in our formulation? What we change with respect to our formulation again we will take if I come here I will take a small piece at a distance x in that piece I will look at equilibrium.

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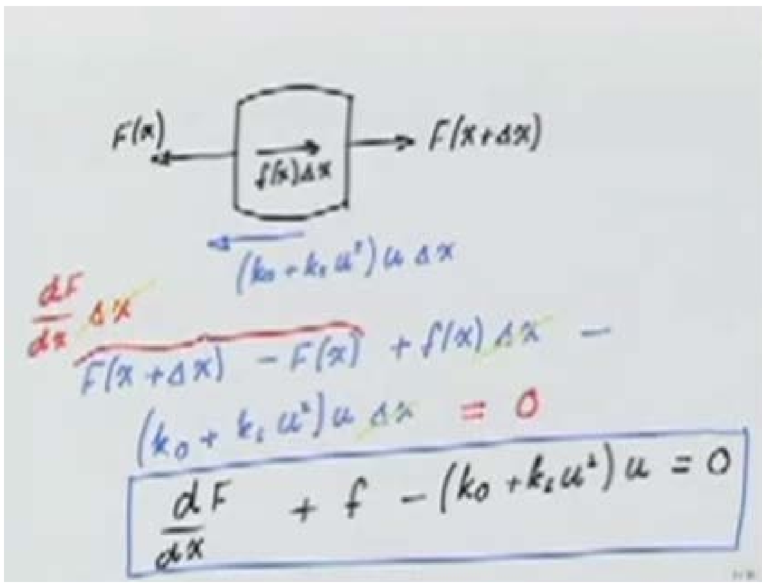

I look at this piece at a distance x here is the force f(x) delta x, here is the actual force f at x plus delta x, here is the actual force F at x and since you have put it the springs on both sides you will only get an actual force of the spring which will be opposite to the motion. Which is given as k_0+ k_1u squared into u delta x, because this is given as, spring constant is given per unit length, that into the length delta x into the displacement of this part. If I look at the equilibrium equation I will get F at x plus delta x minus F at x plus f(x) delta x minus k_0 + k_1u squared u delta x this quantity would become df dx delta x by expanding F(x) plus delta x above x in linear expansion ignoring the higher order terms and this is going to be equal to zero. This is what we will have? You can cancel off or through out this delta x parts delta x delta x. What we get is df dx plus f minus $k_0 + k_1u$ squared into u is equal to zero. This is our non linear problem.

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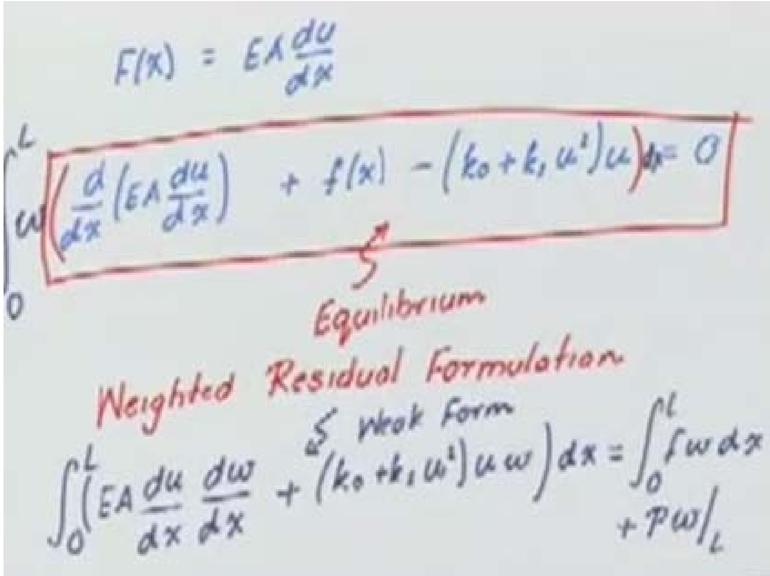

Here F as a function of x is equal to, as we have done earlier EA du dx, in terms of u it becomes d dx of EA du dx plus f of x minus k_o plus k_1u squared into u that is equal to zero. This is our differential equation corresponding to equilibrium. Next what do we do? We would like get to the weak formulation, how do we go to weak formulation? We start with the weighted residual formulation. Weighted residual, here again we do the same thing as we have been doing till now, we take this expression, multiply it with w and integrate from 0 to l put dx here (Refer Slide Time: 09:28). We integrate it from 0 to l again we make the same arguments that here the second derivative of u is sitting. We would like to relax the smoothness requirements on you so we will do an integration by parts, integration by parts will lead to integral from 0 to L, EA du dx dw dx plus $k_0 + k_1$ u squared uw dx is equal to integral 0 to l fw dx plus p into w at l, this is our weak form.

