Finite Element Method
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Module-13 Lecture - 02

In the last lecture, we stopped at the parabolic problem and its finite element implementation. What we have talked about is there are these alpha families of approximations which are possible. Depending upon the choice of alpha we get something called a stable scheme or a conditionary stable scheme when we are doing the time marching. Let us look at these definitions in a little more detail today, so first definition is stability in the time sense. Here the time marching scheme is stable provided that we have two sources of error.

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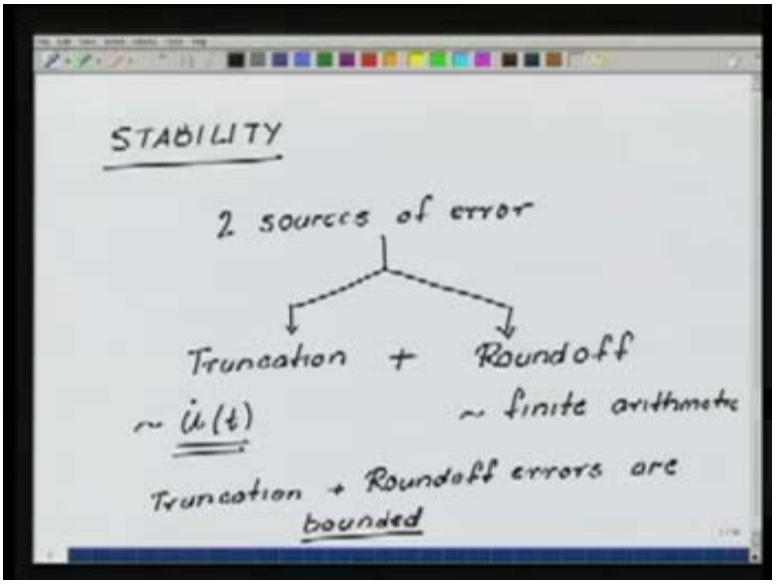

These are one is truncation and the other is round off. The truncation error comes because of our approximation of the derivative with respect to time. We are approximating the derivative with respect to time with a difference scheme which is essentially based on truncating a Taylor series expansion of u dot t about a time with some finite number of terms. Just like we do with the expansion of any smooth function so because of this truncation we are introducing an error which depends upon the delta t that we take from the given time to the next time. The other part of round off error is due to the finite arithmetic that we are doing that is the computer can only represent numbers with certain finite accuracy. It could be a single procession or double procession so these are the two sources of errors and what we say that a numerical scheme is stable, if sum of these two errors, the truncation and the round off errors are both bounded with respect to time.

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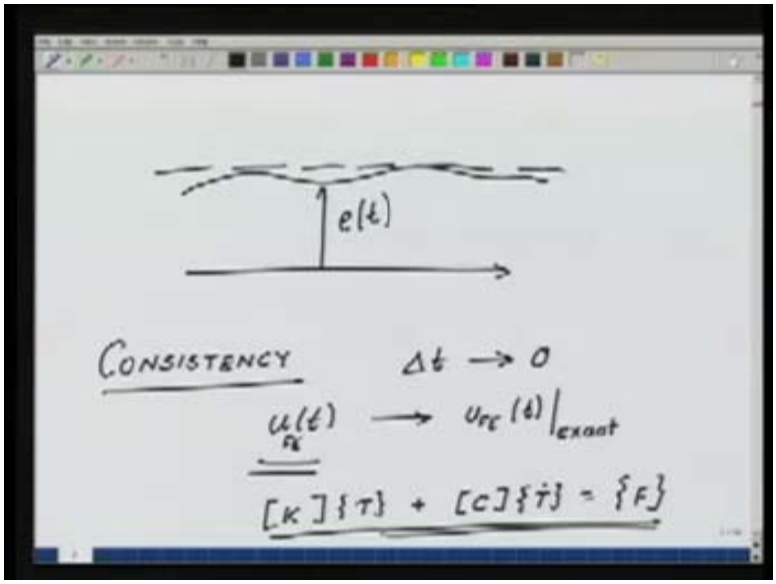

What do we mean by bounded? It means that as this is time, the errors lies within some upper bound. It can go up it can come down. What will happen is in one step the error builds up in the next step it should actually decrease so that it stays within the bounds this is what we mean by stability that these errors do not grow with time, so this is the error, error at time t. This does not ensure that the solution is accurate that is the finite element solution that we have obtained at a given time station is very close to the exact one that we cannot guarantee. It only says yes that the errors, if they are there they are bounded so now let us go to the next definition. Next definition is about so called consistency. The numerical scheme is said to be consistent if as my delta t tends to zero. The solution u as a function of time, u_{FE} as a function of time tends to the exact solution for the system.

We have to keep in mind is that this is not the exact solution of the problem that we have, the actual continuum problem that we have. This is the exact solution of the system of equations that we have created after doing the integration with respect to space. So exact solution is with respect to the finite element discretization that we have put in, so this means as the time steps decrease I converge to the exact solution of the system K as we had written a K T plus C T dot is equal to F. Exact solution of this system is what this will converge to and it will not converge to something else. If it converges to something else which can also happen then the numerical scheme is not said to be consistent. This is a very important point that we have to keep in mind that the solution has a time step converges the numerical scheme should be such that it ensures that I indeed do converge to the exact solution of this system not of the physical problem that we are trying to take care of.

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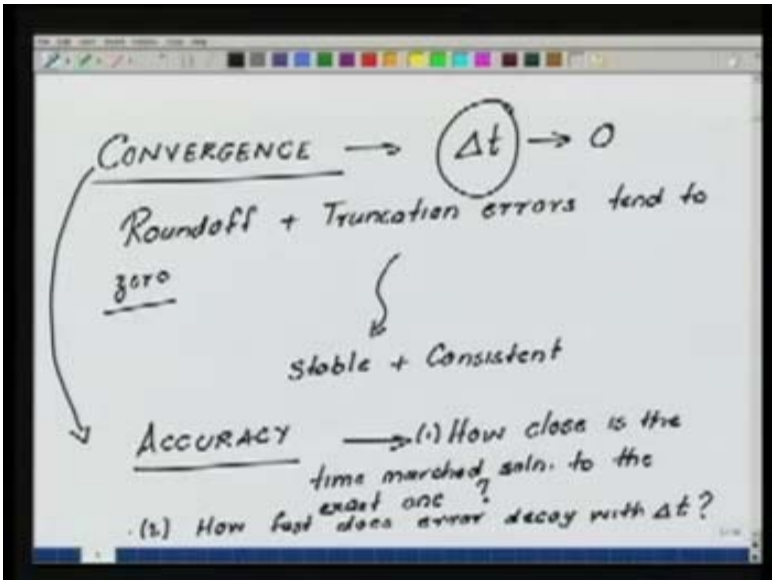

Third definition is convergence, this we have talked about a lot when we were doing the static problem, the convergence we mean that as the time stamp size tends to 0 that as it is made smaller and smaller, the round off and truncation errors for the system of ODEs tend to zero. This is very important as the delta t tends to zero, the round off and truncation errors tends to 0 which means that the solution is both consistent and stable. I get that the solution then converges to the exact point so it has to be stable plus consistent, for it to be convergent. The solution to the system of ODEs converges to the exact solution, for that system of ODE as the delta t tends to zero.

