Finite Element Method
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Module 13 Lecture 1

In the previous lecture, we have talked about the free vibration of various structural numbers that led to an Eigen value problem, which we solved to find the characteristics of the response of the solution of a system in free vibration. We'll now extend our analysis of transient problems to the case of force response we could also do the free vibration analysis using a direct time solution that is we do not do the separation of variables straight away and separate this spatial and temporal part and we solve for those spatial and temporal parts together.

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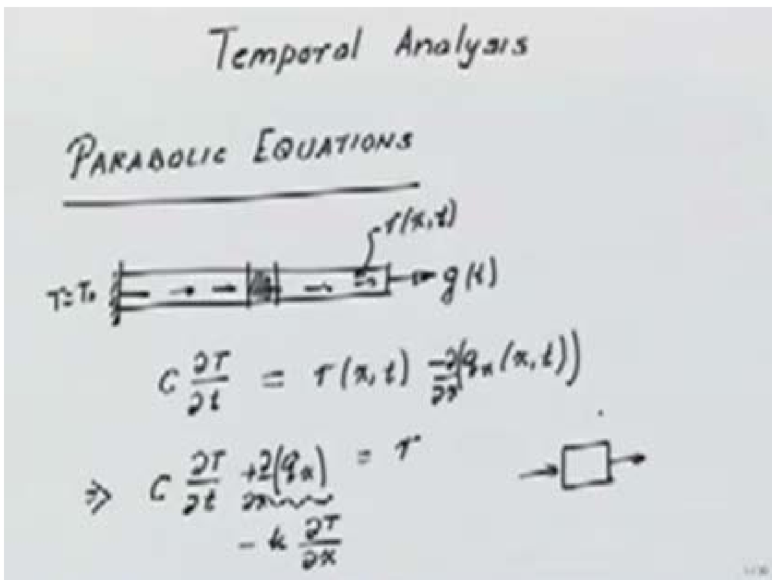

Let us see how we are going to do it, we will do the temporal or the time analysis. Let's consider some typical differential equations corresponding to physical problems which arise in the case of transient analysis. One case is what we call as parabolic equations, parabolic equations a typical example is the heat conduction problem. What we have done long ago as a steady state conduction problem when heat conduction is not in steady state that is it is actually following the result transient then it would lead in a simplified form.

Let's say I have again a member here I specified temperature is equal to T_0 I have in one dimension the source term, the heat source term r as the function x and t and here is my boundary flux term keep fluxing out this is given by g lets say as a function of t. Then it can be shown, that the governing differential equation would be of the following type some constant into delta t. The change in temperature with respect to time this is essentially the buildup of the internal heat with time, this will be equal to the amount which is produced by the source minus the amount which is fluxing out.

This would give implies C the del t del t plus q_x is equal to half, q_x as we have seen in our basic mechanics courses or the heat transfer courses is nothing but minus some constant k into partial of t with respect to x that is heat flows from a hotter to a cooler region. It flows in the direction opposite to the creative end that is what we mean it. It will be del del x of q_x, this will be del del x, because we are talking of the infinitesimal volume. This is what you get, this you can check in any book on mechanics or heat transfer or you can take an infinitesimal piece look at the heat transfer in that infinitesimal piece, flux coming in and flux going out and so on.

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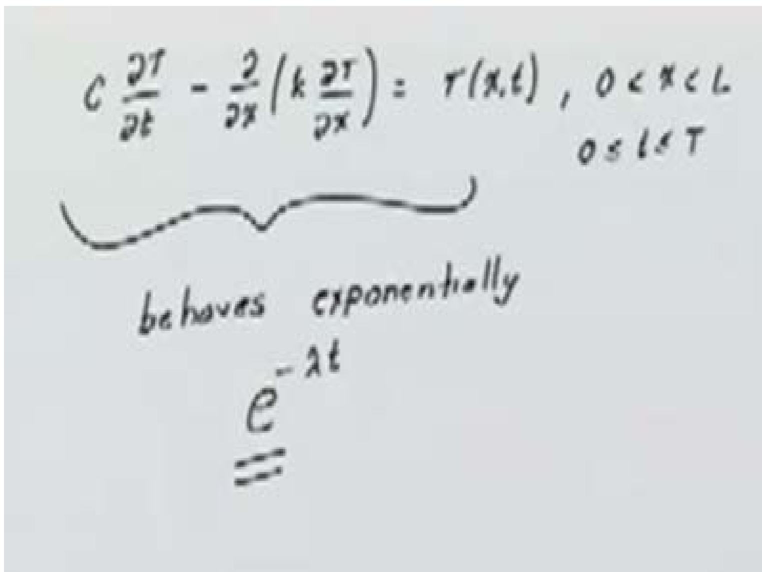

This equation now we are going to work with, I have C del t del t minus del del x of k del t del x is equal to r of x and t r of x and t that you remember that here the source can change with time. How do I find a solution to this problem and for every point in the along the member and for all times from zero to time capital T so here in order to find the solution we will follow something which we have been following till now, that is we are going to talk of the instantaneous weighted residual formulation. We will talk about it a little later. First let us look at the solution if I look at this problem, the solution to this problem in time behaves exponentially that the solution will have e to the power of minus lambda t kind of behavior so it is very smooth in time.

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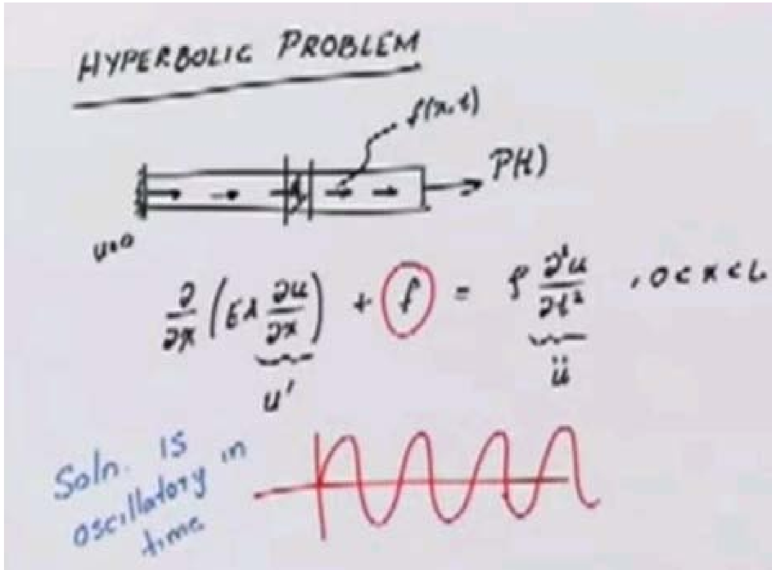

Another type of differential equation that we come across very often is the hyperbolic problem. An example of hyperbolic problem we have been doing this for a long time now there is bar subjected to actual forces f which now can change with time and n force p which can change with time. Here the u is equal to zero for all times, then for this bar if I look at the equation of motion for this bar, again taking an infinitesimal piece and looking at it I will get that del del x of EA del u del x plus f is equal to rho del2u del t squared, wherever I have a derivative with respect to x I will write it as a prime, prime will indicate a derivative with respect to x, wherever I have derivative with respect to time I will also use the dots. So a dot means derivative with respect to time, two dots secondary derivative with respect to time and a prime will mean derivative with respect to x the spatial derivative.