This I am sure all of you can derive with your eyes close, because this has become so standard in all the things that you have been doing till now, you have to start from this step in order to be able to construct a finite element approximation.

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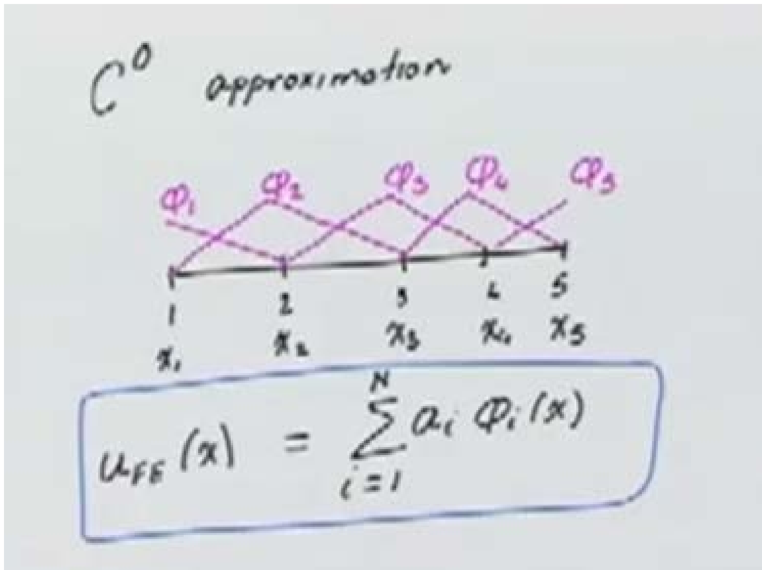

We have the usual requirement that is what is the smoothness requirement on u and w, we want du dx dw dx to be defined, we can do with C 0 elements, C 0 approximation. Again we can construct a mesh of elements with nodes 1, 2, 3, 4. Let's say I have made five nodes coordinates $x_{1,}$ $x_{2,}$ $x_{3,}$ $x_{4,}$ x_5 and again we can make the basis functions. I am taking the linear ones here, you can extend it to the higher order one using exactly what we have been doing till now. This becomes my phi_1, this becomes the phi_2, this is phi_3 phi_4, phi_5. Again my u_{FE} as a function of x is equal to sigma I is equal to 1 to N, in this case N equal to 5. Let's put as coefficients a_i into phi_i(x). This is our representation of $u_{FE.}$

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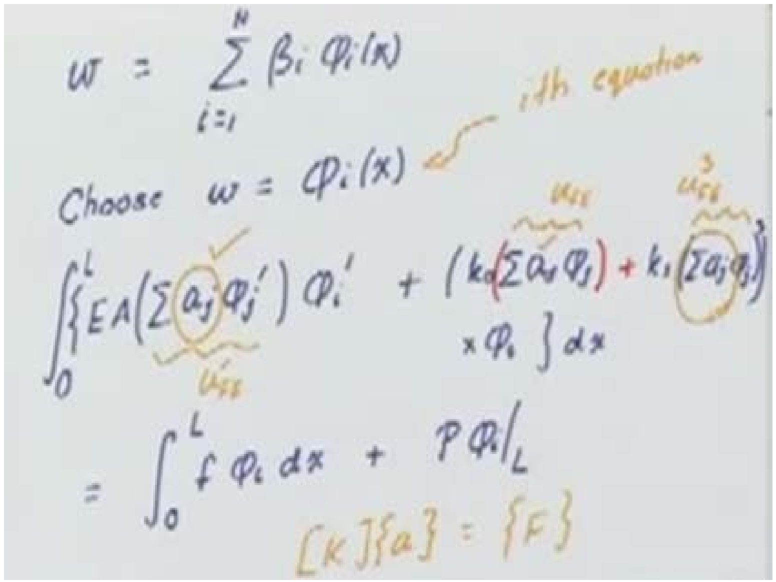