Convergence will come with another added question. How fast does it converge? As delta t goes to 0. How fast does the error in the solution converge that is given by the measure of accuracy. The accuracy of the numerical scheme depends is essentially when we ask for this then we ask the question how close is the time marched scheme a solution to the exact one and this is 1 and 2, how fast does error decay with delta t? These are two very important questions that we will ask which means that by accuracy we mean the rate of convergence.

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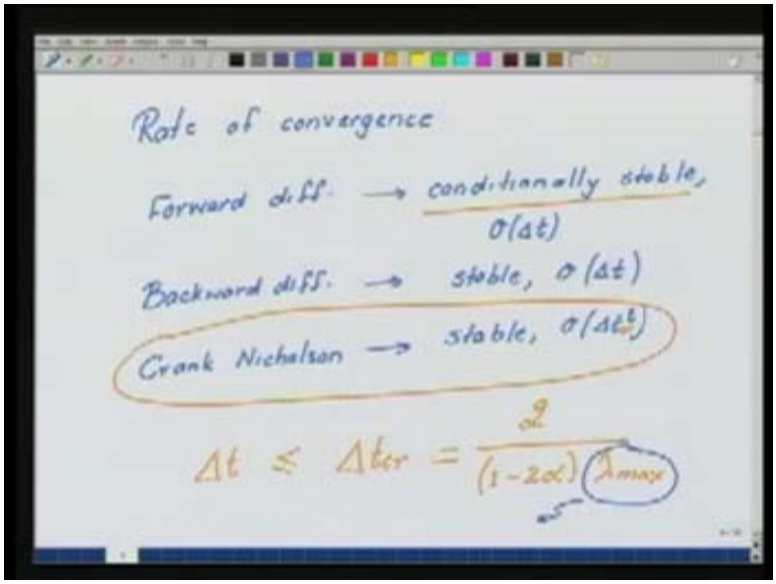

In light of this, what we have discussed in the last lecture? There the forward difference scheme is conditionally stable and it has accuracy of delta t, it is written by O that is the error is of the order delta t, the rate of convergence of the error is to the power delta t to the power of one, so rate of convergence is one. The backward difference scheme is stable and the rate of convergence is order delta t that it is unconditionally stable that is for any delta t, the error will remain bounded and it will converge again at the rate delta t to the power of one.

We talked also about the Crank Nicholson scheme. This is stable and the rate of convergence here is quadratic that is the error converges as delta t squared. What we except is that with larger time steps. I will get more accurate solutions, if I use this one that is what it means. Just like when we use higher p elements with a course or mesh with lesser number of unknowns, we got the desired tolerance because the rate of convergence increases. These are similar approaches, similar philosophies which are adopted for both spatial and temporal solutions. For conditionally stable schemes there is the time step size should always be less than some critical time step size and this is given by critical time step size which depends upon the alpha that we have chosen and the maximum eigen value for the system that we are written here.

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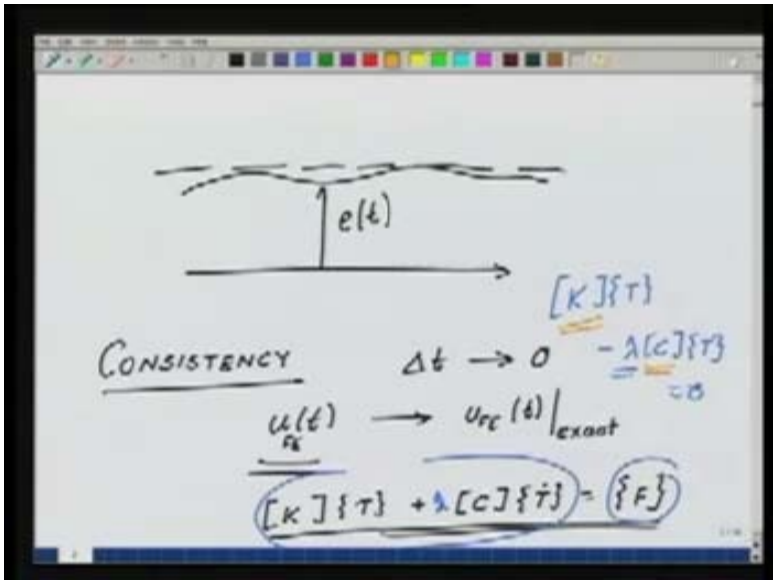

Let me come back for this system of equations. If I do not have the right hand side I put a lambda here then it would I would not put the lambda here. I will solve the system corresponding free problem minus lambda CT is equal to 0 and from here from this system I will solve for this Eigen value lambda. When there is no forcing function then we can use separation of variables and write the temporal part of the solution is e to the power of minus lambda t and the spatial part in terms of the lambda. This is the Eigen value problem that we will get and this lambda max depends on what is the k? And what is the c?

In a way this depends directly on the approximation in the space that is as we keep on adding more and more elements to the finite element mesh. This lambda max is going to change so depending upon that lambda max, the delta critically is going to change. So far a fixed mesh there is a particular lambda max that we are going to get from the stiffness and the C matrix. While, we change the mesh this is going to change, so this is something that one has to keep in mind in when we want to the temporal solutions that the selection of the time steps have to be done in a proper way. Let us now go with all that we have learnt here to the next set of important problems which are hyperbolic problems.