This is the equation of motion for the bar problem. This is the hyperbolic problem and as we had seen that in case we do not have the force this force in function f we do not have it the free vibration problem then the solution behaves both in space and time in an oscillatory manner. That is a solution is oscillatory both in space and time, this is the feature that we have to keep in mind, solution here is oscillatory in time and we had discussed that as far as resolution of the solution in space was concerned that is in order to get the natural frequencies and the modes properly our finite element approximation in space had to be sufficiently fine to be able to represent the mode properly.

Similarly in time when we do the temporal approximation or approximation in time should be sufficiently fine to be able to capture the corresponding oscillatory behavior in time, this is something we have to keep in mind when we go ahead and construct temporal approximations.

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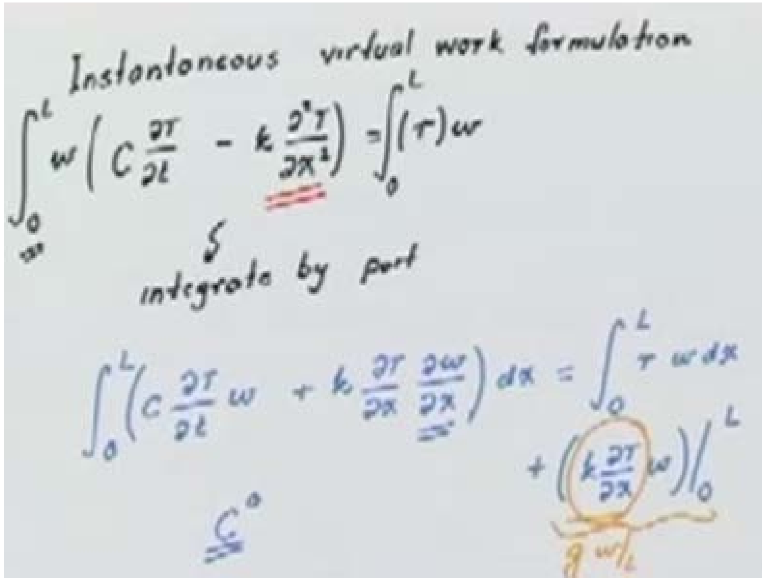

Let us now go and look at first the parabolic problem. If we go back to our parabolic problem this differential equation and try to construct a form which can be used in a finite element formulation so we will take, what will I do? I will take the instantaneous virtual work formulation virtual work or rate adhesive formulation.

We take the differential equation C minus let us say k is the constant and multiply both side with rate function w and integrate from 0 to L. Once I have done this then again I do what we have been doing so long this we do integration by parts, integrate which parts by parts, it is the spatial derivative has to be seen and that has to be integrated by parts, because here we are doing integration over x, we will have to take care of this part. When we do an integration by parts? How do we take it? This will give integration by parts once will give watch this very carefully that this is going to be w plus k del t del x del w del x dx is equal to integral 0 to L r w dx plus k del t del x into w evaluated at 0 and L. Again the same issues arise as far as the choice of w is concerned end usability of w, it should be such that the first derivative of w is defined similarly the first derivative of t should also be defined I use C 0 approximations for both t and w and you see the boundary condition here this is term which is given at the two extremities of the member for all times remember that this condition has to be expressed for all times.

The way we have drawn a figure that at x equal to 0 the temperature is maintained if you see from the figure is maintained to the value T_0 for all time, which means that at that end w has to be chosen to be equal to 0, because temperature at any instant is known at that end, at x equal to 0 w is 0, that part gets cancelled out. At the other end at x equal to L we have that k del t del x the flux condition is given equal to 0. We have exactly the same boundary conditions that is the flux at the end x equal to L and temperature at end x equal to 0. We say that this quantity is given at the end x equal to L and we will say that this whole thing will be equivalent to g into w at L.

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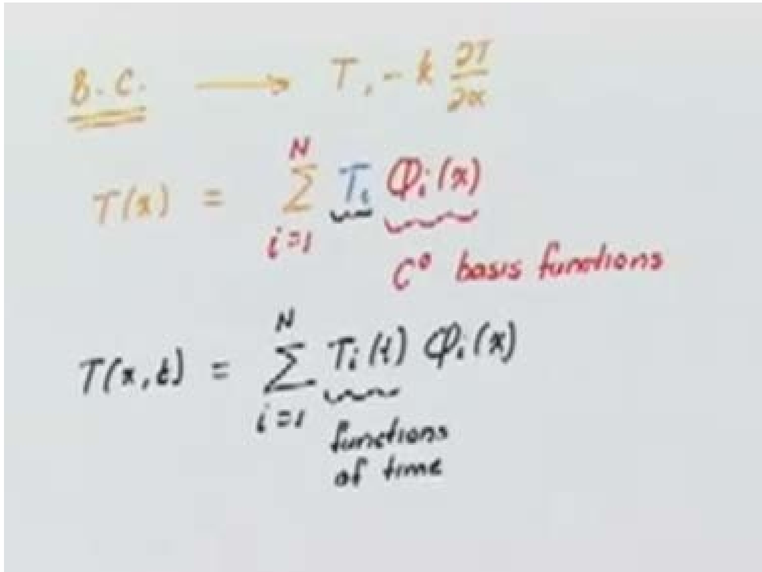

Look that here in this problem we have boundary conditions coming in just like if doing it till now this will be either temperature is specified or the flux is specified. This these two quantities either or have to be given at each end, for all times now we ask the question that fine I have this formulation then what do we do? How do we use this to construct a solution? The basic idea is that I will take the same representation in space for t as we have been doing for the static problem, t as a function of x for the static problem of sigma I will put some coefficients, coefficients it will be T_i phi_i x i is equal to 1 to number of degrees of freedom n where phi_i our usual C 0 basis functions. You could use the add functions, you could go ahead and use the lagrangian shape functions that we had talked about over and over again.

When I want to use this representation to write the solution as the function of space and time we use variation of parameter approach that is these parameters T_i the coefficients will become functions of time, in our representation we are going to now use the variation of parameters approach. These are functions of time. This is the approach that we are going to take in a representation and you see that very nicely this fits into the scheme of things the way we have been following it.