Now what do we do? We put this in the approximation or in the form that we have obtained the weak form and we will also put w is equal to of the same type sigma $beta_i$ $phi_i(x)$. You said since w is under our control this is the representation of w, we choose w equal to $phi_i(x)$ and put this along with u_{FE} in the weak form that we have obtained. I will get here EA sigma a_j phi_j prime into phi_i prime plus I will get k_0 into sigma a_j phi_j plus k_1 into sigma a_j phi_j whole cube, whole thing multiplied by phi_i integrate the whole thing dx, this is equal to 0 to L f phi_i dx plus P phi_i at l, this is what I would get if I simply went and did a substitution. This is u_{FE} prime the derivative of u_{FE}, this is $u_{FE,}$ and this is u_{FE} cubed. You see the problem is that if I go to this i th equation, because remember when I choose w equal to phi_i, I will get the i th equation. This i th equation is no longer a linear in terms of this coefficients a_j it is now, here it is linear, this part is linear, but here it will be a cubic in terms of the a_js so this is non-linear in terms of the a_js. What we have been doing till now, we were solving K into a is equal to f, because of this term being present there I cannot write it in this form directly.

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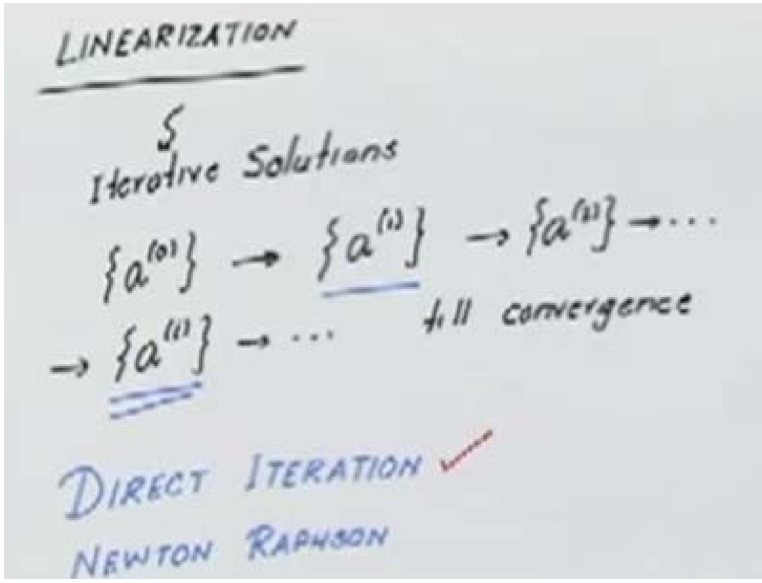

We have to do something, what do we do? We essentially do linearization, so what do we do? We do a linearization. What is the basic idea here? We will use an iterative solution technique in this because it is it cannot be written in this form directly. You cannot get solution vector a by simple inverting k, because now k will also be a function of EA itself. We have to make a guess start from the guess go to the next solution, we have to solve iteratively. So iterative solution, idea is I start with a guess value of the vector a just like we have done in that time integration, go to the next value a 1, go to the next value a 2 and so on to a i then convergence. Convergence means we hope that the new vector, that I get starting from the previous guess vector is an improvement over the previous vectors and finally the vectors that we get do not change much in some measure. We have to give what is the measure that we are talking about. We have various ways of doing this non linear solution the 2 primary ways which are quite often used are 1 is the direct iteration and the other is what we know as the Newton raphson. What we are going to do discuss now in this lecture is the direct iteration method.

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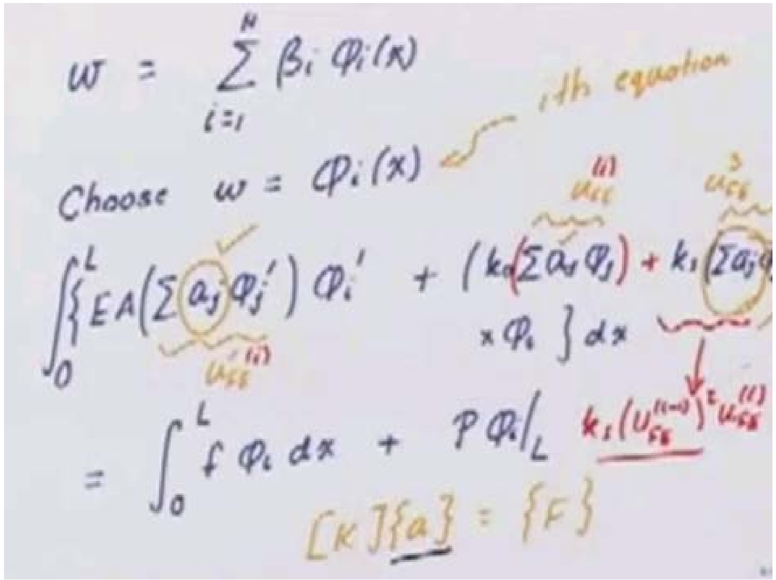

In the direct iteration what do we do? We say that if I go back to my weak form what we will do we will write this quantity, this part in the direct iteration as k_1 I know the solution at the previous step i minus 1 FE whole squared into u_{FE} at i, this will be u_{FE} at i this is will be at u_{FE} at i. What we are doing is we linearize it about the previous solution? That is we assume that this part is known from the previous solution. This is an approximation this will not be the exact solution.