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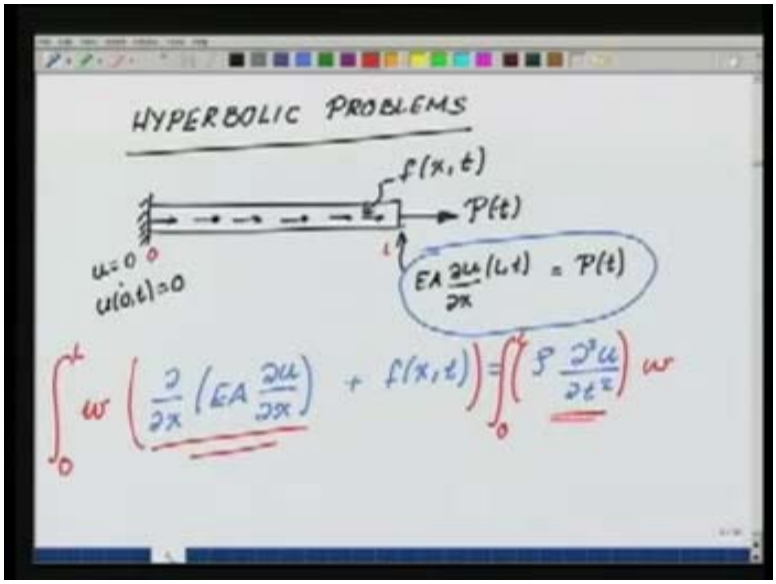

As an example, here we will take the bar that we had the motion of the bar under an applied load. I could have an end load P which is a function of time and an action load f it is a function of space and time. Let me be more specific u at x equal to 0 for all time is equal to 0. What will I get as a boundary condition here? Here I will get EA del u divided by del x at (L, t) is equal to P (t). This is the boundary condition here I will not go into this detail because we have done enough of this and we can derive these things in your own now. What do we do? We again go from the differential equation? Multiply by the rate function at the given instant of time t and then integrate by parts so we will do that again, so we had if we remember del divided by del x of EA del u divided by del x plus f (x, t) is equal to rho del two u divided by del t square. I will multiply this by a w, multiply this by a w, integrate 0 to L this is the L, this is 0 then integrate this part by parts remember we are not going to do anything to this because this does not have a spatial derivative sitting here.

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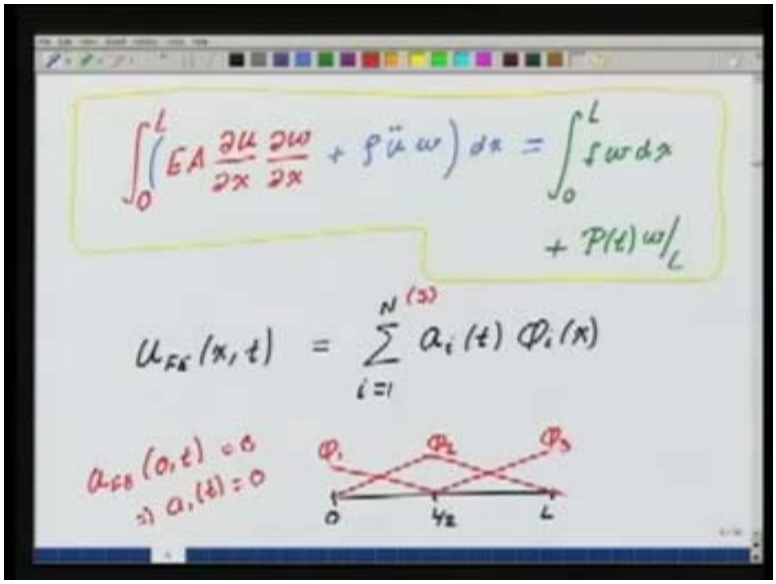

We will do integration by parts and if I go to the next one integral 0 to L, EA del u divided by del x, del w divided by del x plus rho u double dot w dx this is equal to integral 0 to L f w dx plus P(t) w by L. This is our instantaneous weak form that we get corresponding to this problem. We go ahead and do our approximation. How will we do the approximation? Using the variation of parameters that is u finite element as a function of x and t will be equal to sum of i equal to 1 to N. I will put it as coefficients a_i now becomes function of t become functions of t into phi of x. For example, I may use again just like we are done in the last lecture a two element mesh.

Let us say uniform then this phi's will be the standard basis functions that we have this is phi_1, phi_2, phi_3 and u_{FE} will be equal to a_1 phi_1 plus a_2 phi_2 plus a_3 phi_3. In this case I could put n equal to 3 and what will be the a_1 be, a_1 will come from the value of u_{FE} at 0 is equal to 0 implies a_1 as a function of time could be 0. This we will not impose at right away we will do it when we have constructed the matrices then we will go and remove this part. The way we had been doing, imposing the boundary condition in the static part. We will continue with this representation and then do the rest.

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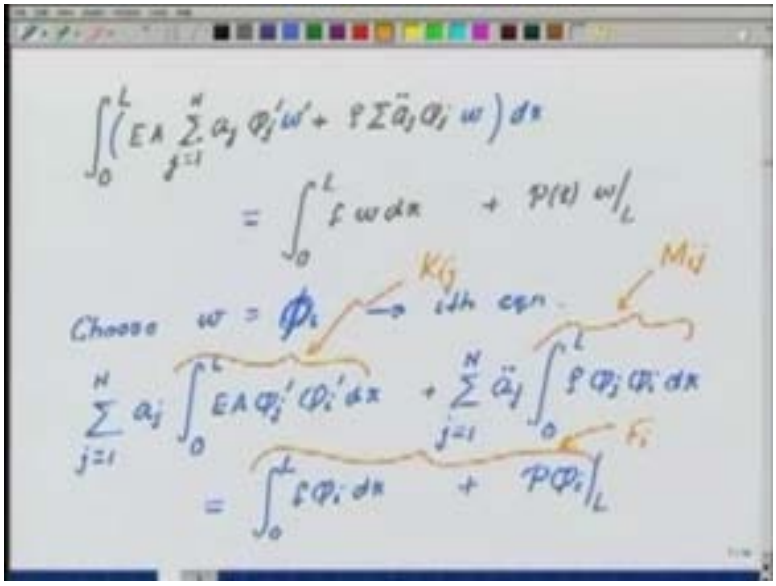

If I put it back in here I will get integral 0 to L, EA sigma 1 to N. I will put now as j 1 to N a_j phi_j prime, prime means derivative with respect to x plus rho sigma a_j phi_j and here I will have actually into w prime this into w dx. This is equal to integral 0 to L fw dx plus P (t) w by L. We choose construct the ith equation choose w is equal to phi_i. This actually I should be doing with phi_i. Again this will give me the i th equation in terms of a, a dot, a double dot and so on. Here I should have a_j double dot from what we have done in the last one, a_j yes double dot will be sitting here, a_j is sitting here and phi_j, phi_j prime. When I do this then I will get sigma j is equal to 1 to N a_j integral 0 to L EA phi_j prime, phi_i prime dx plus sigma j is equal to 1 to N a_j double dot integral 0 to L rho phi_j phi_i dx. This is equal to integral 0 to L f phi_i dx plus P into phi_i integrated by L.