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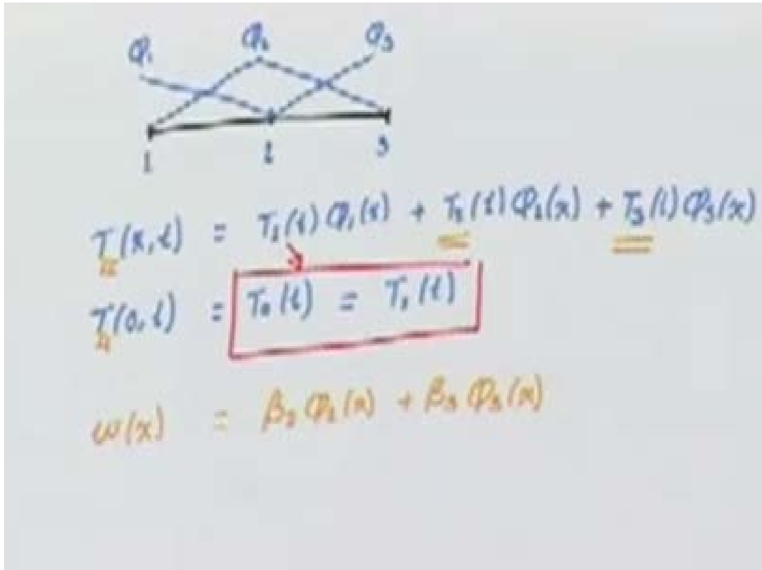

Once I have this. Let us take an example, take a simple example that I take a mesh it just 2 linear elements. What will the basis function look like? They again look like this. If I look at this is my ϕ_1 as a function of x ϕ_2 as a function of x ϕ_3 as a function of x so this is node 1 node 2 node 3. The T as a function of x and t is equal to T_1 as the function of t ϕ_1 of x plus T_2 as a function of t ϕ_2 of x plus T_3 as the function of t ϕ_3 of x. Now what is the boundary condition? T at the point zero for all times is equal to T_0 of t, this will be equal to what for x equal to 0 ϕ_1 is equal to one ϕ_2 is equal to 0 ϕ_3 is equal to 0 so this is going to be equal to T_1 of t.

What we have obtained is that the value of this coefficient is obtained from the given boundary condition for all time. Then it becomes relatively easy to see what we are trying to do this and this coefficients T_2 and T_3 are the unknown coefficients that hereafter as far as the function of x what will happen here, how will I have the w instantaneously will be equal to β_2 ϕ_2 of x plus β_3 ϕ_3 of x. This is how we are going to choose the representation of the finite element solution. I will do FE, this is FE. This we have to keep in mind that this is the finite element solution.

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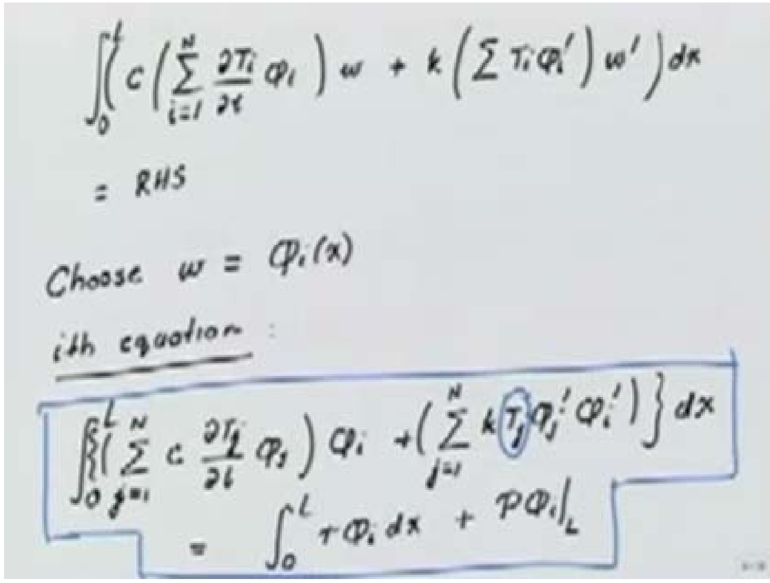

If I take this representation of the approximation for temperature t, we put it back in this formulation that we have done. We will get integral 0 to L c in terms of the derivatives I will get I is equal to 1 to 3 del T_i del t phi_i for all particular problem in the general case we will replace this by n, let me replace it with n in the general case, N phi into w plus k into sigma T_i phi_i prime, prime means derivative of phi_i with respect to x. So whether the prime derivative is concerned I will take the derivative of the coefficients where as the spatial derivative is concerned I take derivative of the basis function. This into w prime dx is equal to the right hand side.

What do we do? What it's quite easy when again we choose to construct the required number of equations, how many equations do we need? In the special case that we have taken we need only two equations, because there are two unknowns T_2 and T_3 as the function of time. In the generic case we need if it is for a mesh or say n-1 elements we need only n-1 equation because the first equation as the first previous elements known choose will in a generic way we will choose w is equal to phi_i of x. I will get the i th equation, which looks like interval 0 to L I will put sigma i equal to 1 to N c del t not i I will now put it j, because del t del t phi_j into phi_i plus sigma j equal to 1 to N k T_j phi_j prime phi_i prime dx is equal to integral 0 to L r phi_i dx plus p into phi_i at L.

This is essentially our equivalent of the weak form that we had obtained for the static problem here. Just let's look at it, here the solution is in terms of this unknown coefficients T_j and their time derivative, the first derivative coefficients that's why it's the parabolic problem because the first derivative of the unknown coefficient with respect to time is setting time.

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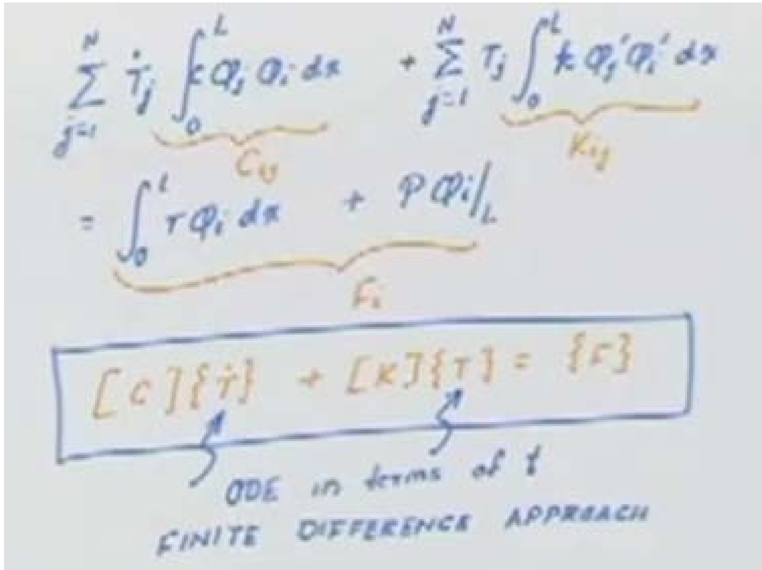

If I write it now this will be equal to writing it as sigma T_j dot J equal to 1 to N integral 0 to L phi_j phi_i c dx plus sigma j equal to 1 to N T_j integral 0 to L k phi_j prime phi_i prime dx this is equal to integral 0 to L r phi_i dx plus p phi_i at L. I will call this whole thing this integral I will call as a big C_{ij}. This integral I will call, which we have already seen as the big K_{ij} and this whole integral is F_i then I can write this problem rewrite it in the following form that c into T dot whether vector t dot which contains its components T_1 T_2 T_4 plus K into the vector T is equal to the vector F. This is the matrix form of this problem. You see that what was happened is we have now written this in terms of partial or time derivatives of these unknown coefficients and the values of these unknown coefficients. This becomes an ordinary differential equation for this coefficients T_i in terms of time. Our job is now to find how this coefficients change with time or evolve in time. What do we do this ordinary differential equation? This is actually a first order differential equation now can be solved using finite difference approach.