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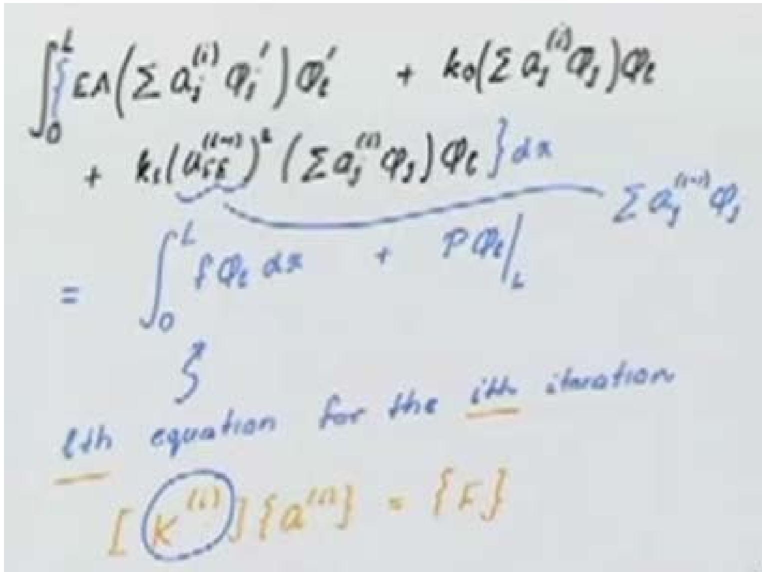

What we do here is we are going to re write this weak form into integral from 0 to L EA sigma a_j for the step i phi_j prime whole thing into phi_l prime for the i th step you are doing plus k_0 sigma a_j i phi_j into phi_l plus k_1 into u_{FE} i-1 whole squared into sigma a_j for the i th step phi_j into phi_l. This is equal to 0 to L f phi_l dx plus P phi_l at x equal to L. This is the l th equation for the i th iteration. If I write it like this really this quantity if I look at this one what would this be? This would be sigma a_j from the step i-1 phi_j that has to be evaluated, this is the l th equation for the i th iteration.

Can I write the stiffness entries, stiffness entries will be, I will have a K, I would put it has lj for the i th iteration, this I would put it like this k for the i th iteration into a for the i th iteration is equal to F as such. What is happening here? This we have now obtained from the linearization, because if you see that these coefficients now you see that this equation is linear in terms of the coefficients a_j for the i th step.

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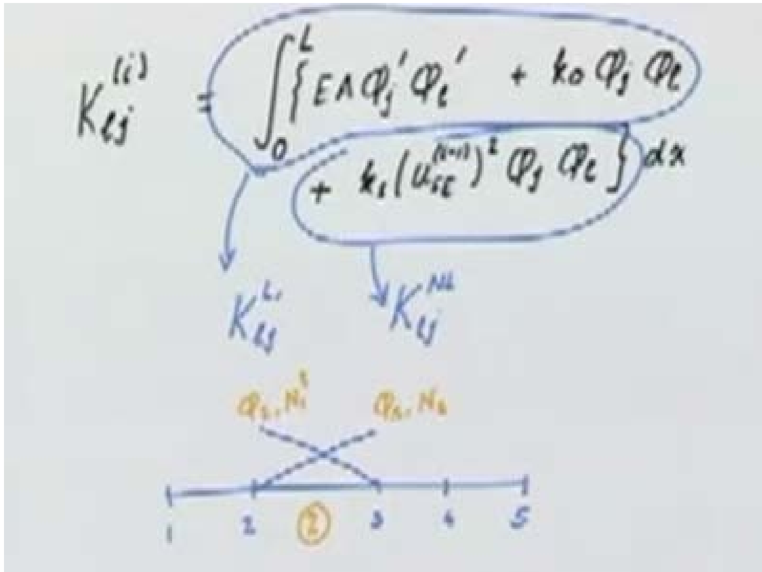

Where now we'll have K_{lj} for the i th iteration is equal to integral from 0 to L EA phi_j prime phi_l prime this remains unchanged plus k_o phi_j phi_l this also remains unchanged, because this is the linear part here there is no non linearity plus I will have k_1 into u_{FE} from step i-1 whole squared into phi_j phi_l. This is dx. I can write this part, this part of the integral as K linear lj and this part of the integral as K non linear lj, this part always a stiffness match works will have the liner part and the non linear part.