Again this is in the matrix form and these quantities are nothing but this is the K_{ij}, elements of the global stiffness matrix. This quantity is nothing but elements of the mass matrix and this quantity is nothing but element of the load vector at the given instant of time. Another thing is if EA is a constant with time then the stiffness matrix does not change with time. It could change for certain problems specially, where I have plasticity or when I am doing a finite elasticity problem where the area is changing with time when in that case EA could change and because of that the K could change. Here the mass matrix again similar thing, the density could change as the material is deforming for finite elasticity. The way we have done things in this problem, the density does not change even this part the M also remains constant with respect to time.

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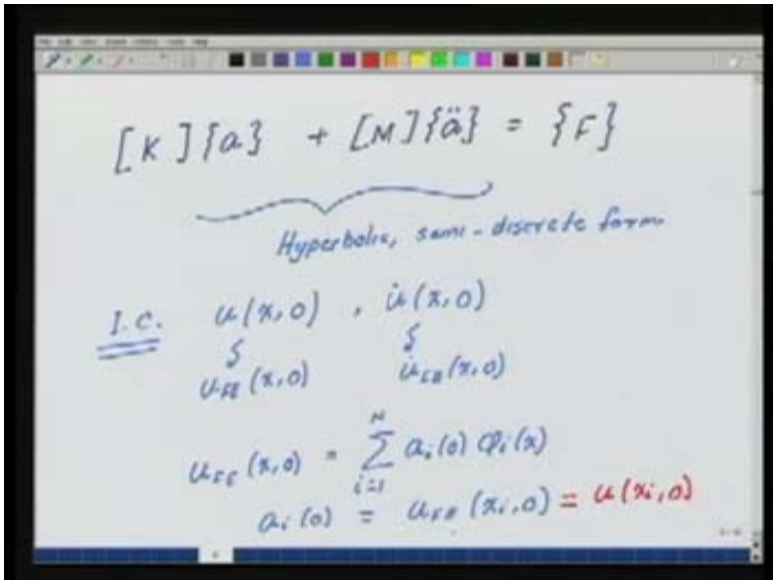

Let us now write it in a form which we know how to use so this will be K into a plus M into a double dot is equal to F. This is our hyperbolic semi discrete form. What are the boundary conditions? As before the boundary conditions are either the displacements or the forces at the ends. This problem will need initial conditions. The initial conditions for such a problem, for the second order problem in time will consists of defining what is the displacement at time t equal to 0? And what is the velocity at time t equal to 0? Before we go ahead, let us look at how to impose this? That is we would like u_{FE} at x equal to 0 and u dot FE at x and 0.

u_{FE} would be at time t equal to 0 will be sigma i is equal to one to N $a_i(0)$ phi_i (x). If I am using the Lagrangian representation then what are this a_is, these a_is are the value of u_{FE} at a corresponding node. So a_i at 0 is equal to u_{FE} at the corresponding node x_i at time 0. We are going to now impose that this is equal to u at the node x_i and 0.This is one way of imposing the boundary conditions so what we will get is essentially we will say that the initial solution is obtained by interpolating the given initial displacement using the finite element basis functions.

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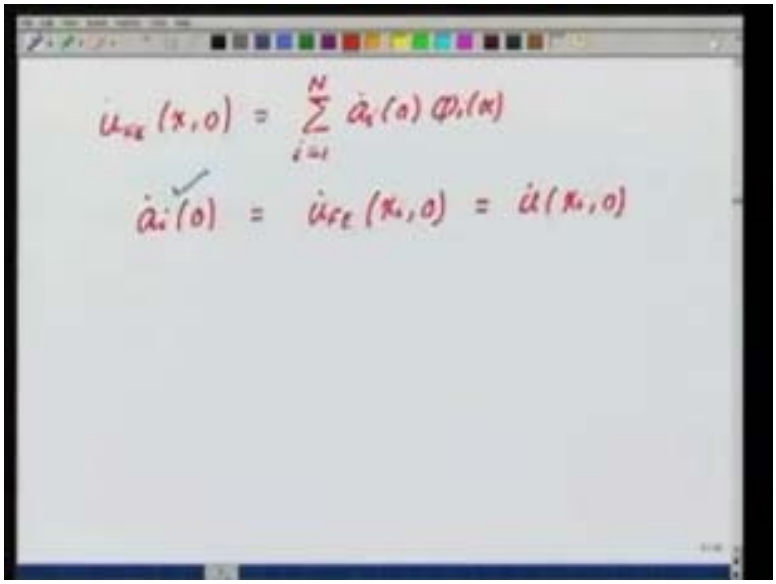

Similarly, we will get u_{FE} dot at x and 0 is sigma i is equal to one to N, a_i dot 0 phi_i (x) and again a_i dot at 0 is equal to u_{FE} dot at the point x_i 0 and we are going to force it to be u at the point u dot at the point x_i and 0. We are going to interpolate both the velocity and the displacement in order to get these coefficients a_i dot and a_i at the initial time.

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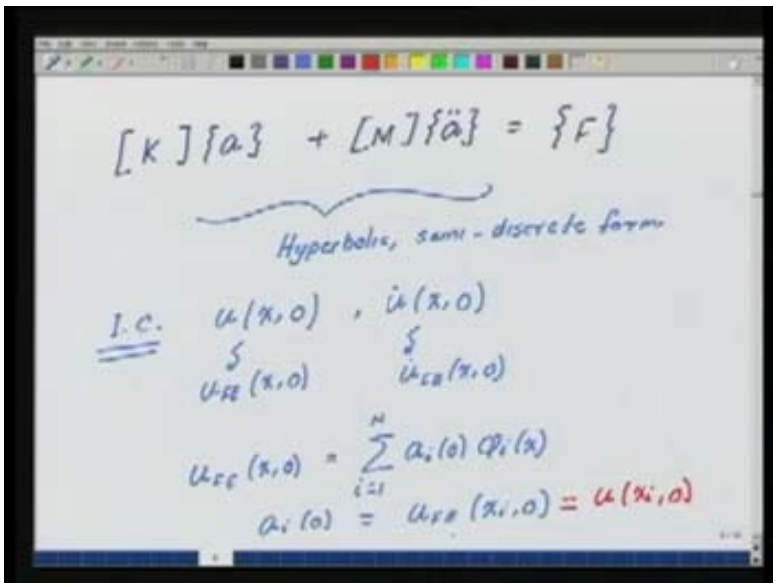

If we see that, we have the values of these coefficients the coefficient a, or the unknown displacement coefficients a, at the time 0 and a dot at time 0.