As far as taking care of the part in the variation with x we use what we have been doing till now in the standard element finite form is in the basis functions, we integrated the x part out we got equations in terms of this unknown coefficients T_i and the derivatives.

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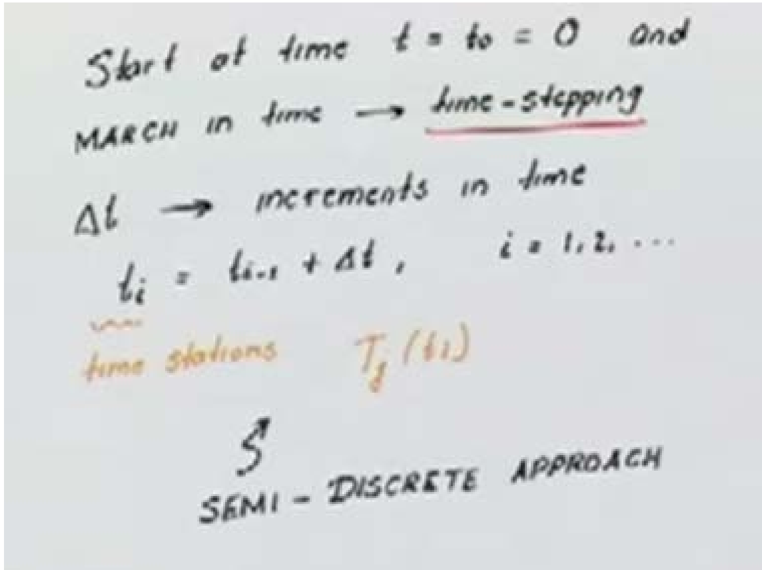

This leads to a first order system in terms of these T_is in the derivatives which we have to solve. What is the idea of the finite difference approach? That I start, start at time t equal to t_0 which is equal to zero and we do a marching in time which we will call as time stepping. That is we take its increments in time as delta t its increments could different for various time stations increments in time. t_i is equal to t_{i-1} plus delta t, i equal to 1, 2, till we reach the desired final time t. It could be the time when the solution reaches steady state it could be some cut off interval for obtaining the transients it could be the long time behavior, all sorts of requirements about, how far we want to march with respect to time? And what we are doing is we are trying to construct solution at each of these, what we call time stations?

We would like to obtain the value of this coefficient at each of this time station T_j that is the coefficients at the time station T_i is what we want to construct. Given this coefficients at the station T_i we can construct the whole response of the heat the temperature profile has it changes with respect to time response over the whole x as it changes with respect to time. That whole picture we can construct. This approach that we have taken is called as semi discrete approach.

Why it is semi discrete? Because we integrated with respect to x who convert the equations to this discrete equations ordinary differential equations in terms of time in terms of this discrete variables T_i. This looks very much like the motion of a multi degree of freedom spring mass, (28:07) spring mass system. Where each mass we are interested in its motion with respect to time. This coefficient represents something similar.

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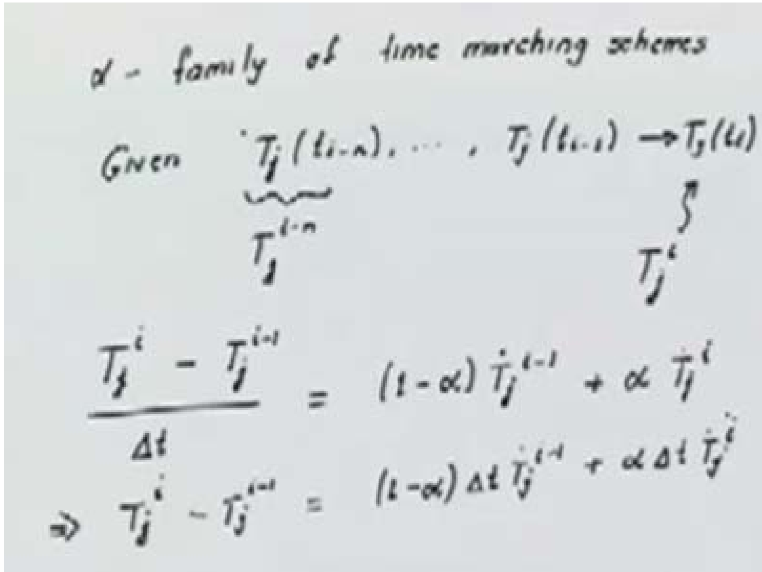

There are standard ways of doing this time marching. What we are going to use is we approach the alpha family of time marching schemes. Basic idea here is that if I am given the coefficients T_i for time intervals i-1, i-2 because if I know i-1, I know the values of this T_i at all-time intervals up to i-1 I know for i-2, i-3 and so on and using that I construct the solution at the time interval T_i. Given the value of this T_js at time station i minus some small n up to T_j at time station t_{i-1} I would like to obtain T_j at the station t_i, in our standard permutation we will call it T_j at station i-n. This means that it is that time station i-n and this I will call as T_j at station i. Given these I would like to obtain it, the alpha family uses the value at previous time station.

How does it go about doing it is the following? We have that T_j i let us say I know the value at i-1 - T_j i-1 divided by delta t, if we are using uniform time step if not we use the variable time step this is equal to 1 minus alpha into T_j dot at station i-1 plus alpha T_j dot at station i. If I am given the solution at time i-1 and its derivative temporal derivative time i-1 then I can construct the derivative at time i in terms of the unknown value of this coefficient at time i and the values at i-1 the derivatives at i-1. This is the alpha family of temporal approximations. The question is if I do this then we can see that this will imply that T_j i - T_j i-1 is equal to 1 minus alpha delta t into T_j dot i minus 1 plus alpha delta t T_j dot at station i.