This part is very easy we have already done it many times in our 1-D analysis, how do I take care of the non linear part? Remember that now this will reduce to an element wise integration procedure, because we are going to now write everything in terms of a element equations. The global basis functions will reduce to element shape functions will lets say I have these shape functions.

In the element, let's have has this 1, 2, 3, 4, 5 this is my element (Refer Slide Time: 24:50). This is now my global phi_2 and phi_3 and in the element level N_1 of element 2, this is N_2 of element 2. When I am doing this integration, I am going to construct these element matrices first and by summing them together by using our assembly procedure exactly the same way as we had done for the linear problem, we will get the non linear equation.

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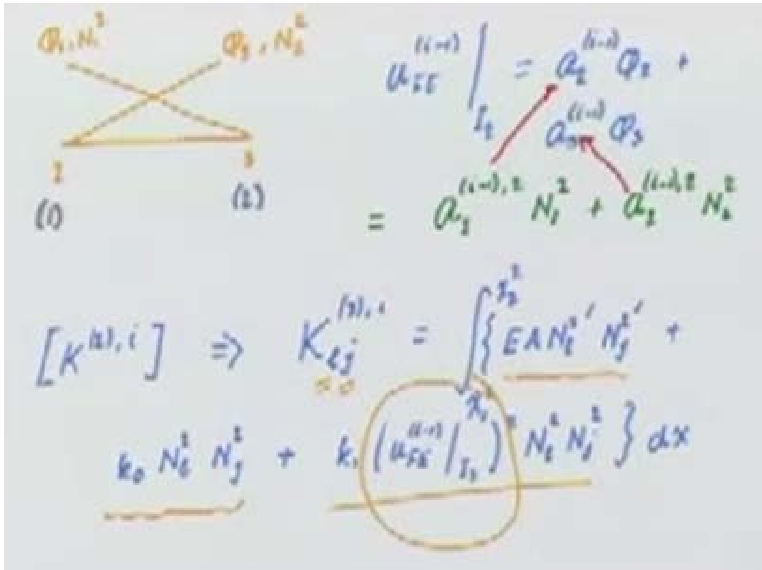

For the element 2 lets say I want to now get the element matrices. This way if you remember this is phi$_2$ N_1 of 2, phi$_3$ or N_2 of 2 this is node 2 node 3 globally locally it is node 1 node 2, let me put the local one with a different color. If I have to now construct the u_{FE} i-1 in this element 2, it will be equal to, what will it be, it will be a_2 coming from step i-1 phi$_2$ plus a_3 coming from step i-1 phi$_3$.

This is how we can construct the finite element solution coming from the previous step for the current step. This would be equal to a_1 i-1 for the element 2, N_1 of the element 2 plus a_2 from the step i-1 for the element 2, N_2 of 2 where a_1 for the element 2 is this, a_2 of the element 2 is this, N_1 of 2 is this and so on. This is the representation of the finite element solution coming from the previous steps. I have to store these coefficients a_1 to a_n for the previous step in order to be able to construct the solution in the element at the current step.

This is one thing, then in the calculation of the element stiffness matrix, I will have by the same token K for the element 2 corresponding to the step i will be equal to will have the components K_{lj} for element 2 corresponding to I, step i this will be equal to integral from x_1 of element 2 to x_2 of element 2 where x_1 is the global x_2, x_2 is global x_3 of EA and l of element 2 prime and j of element 2 prime plus k_0N_l of element 2 into N_j of element 2 plus k_1 u_{FE} i-1 in the element 2 whole squared into N_l of element 2 N_j of element 2. F calculation remains the same as in the one day in the linear case, here this is the crucial difference for what we have done till now.

This part we had already handled and if I can do this, how I will do the assembly? Again I will go to which what does this l correspond to globally, what does this j correspond to globally those that row and that column I am going to assemble it. For example l equal to 2, l equal to 1 will to correspond to global 2, j equal to 2 will correspond to the global 3. Everything remains the same, the only thing is that I have to carry this information from the previous step and use it here.