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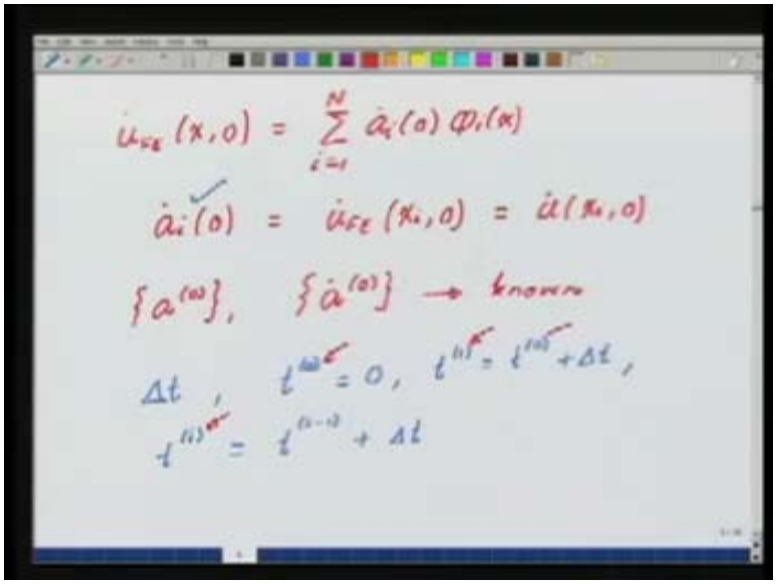

Essentially in the time marching scheme, the vector a, at step 0 and the vector a dot at step 0 are known. Once these vectors are known then we can march in time that is we will again take time step delta t. Such that t (0) is equal to 0, t (1) is equal to t 0 plus delta t. I am taking uniform time steps but we could change the time steps and the solution shows very little activity that is it is very smooth with respect to time in a particular part of the time interval. I can use large time stamps but if it is showing some very sharp temporal behaviour in some time interval there I should use smaller time steps so in an adaptive way, we could choose the time. But here we want to be simple and keep the time steps constant. This way I can again have the time at station i is equal to the time at station (i-1) plus delta t and we want to find the solutions at each of these stations a, a dot and all those things at each of these stations. Just like we have done in the case of the parabolic problem, we have to propose a different scheme to represent the acceleration term here, acceleration term in terms of the values of displacements and the velocities at previous times. There are many methods available primarily the ones which are very popular are the Houbolt's scheme.

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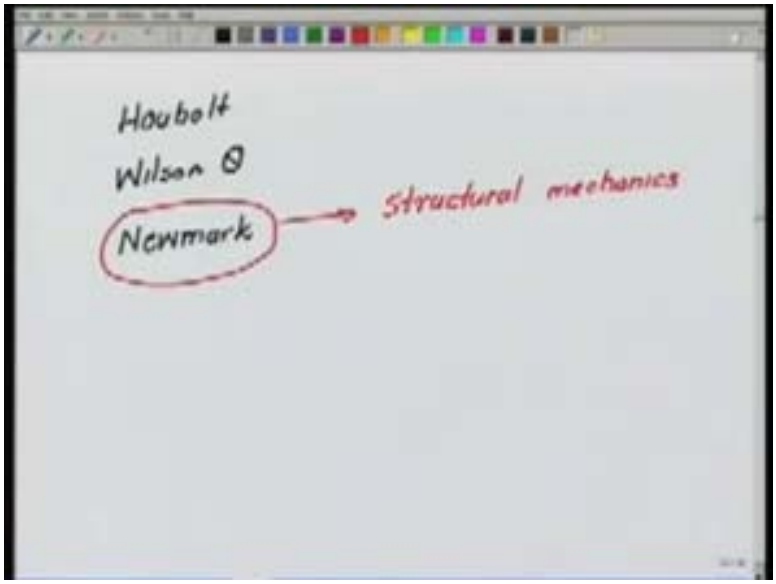

There is one called the Wilson theta scheme, there is one called the Newmark scheme. What we are going to follow is the Newmark scheme because in structural mechanics, this is the scheme which is very popular. One need not use only the Newmark's scheme that is not the only scheme which is available, we could have many such schemes depending upon what accuracy we want? What kind of efficiency in terms of solution of the system of the equation it needs to etc? Whatever is the scheme, if I go to the system that we had obtained here, this system will have its own features with respect to time. What kind of features that the solution here with respect to time is going to be oscillatory?

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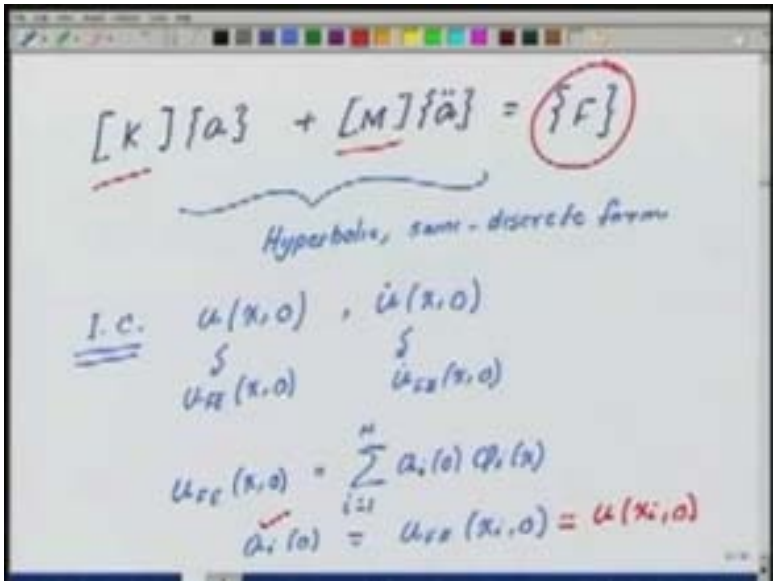

If I think of it as a spring mass system this is the spring, this is the mass. It should have an oscillatory behavior in time, if the F is 0. If F is nonzero we will have one part which is the standard homogenous solution and the other part which comes due to the F. The force part of the solution is something that we have to keep in mind when we are constructing the numerical solutions to the systems.