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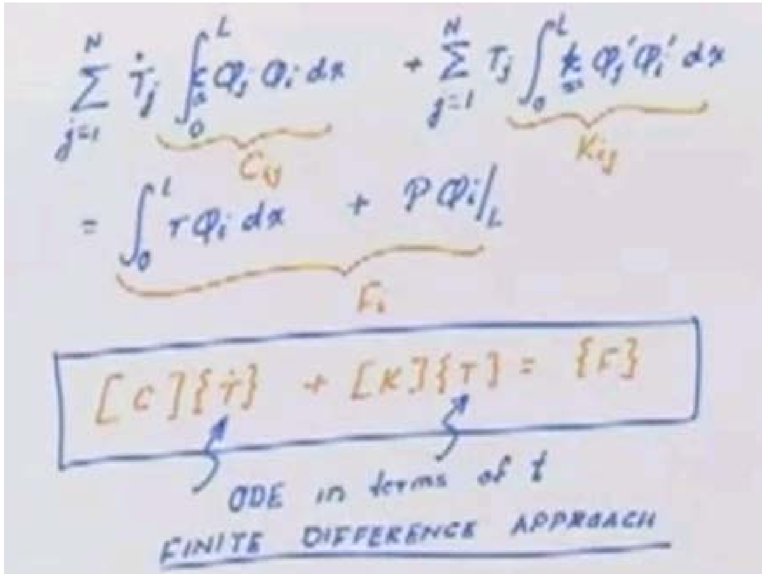

What can we do? We have if we remember our differential equation if I go back to the differential equation that we have created you see if our c is let's assume that c is not a function of this small c is not a function of time, similarly the k is not a function of time. In that case we can say that this matrix C_{ij} the component of the matrix are independent of time, the components of the matrix K_{ij} are also independent of time, in case they are it is not a problem then these components will also become functions of time and we evaluate them at each time station. For simplicity we will keep the C and K independent of time.

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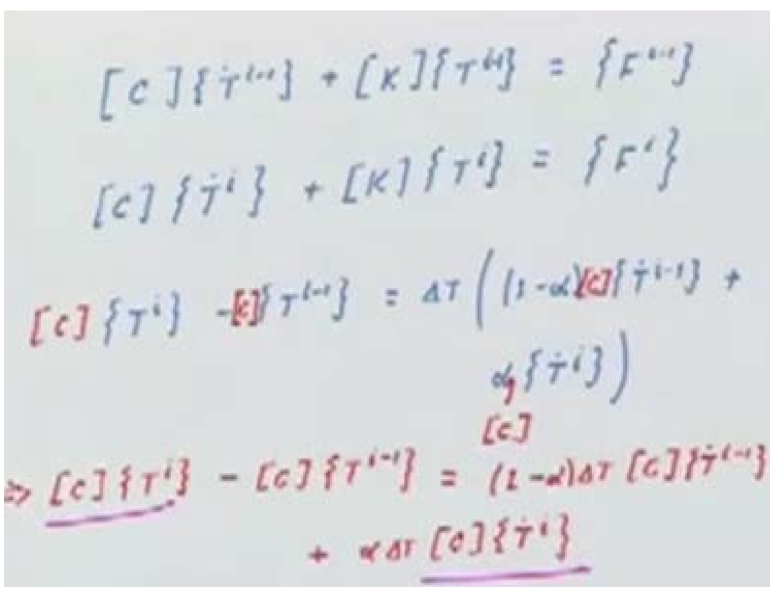

We have here given this then the equations will become c at time station i-1, I will get C into T i-1 dot plus K into T at the station i equal to F at the station, T at i-1 F at station i-1. This is at the time station i-1, because this equation as to be true at each time station similarly I will also have c at station i T dot at station i plus K into T at station I, this is equal to F at station i. These things are known I mean these are the equation that we have, we come here in this what do I do? I am going to take this equation and say that here I will operate on this with c, that is $C_{ij.}$ What will I get this equation will be equivalent to writing the following that the vector T at station i minus vector T at station i-1 is equal to delta t into 1 minus alpha into vector T dot at i-1 plus alpha T dot at i.

There I wrote it for the component here we writing for the whole vector. What I am going to do is operate on both sides with c. I will have c, c here is c and here also I am going to have c. If I operate on both sides with c, I will get implies and then we end up getting c into T at the station i minus c into T at the station i-1 is equal to 1 minus alpha delta T into c operating on T dot at the station i-1 plus alpha delta T c operating on T dot at the station i.

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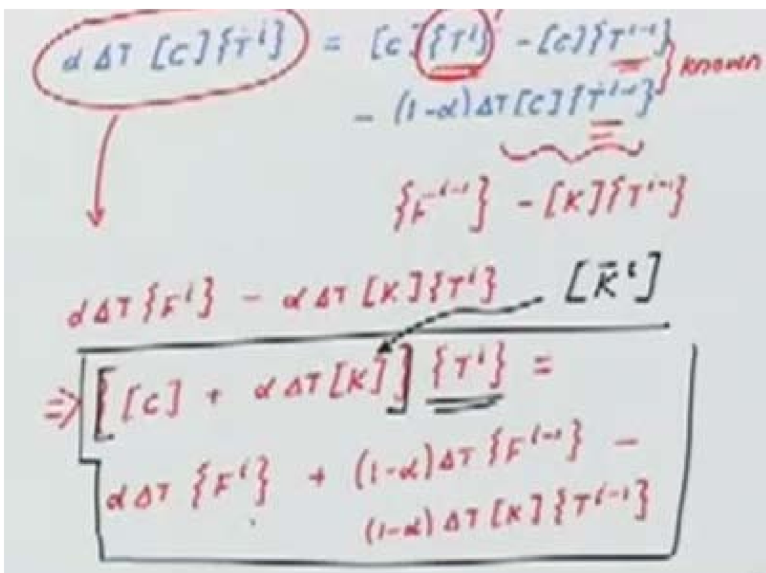

Remember that T and T dot at i are not known. What we have done by doing this we have expressed this at time station i in terms of c into T_i at the time station i. This will give me alpha delta T c into t dot at time station i is equal to from what we have done c Ti minus C T i-1, C at the time station i minus C into value of T at time station i-1. In minus, minus one minus alpha delta t into T dot at i-1.

This is what we will have as a representation of this part in terms of this and this and this both these things are known, from the previous step. This is the unknown and we want to solve from this. We are going to use this expression where in our equations here and equations. What do you see that c into T dot at station i-1 equal to F at i-1 -k at k into T at i-1, we can write like this. This is equal to essentially I can replace this in terms of what we have already written what I will get is I will get an equation in terms of T at the station i. In the following we are going to arrive at that expression. This now from the first of the equations that we had written will be equal to F at station i-1 minus K into T at station i+1. This substitution I will do. Now from the next equation if I multiply both sides with alpha delta T alpha delta T, into alpha delta T. What I will get as c T dot at station alpha delta T c t dot is equal to this point of t this minus this. I can do the following that this will be equal to alpha delta T F at station i minus alpha delta T K into T at station i.

What we will have implies we will collect all the terms corresponding to T i on one side. We'll have c plus alpha delta T K into operating on T i is equal to alpha delta T F i plus 1 minus alpha delta T F i-1, minus 1 minus alpha delta T K into T i-1. This is essentially a system of equations that we have setup and you see that this is all in terms of the unknown T I, this I will call as K bar at station i. This will become K bar at station i into operating on the t unknown t at station I, in terms of the F at station i plus minus terms which are obtained from the previous stations. This whole thing will become as F bar at station i.