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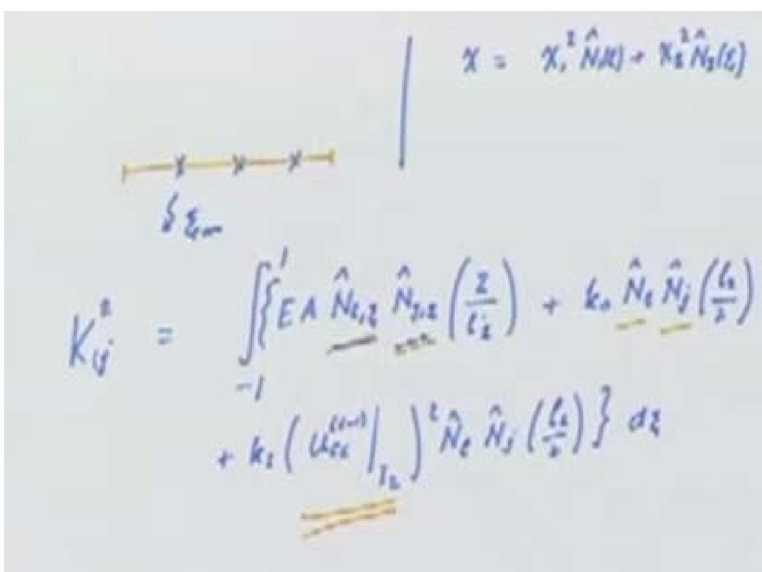

This again you do using numerical integration, what will we do? Let's say I have the element and here I have some gross points in the master element, again you will go to the master element these are xi_m are your integration points, in the element K_{lj} for the element 2 will become integral -1 to 1 by our standard mapping, mapping remains the same that is x equal to x_1 of element 2 N_1 the linear N_1 of element 2 plus x_2 of element 2 N_2 of element 2 actually this will be written in terms of psi, N_1 hat in the function of psi, N_2 hat which is the function of psi.

Everything the rest of the procedure remains exactly the same, I will have this one has EA N_l derivative with respect to psi, N_j derivative with respect to psi into 2 by length of element 2, which is the jocobian plus k_0 N_1 hat N_j hat length of the element 2 divided by 2 plus k_1 u_{FE} i-1 restricted to element 2 whole squared N_l hat N_j hat l_2 by 2 d psi. Everything else is exactly the same as we have done. Now what will we do? We will replace it by summation. We will evaluate these quantities at the Gauss points, sum them up multiply by width, this also has to be evaluated at the Gauss points by using the representation that we are here u_{FE} at the Gauss points. How will you evaluate at the Gauss points xi_m, I have to obtain the value of the shape functions at this Gauss points find multiply with this coefficients find the value of u_{FE} at that Gauss point put it in this expression. It will be a number which I have to put in this expression. If I do this then I get the element matrices.

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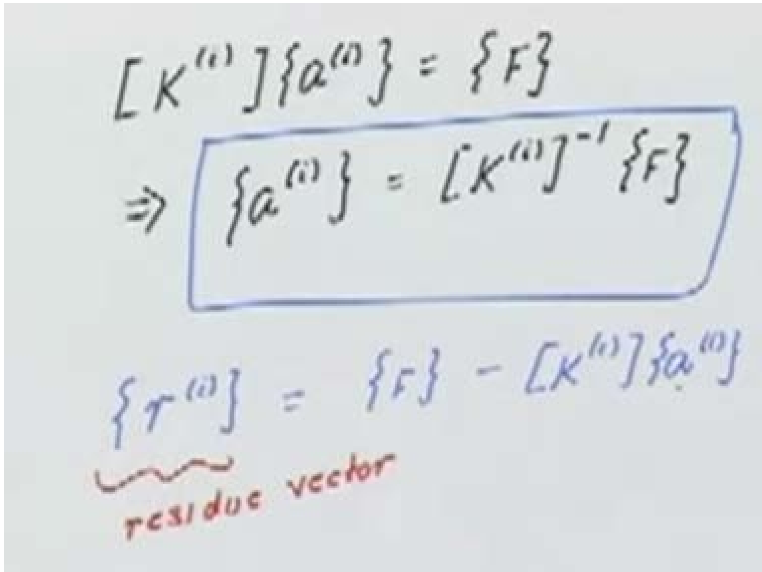

Then what will I do? I will have K i a i is equal to F implies a for step i equal to K i inverse into F, question is how long do we continue? Once I have these coefficients then I have to check. To check what do I do? I find the vector r i is equal to F – K i into a i. What is this vector called? This vector actually gives me the error in the solutions. This is called the residue vector. This is called the residue vector, when I have this residue vector then if the solution is exact solution to the problem for the current step, then I should get the residue zero, which means that as the solution converges the size of this vector, the residue vector goes closer and closer to zero.