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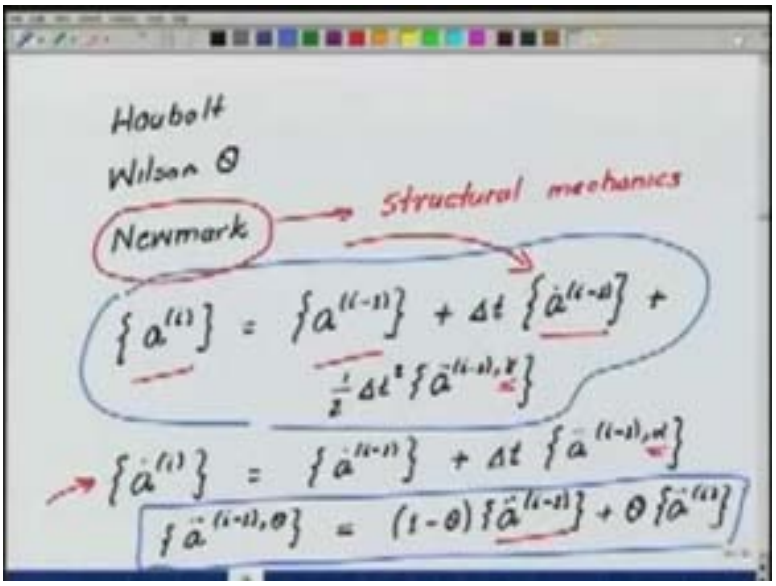

Let us go to the Newmark's scheme, it is quite similar to what we have done in the alpha family of approximations earlier. The Newmark's scheme works on this principle that the displacement vector, it is essentially component wise at station i is given in terms of the displacement vector at station (i-1) plus apart due to the velocity at the station (i-1) plus. I will explain this later what does this mean? Similarly, the velocity at station i is given in terms of the velocity at station (i-1) plus delta t, it should be acceleration here so a double dot at station (i-1) comma beta comma alpha. I need to put it like this, the displacement is given in terms of the displacement at the current station in terms of the displacement at the previous station and the part due to the velocity at the previous station plus the part due to this acceleration term. The velocity is given in terms of the velocity at the previous station plus delta t into a part due to the acceleration term. We have a double dot (i-1) comma theta in a generic way is equal to one minus theta into a double dot at station (i-1) plus theta a double dot at station i.

We have this kind of an interpolation for the acceleration and we see that in the interpolation when it is used for the displacement term. I have a separate value of theta when I am using interpolation for the velocity I use a separate value of theta. Based on the choice of the alpha and the gamma, I will get different families of the Newmark's scheme. How do I choose these families? A member of this family for the implementation, stability and accuracy are the two important criteria's based on which I will have to choose. We see that I can write a (i) in terms of a (i-1) a dot (i-1) and the acceleration at the step (i-1) and acceleration at the step i.

In this representation, I can write by substitution the acceleration at the step i in terms of the displacement at the step i minus one, velocity at step (i-1) and acceleration at step i minus one. I can write a double dot I in terms of the unknown a (i) and knows at the step (i-1). Here we assume that the solution is completely known at step t (i-1). Similarly, the velocity at step I will be given in terms of the velocity at step (i-1) plus the part due to the accelerations. This is how we are going to update the velocity. Here in this the way we have done things explicitly, we do not need to use velocity in the formulation but the velocity has to be updated so that it can be used here so if I want to go and do the implementation of this what will I do? I will take this formula here expand it out. I will take it expand it out and then put it back in the equation that we got in a semi discrete form and I get the system of equations which will be used to get me a (i) in terms of what is known as the previous time step. Let us look at some choices of the alpha and the gamma which will leave to different families of approximations.

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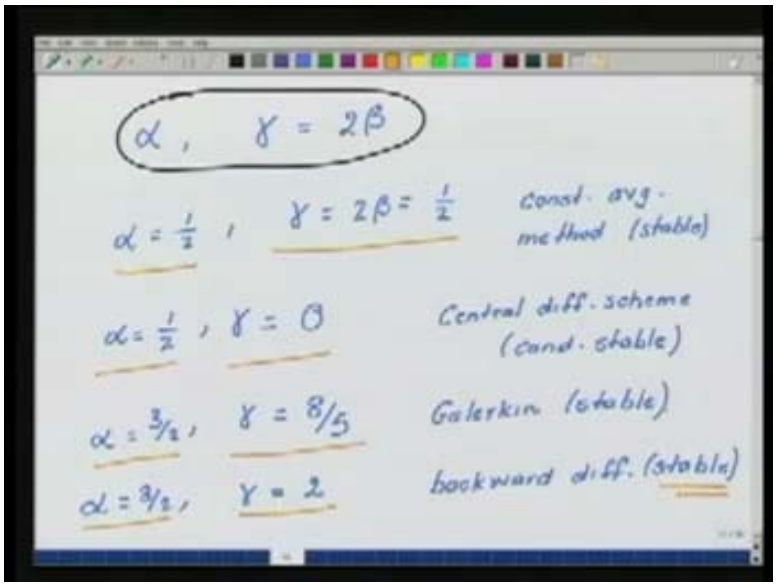

Here I have written that I will choose alpha and gamma will be written in terms of 2 beta. Gamma is equal to 2 beta and if I have alpha equal to 1 by 2, gamma is equal to 2 beta is equal to 1 by 2 this is called a constant average method and this scheme is shown to be stable. If the alpha is equal to half, gamma is equal to 0 then I get the central difference scheme which is said to be conditionally stable. When alpha is equal to 3 by 2, gamma is equal to 8 by 5, gamma equal to 2 beta, remember that it is always 2 beta. We get the standard Galerkin method which is shown to be stable. Alpha is equal to 3 by 2, gamma is equal to 2 then we get the backward difference scheme which is also stable. For the conditionally stable schemes where gamma is less than alpha and alpha is greater than equal to 1 by 2 which essentially corresponds, we can take the central difference scheme falls in that category.