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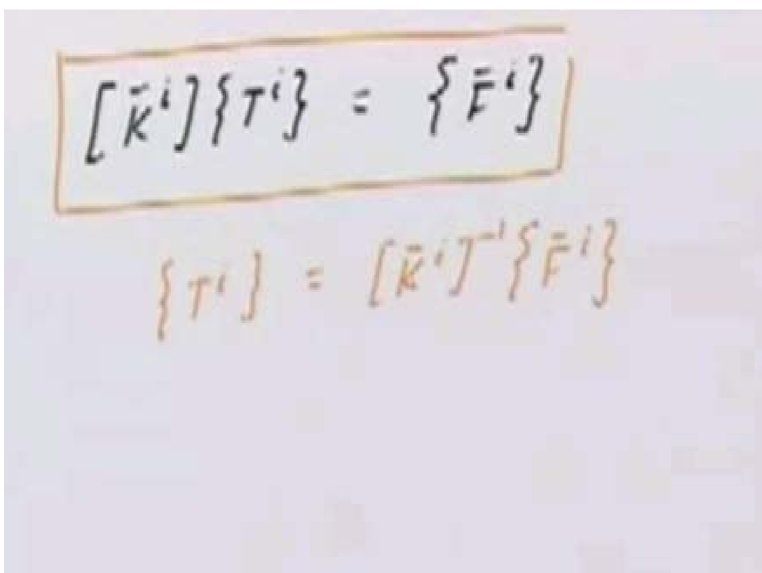

We will write K bar at station i into T at station i is equal to F bar at station i. This is a system of equations we have to solve and then T at station i is equal to K bar at station i inverse into F bar i. This is how we are going to solve the problem. We didn't say anything about the choices of alpha that we can take. Depending on the various choices of alpha here we get different types of schemes.

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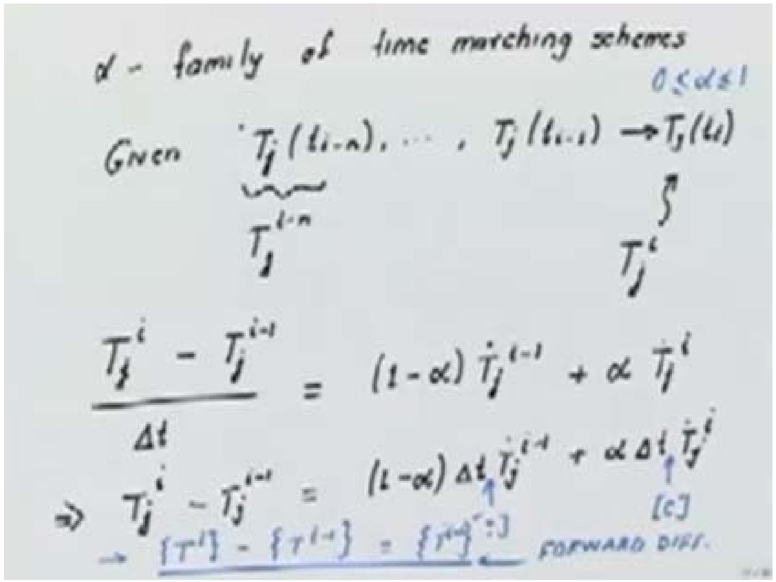

For example if I go to our definition of this alpha family, we generally what we are going to take is alpha is between 0 and 1 if alpha is equal to zero. If you see later it is very interesting thing if alpha is equal to zero then we will get T_j at i minus T_j at i-1 is equal to T dot at i-1 alpha equal to zero gives me a choices if I put alpha equal to zero t at the vector t at i minus vector t at station i minus one is equal to vector t dot at station i-1 at station i-1, this approach is called the forward difference scheme which means that in taking the time derivative at a current instance I used the value ahead of the current instant to get the derivative, the value difference will be into delta t. If I take the value ahead of the current station minus the value at the current station divide by delta t that becomes my approximation of the derivative at the current station.

This is the forward difference scheme; this scheme as its disadvantages what happens is if my time step this delta t that we have chosen is bigger than a critical time step size for the representation that we have taken the discrete representation that is for the choice of degrees of freedom n or the choice of the mesh. If this is bigger than a critical time step size then my temporal solution when I do the time marching solution will diverge. It will go away that is the solution in this case is set to be conditionally stable that is the stability or the propagation of the error in time is dependent in this case on what is the size of the time step we have taken.

If now I choose alpha is equal to one, if I choose alpha is equal to one then it quite easy to show that the T_j at i - T_j at i-1 is equal to delta t t dot j at i that is the temporal derivative at the current station is given in terms of the value at the current station and the value at the previous station. This is called the backward difference scheme that is the derivative is obtained by taking the value behind the current temporal station. I could take various other values of alpha. If I take alpha equal to half or two third then we get the crank nicholsons scheme and the galorican scheme, different choices of the alpha will give us different solutions and different characteristics of the temporal solutions.

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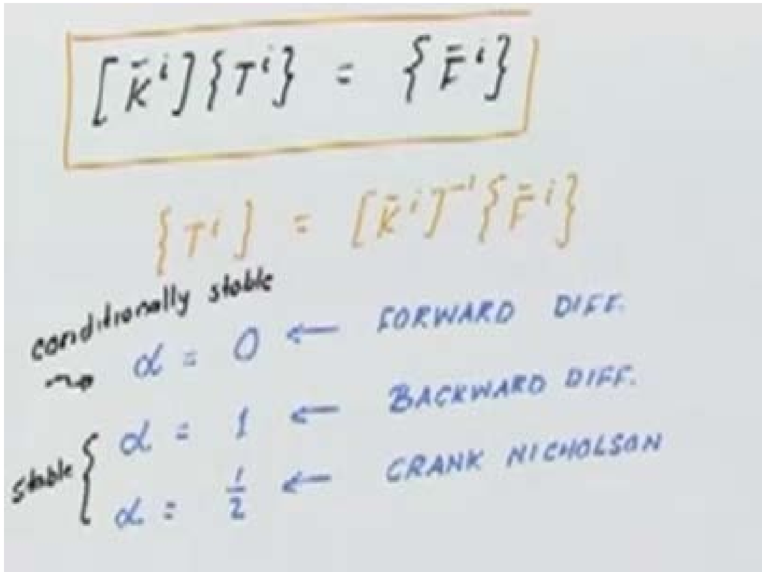

We’ll look at the choice alpha is equal to zero, this will be forward difference alpha is equal to one this is backward difference scheme, alpha is equal to half is the crank nicholsons scheme. This one is said to be conditionally stable and these two are both stable schemes. Whenever we are choosing this time stepping delta t we are doing approximation in time instead of solving the differential equation for all times you have solving it at discrete time stations then obviously that brings in errors, errors due to the representation of the function in time. This the truncation error that you are going to get, there are also because of the meshing precision involved that is we are using single precision arithmetic or double precision arithmetic especially when you are using matlab or mathematical unless you specify the precision it will work with single precision.