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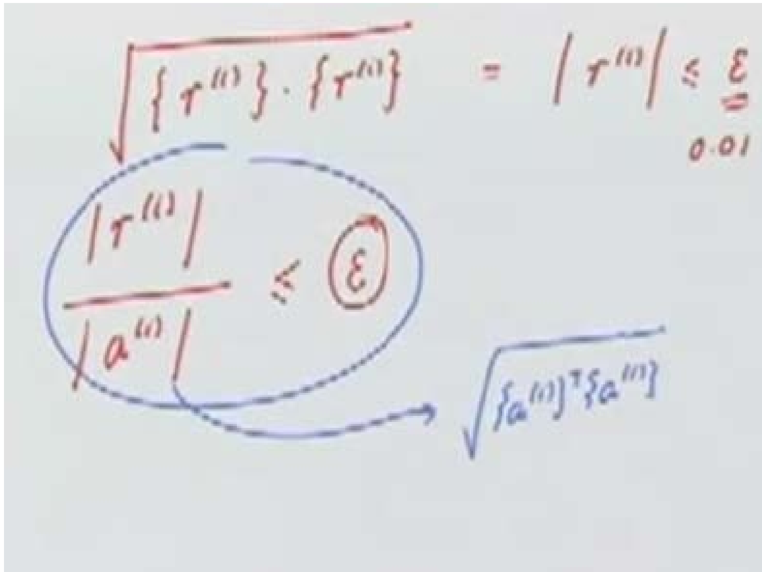

One way of checking for convergence is I will find the residue vector dotted with itself which is nothing but the length of the residue vector. If I take the length of the residue vector and I say that this length is less than equal to some tolerance which I set may be it is 0.01, for example I am happy. I could also use the normalized length, because if the displacement vector is small as for given problem then obviously I except the residue vector to be also small. For that small thing if I try to see this residue I will get convergence very quickly but in that small displacement vector my errors could be large. I could use the scaled residue vector divided by the length of the vector a i is less than equal to the tolerance.

This is one more thing which is a better measure, will be more demanding measure which ensures that the error in the solution is under control and I can set this tolerance to as small as I wish depending on the accuracy that I desire from this solution. I could have other measures for the stopping criteria but this is a good enough measure. The length of the vector a i would be square root of vector a i transpose a i, with this we can check if I can get convergence or not. What does this method really do?

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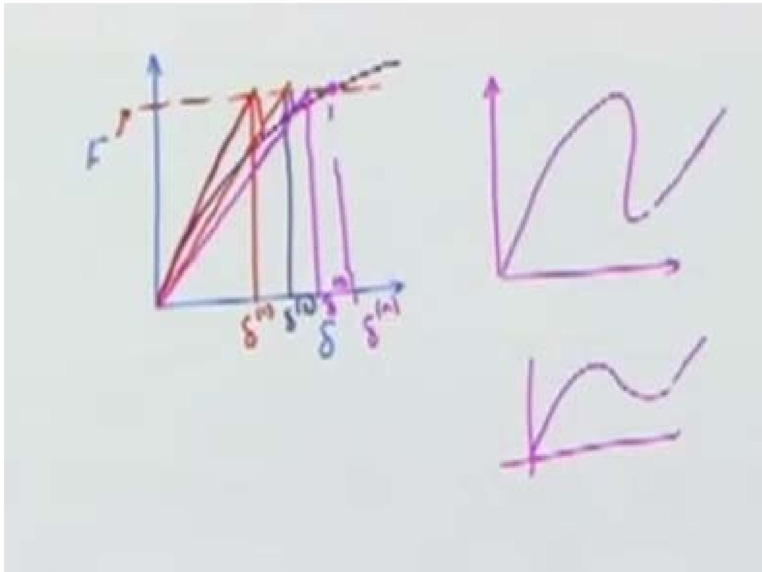

This method actually achieves convergence in the following way. Let's say this is my F or forcing function and this is my deflection delta, may be deflection. This non linear problem lets say it goes like this. Let's say what this method does? Its start with a tangent here. When you have the initial guess which starts with the tangent at this point at point x = 0.