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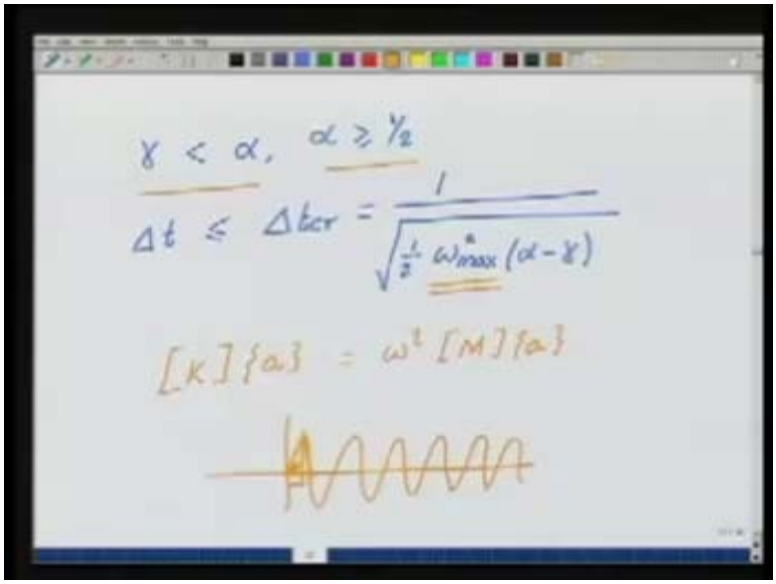

In that case, again as we had in the case of the parabolic problem here in the hyperbolic problem we have a critical time step size such that delta t has to be less than or equal to delta t critical. What defines the delta t critical? It is the natural frequency of the system that we have. If I solve this problem K, a is equal to omega squared M a. So for this system the maximum natural frequency squared that defines the time step size. The bigger is a natural frequency, the smaller is the time step size that we have to take. Why should we do that? Because we see that the higher is the natural frequency, the more oscillatory is the solution in time as well as in space. In time if it is oscillatory then I need at least the time step size to be this much, in order to be able to capture this behavior. The time step size is essentially governed by the worst most oscillatory part of the solution that the system is trying to represent.

This is the philosophy that we have to keep in mind, remember that as the mesh gets refined further and further, the maximum frequency increases. Omega max increases, I can now resolve in space higher and higher frequency terms. If I can resolve higher and higher frequency terms in space, it tells me that the omega max because as the size of the increases omega max also increases and as omega max increases I need to use smaller time steps in order to resolve these frequency components. Otherwise the temporal error will become large, it may blow up and then the entire job that we have done in trying to do a very good approximation in the spatial region will go out of the window. We will not be able to get accurate approximations in that case. Remember that if I am doing something in the spatial region that is I am trying to refine the approximation there I have to do a corresponding job in the temporary region.

Further there are issues that in the spatial region, I have a rate of convergence if I do not get the similar rate of convergence in the temporary region, I have to ensure the time steps as small enough that in the temporal error, the method may be stable but the temporal error is essentially at the same order of magnitude as the spatial error because

otherwise the total error in the solution is so large that it will get essentially over powered by the spatial error or a temporal error and then the solution is of no use. This is very delicate inter plane has to be done. If possible one can think of automatic methods of predicting the sizes of the error and then controlling the time step in the mesh in such a way that these do come under control.

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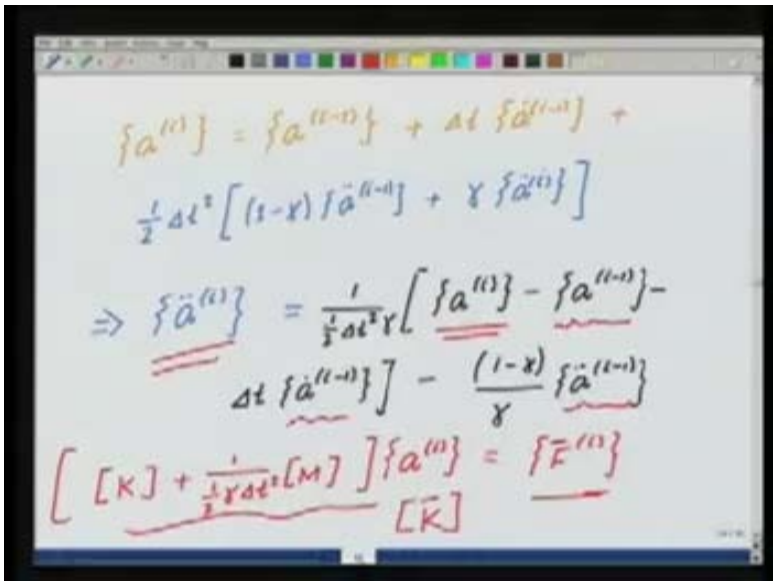

Let us go and look at an implementation in generic way of this approach. Let us start off with this a at step i is equal to a at step (i-1) plus delta t a dot at step (i-1) plus I will put half t delta squared into we had a double dot gamma (i-1) gamma. I will put it like this into one minus gamma a double dot at step (i-1) plus gamma a double dot at step i, so from here if I see I will get implies a double dot at step I is equal to from this approximation one by 1 by 2 delta t squared gamma into a (i) minus a (i-1) minus delta t a dot (i-1), this minus one minus gamma by gamma a double dot i minus one, so this a dot is given in terms of the unknown a (i) at the current time and the known a (i-1) a dot (i-1) and a double dot (i-1).

I put this back in the differential equation that we have so I will get essentially collecting terms I will get differential equation at time station i K into K plus, I will get here one by 1 by 2 gamma delta t squared M into a at station i is equal to F bar at station i. F bar at station i will be F at station i plus the contribution of all these things that is a simple algebraic job that has to do. For example, it will be plus 1 by 2 delta t squared gamma into M operating on a (i-1) plus one by 1 by 2 delta t gamma into M operating on a dot at i and plus M one minus gamma by gamma into M operating on a double dot at i minus one. This will construct, this will give me the load vector here and this and then I invert this matrix, this matrix is called as K bar.