In that case we are introducing further round of errors, that is the number which should have been two is seen by the machine as 1.999, these introduce round of errors they also propagate can propagate in time. When the round of errors and the truncation errors are always bounded that is they do not glow in time the method is set to be stable for if it is for all sizes of delta t. It does not mean if the method is stable that the solution is accurate that is depending on the delta t my error can be large but it will not grow in time that’s all its says accuracy depends on how fast my solution converges to the exact solution for all times as my delta t becomes small okay we have another definition here which is called consistency. Consistency means that as delta t comes down the approximate solution reaches or converges to the value of the exact solution for this system at that time at any time T.

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These are certain new definitions which are been inherited from the finite difference terminologies that are sitting there, but we discuss those in detail that later. If I now take the forward difference scheme the advantage of the forward difference scheme is that in this scheme of thing that we have obtained. If I go back to our representation here what will happen is this part will get knocked off, I will get c into Ti is equal to this will now get knocked off delta t F i i-1 delta t k into T i-1.

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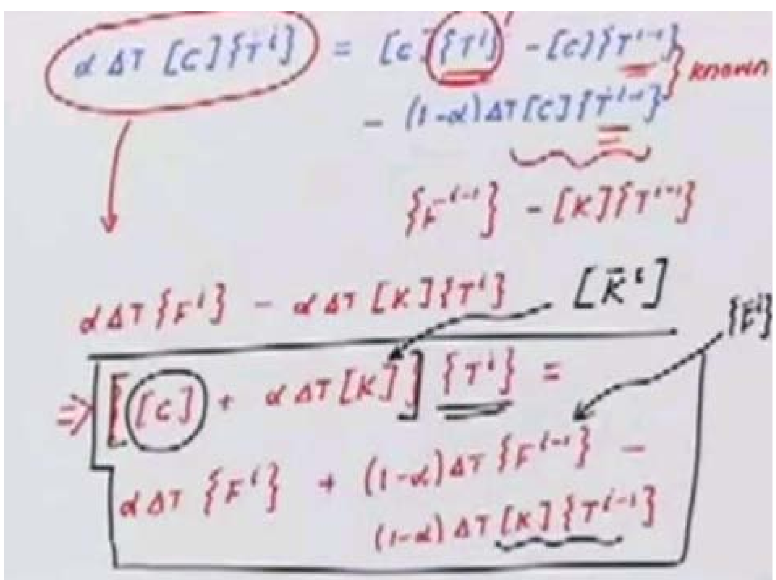

This is a matrix vector multiplication. All we need to obtain is K into Ti, we do not have to invert K why in many cases c can be reduced this matrix c, because this will get knocked off can be reduced to a diagonal form. We can do lumping of the elements. This we can see in a book on vibrations there is a lot of literature available on this. We can reduce it to a diagonal form and in many times it will come out the way you choose the basis function and so on it could come out to be diagonal. In that case if this matrix c is diagonal, then you can obtain the values of Ti or Tj at the station i explicitly because you do not have to do sophisticated inversion of c because inversion of c will give you one over the diagonal terms in the diagonal.

That leads to an explicit solution at the time station i without having to invert any matrices. This is something which is very attractive about these schemes, because when I want to solve the problem the behavior of a system in time have to solve from many times in times and if I have a tool with which at a time step I can very quickly get the values of this coefficients without having to invert, this matrices inversion takes a lot of time n cube the operations in the case of standard Gaussian inversion Gauss (51:58) inversion.

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Then in that case if this can be avoided then we get the very quick solution and we can do very efficient job of marching in time. This is what the forward difference scheme is called an explicit method with understanding it is really explicit when I do not need to involve the matrix c. If again my alpha is equal to one, if I go to the backward difference scheme, if alpha is equal to one then you see that here K is also sitting, k in general will not be diagonal. It will make this whole matrix K bar as non diagonal, then we have to do the inversion. When you have to do the inversion then the backward difference are any other scheme where this matrix is have to be inverted are called assert to be implicit-implicit schemes.

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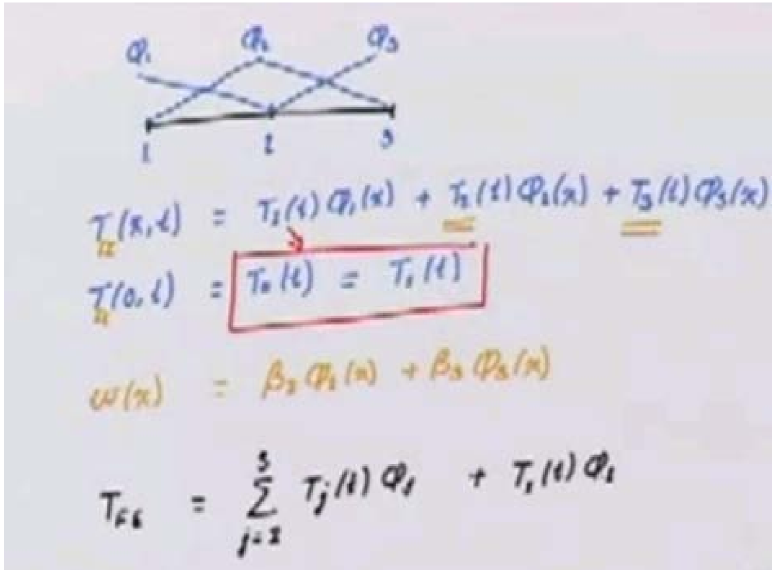

If I go back to a 2 element problem that we are posed right in the beginning here where we had this kind of representation and we have said out T_{FE} in this case will be equal to sigma j equal to 2 to 3 T_j as the function of time phi_j plus T_1 as the function of time phi_1. I will deliberately put this out, because this will be imposed to the boundary condition.

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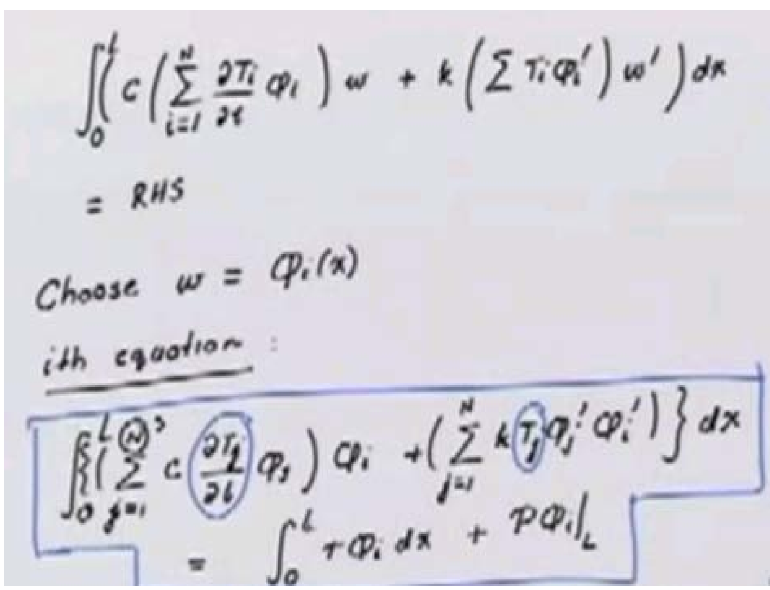

Then I can put this in the formulation, that we have taken that is here n will become, you can take n equal to 3 in the various things we have done here n equal to 3 we can put.