Let's say the size of the load is this, the load value that we want to go up to is this, so it goes up to this solution gives me this. This is my delta one, once it does this then it finds well there is this much gap, between what I am getting and what I should get. Then now it takes this solution and creates this straight line passing from the origin to this solution difference of the current solution with the zero solution, it creates that then it reaches this point. Once it reaches this point then its sees well I have come down this is the gap. If I come down this becomes delta 2, once I get this delta 2 then I would say well now let me start with this I create a line through this point.

You see that I am coming closer and closer to the curve and I hit this point, find this gap this is the delta 3 and I continue this way till I reach this point which is delta m. In some iterations I reach this point, because it is not I mean it is not true that always I will be able to converge I may diverge also but in general for such problems this will leave to convergence.

One thing that we should note is that as I am coming closer and closer to the exact solution corresponding to this load, the reduction in the error becomes slower and slower, the convergence of this method falls off as it become closer and closer to the solution of the problem and it ends up taking many iterations to converge to the solution. Many times in non-linear problems you may have this kind of a situation.

This is essentially a snap back situation, you may have this kind of a situation snap through situation where this method will fail. Anyway before going into the merits and demerits of this method, let us also go back to what we have done here. As far as solution is concerned and remember that at every iteration, I have to enforce the boundary conditions for the vector a i. For example in this problem you will always have a_1 i equal to 0. We have to play the same game to the stiffness matrix that we have been playing till now in order to enforce a_1 i equal to 0 by taking care of the rows and columns corresponding to the first unknown displacement.

This remember we have to do that every iteration let's come back to our discussion here. This method has its problems but nevertheless this is quite an effective method and it can be used for a large class of non linear problems, where the non linearity is not too far away from the linear solution. What is my initial guess? A good initial guess in this problem for example in the problem that we have taken, will be the solution of the linear spring. It would be the case where I solve this problem with k_1 equal to 0. Solve the linear problem where then if I solve this linear problem, let me say I will get K linear into a_0 equal to F.

This linear problem is what we solve first get the initial guess, start with this a_0 and then proceed to a_1 a_2 a_3 a_4. Let's look at how to take care of this anomaly? How to improve the rate of convergence here?

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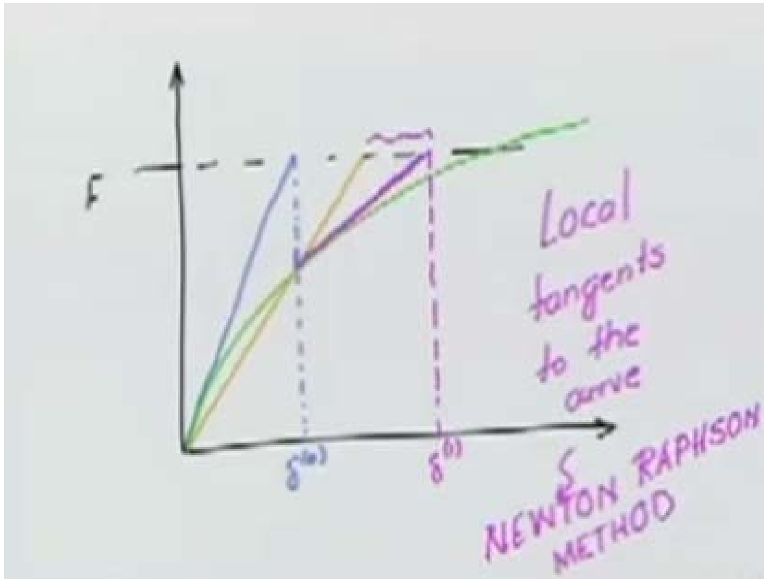

Let’s go back and make the picture on a bigger scale, here is my level for F and here is my actual solution, exact solution to the problem and I start off first with this approximation like the linear solution, I come to this. I will get, the way I have done this will be my delta zero. Once I have this instead of going with this line to the next solution what if we took this line which is the local tangent at this point. If I took the local tangent you see that I have improved so much, the gap has come down. Instead of taking this line I took the local tangent at this point and I went along the local tangent and I got this solution.

If this is my delta 1, you see that I am very quickly very close to the exact solution of the problem. This kind of an approach where I use local tangents to the curve, this is called the Newton raphson method. What we are going to do in the next lecture is we are going to take the Newton raphson method and describe how do we use it in the finite element setting for this generic problem that we have taken and exactly the same procedures have to be used when I am doing any other non linear problems. I will stop.