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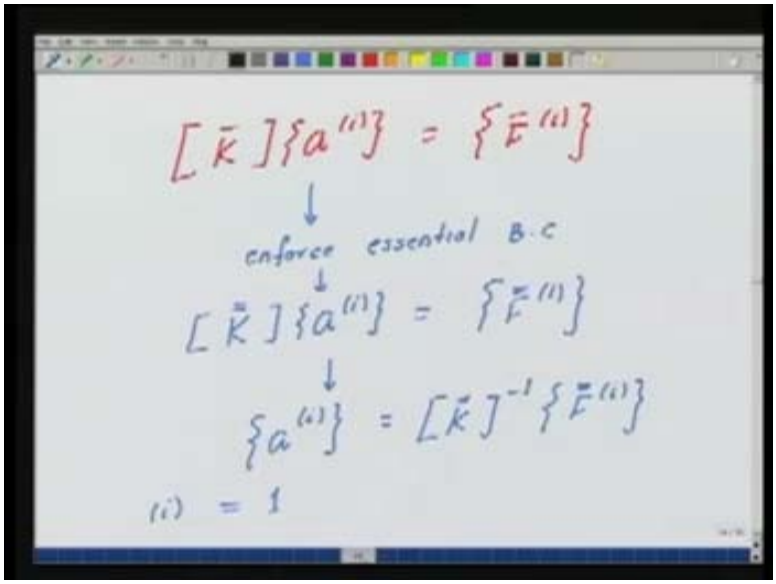

Essentially, once I have this K bar a (i) then I go and apply the boundary condition that is according to the problem, I want to force a one at the station i to be equal to 0 because u at station at time t is equal to 0 at x equal to 0, so I will force exactly the way we had been doing in the static problem and we had discussed for the parabolic problem in the last lecture. Enforce the boundary condition let us say enforce essential boundary condition to get K double bar a (i) is equal to F double bar i. We will get a (i) is equal to K double bar inverse into F double bar at station i. I can construct the solution at time station i using all the information at station (i-1) and the load vector at station i itself. So this I can do, now we see there is a small problem here when I am talking of station 1 when i is equal to one, for the station i equal to one I need to obtain, when i equal to 1, I assume that I know a at 0, a dot at 0 and double dot at 0 this I know, this I know, this I do not know.

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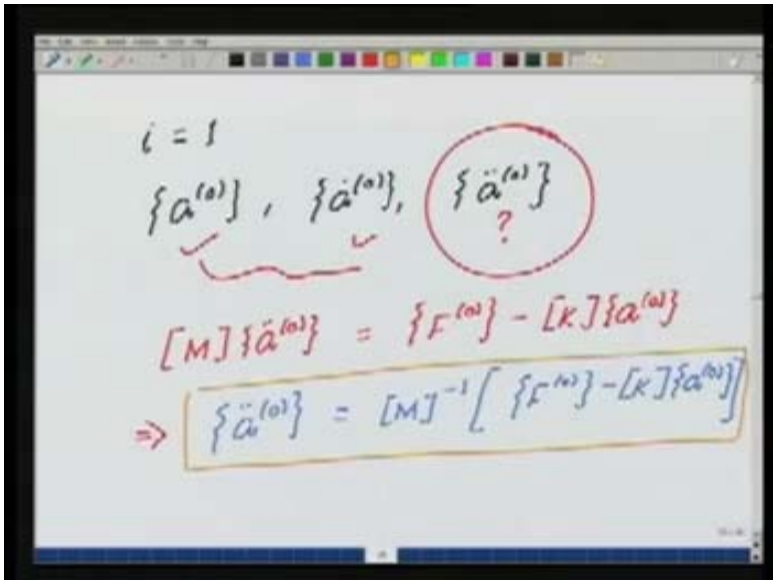

These two have come from the initial conditions this is something which I do not know because the initial condition does not give me the acceleration at time t equal to 0. I have to start with something, in order to start with something we do a very simple thing that we say that M into a double dot at station 0 is equal to F at station 0 minus K into a at station 0 because a is known, implies a double dot at station 0 is equal to M inverse into F at station 0 minus K into a at station 0.

I have this formula which I am going to use in order to obtain the double dot at time t equal to 0 and this becomes the starting value of a double dot, use this progress in time, get the solution. Remember that the time step size has to be small enough to resolve the temporal behaviour of the solution. Imagine I am talking of a free vibration problem I do not do the separation of variable solution.

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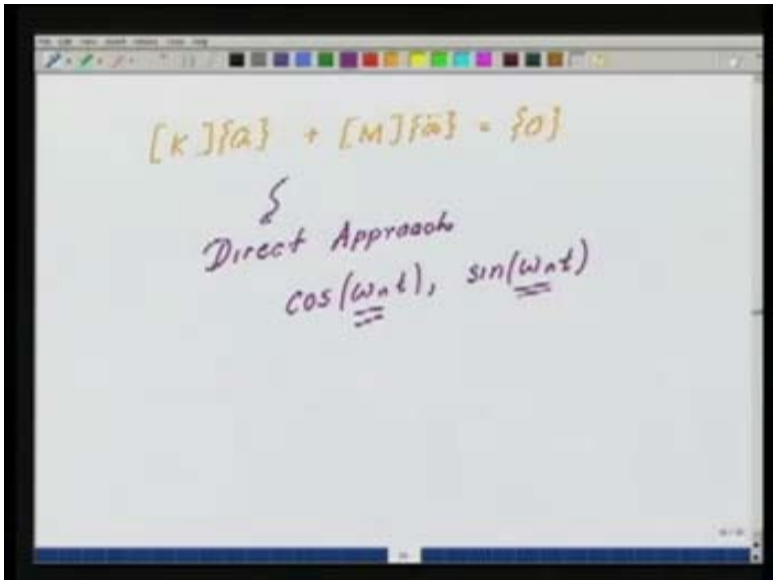

I use the direct approach that is I want to solve K a plus M a double dot is equal to 0. If I want to solve this problem using what we have just done which is the direct approach, I should get the oscillatory nature of the solution given the initial conditions and the oscillatory nature of the solution should be able to represent the frequency content which are there that is the solution should be in terms of linear combinations of the natural frequencies for this system, this we know from our basic vibration analysis courses. I should be able to resolve in time, all the frequency components because the solution in time is going to be cosine $omega_n$ t in terms of this sine $omega_n$ t. Depending on the highest frequency term which is there, I should be choosing the time step such that this also resolved.

With this, we will finish the temporal part of our approximation. We have seen how to construct the semi discrete formulation, apply the various finite difference approaches to do the time marching in order to get solutions at various times. We have discussed the stability convergence and consistency of these schemes and we have clearly given some possible schemes which can be used in a particular temporal solution methodology. In the next lecture, we are now going to shift from this set of problems, we are going to look at one set of problems that we have not discussed till now which are again very useful to all the users of finite elements methods. Users are the people who are in engineering mechanics or in structural mechanics, fluid mechanics or in heat transfer. One set of problems which are important are the non linear problems. We are not going to deal with all the non linear problems, we are going to look at again a sample problem and through this sample problem show how we can construct the solutions to the nonlinear problem. This is what we are going to do in the next lecture and that is what we will be covering in this course.