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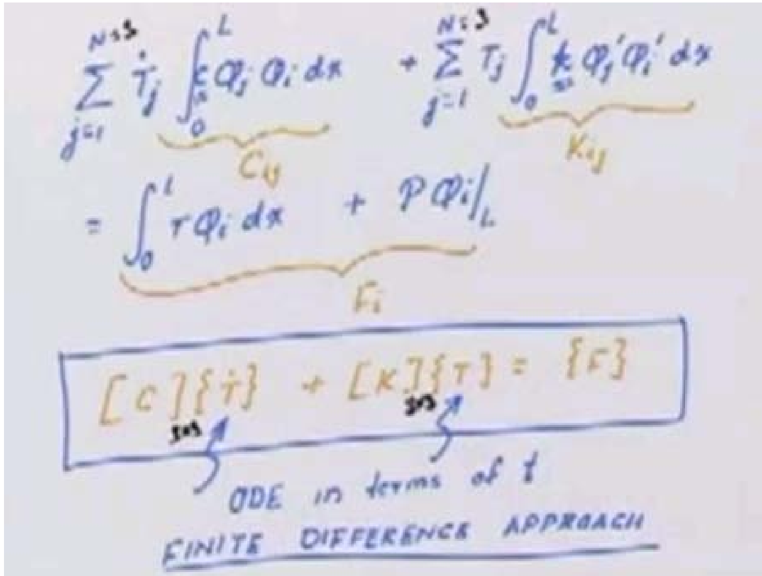

And we can go through the steps everywhere n equal to 3 get to this matrix which will be a 3 by 3 matrix.

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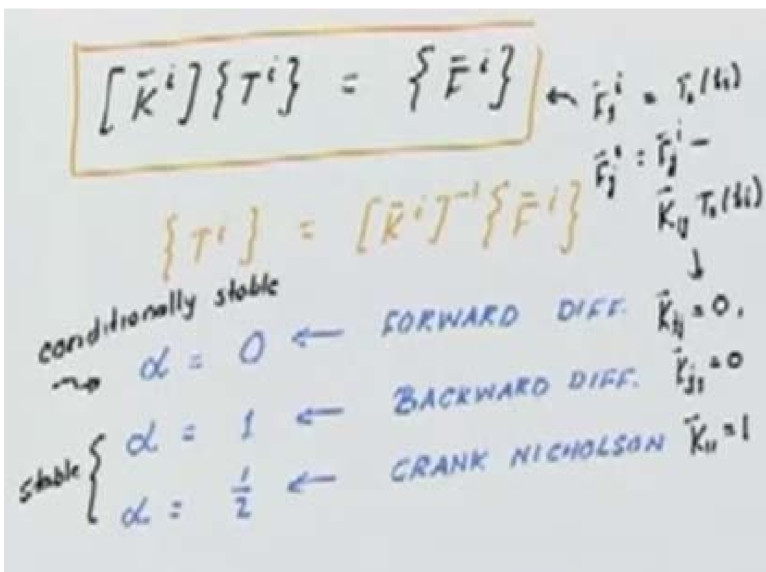

The bottom line will be that when we do all this business of inversion and having this manipulation then we have to apply the boundary condition, that is again you have to force the T_1 i to be equal to the known value at that station which again comes through the K hat matrix, that in the k hat matrix I am now going to force the first diagonal term to be one all other this thing term is in the first row to be going to zero all terms in the then I will modify the load vector in such a way that the first load vector entry becomes equal to Ti T_1 at time t, at time Ti and all other load vector entries are suitably modified.

If I have to do it here I will put F_1 at station I will be equal to T_1 at the time Ti. Then F_j for all j's at time station i will be equal to F_j at time station i minus K bar one j into T_1 at time station Ti. This I will do and then I will set after this set K_{1j} bar equal to zero K_{j1} bar equal to zero and K_{11} bar is equal to one. I will do the same things that we have been doing till now for the static problem in order to enforce the boundary condition, that the temperature at time equal to zero is given.

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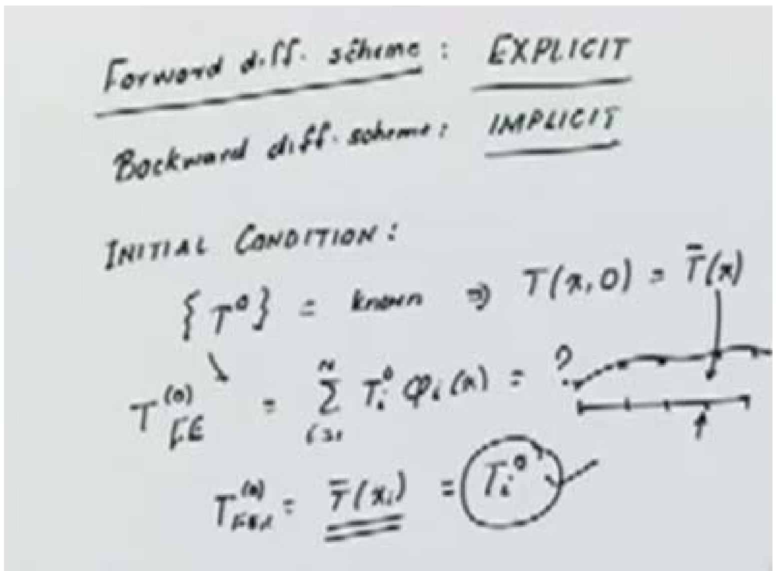

We didn't say how we are going to start, till now we have talked about getting the solution at station i when the solution at i-1 is known. We have to obtain the solution at time station zero, which is the initial condition. The initial conditions have to be specified that is this will mean T 0 is known, this implies t at all points at time t equal to 0 is equal to some function T bar of x. How will i obtain this? If you remember my T_{FE} at time station zero is equal to sigma Ti zero into phi_i of x is equal to i equal to 1 to N this is equal to, what we will do is we will say that at the nodes T_{FE} 0 is equal to T bar at the nodes x_i.

This is nothing but Ti 0. The nodal value of this given function at the nodes is taken to be the values of this coefficient, because that's what this function will be at the nodes. Initial condition gives me the values of these nodal temperatures at time t equal to 0, from there I start marching in time, I construct all the solutions specially in the Lagrange's setting I can have this nodes simply go read of the value of the function at this node, read of the value here, read of the value is here, here that becomes the value of this coefficient, put it in I get my initial temperature profile and from there I start marching in time. In the next class we are going to look at what do you mean by stability, accuracy convergence, consistency and all these issues and then we go to the hyperbolic systems.