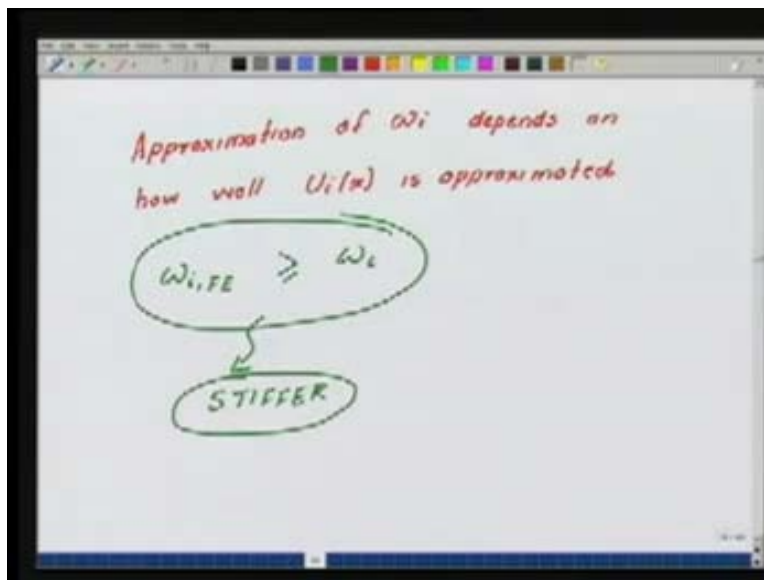


Finite Element Method
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Module – 12 Lecture - 01

We have been talking about the vibration problem till now. So let us continue our discussion further. We had said that the certain properties of the Eigen value problem that we formed in the free vibration case.

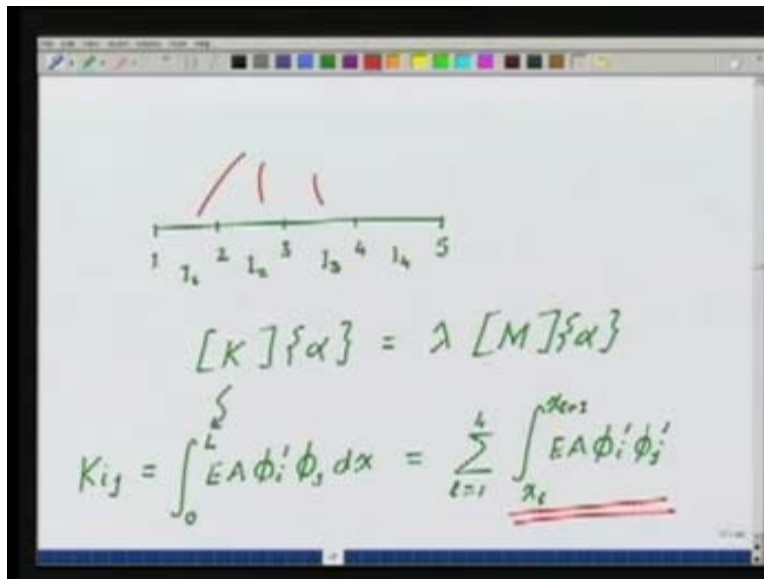
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One is that the finite element that the Eigen values or the natural frequencies obtained from the finite element solutions are higher than those obtained by the exact solution or the actual modes. This is sometimes referred to as the finite element system being stiffer than the exact one. This is some feature of the finite element solution that one has to keep in mind. Why do we have to keep it in mind? Because, if I am writing a program and to do the computation of the natural frequencies of a system, which is the very simple extension of what we had been developing till now. Then as a benchmark, I should test the program for specific problems for which we can construct the exact solution. For example, the problem that I have been considering, here in this case, when I go and use the finite element approximation and get the natural frequencies, I

should get the natural frequencies to be higher than that obtained from the exact one and as the mesh gets refined, they should converge to the exact one from above. These are certain features which have to be kept in mind. If it does not happen then something is wrong in the way I have done the programming or in the way I have developed the finite element approximation.

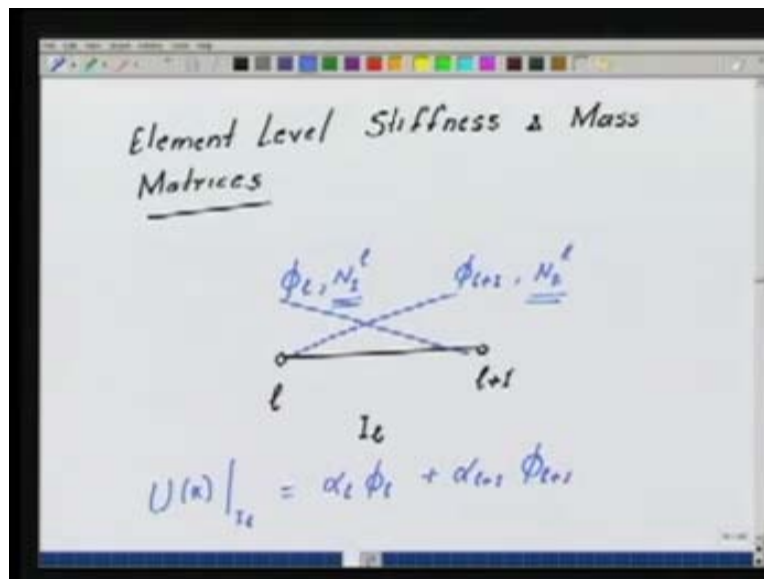
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Let us also look at some implementation issues. We had said that here I have taken this example of a 5 node system and we had let us say that we are taking a piece wise linear approximation. I can again take higher order approximations just like we have done the standard finite element computation. Here this is the element 1. This is the element 2, element 3 and element 4. So just like we did the procedure for static analysis, here we have to obtain the global matrices K and M. We have to obtain these global matrices from our finite element computation. Again I will say that look what is this K was as I had done '0' to 'L' integral of EA ϕ_i' ϕ_j' dx was K_{ij} and this can now be written as we have done earlier. That is L is equal to 1, 2, in this case 4, the number of elements ' x_1 ' to ' x_{l+1} ' EA ϕ_i' ϕ_j' . I can break the integral into sum of the integral over each of these elements.

Which of the ϕ_i 's are active in the particular element? The only ϕ_i which is active in this element, let us say element 3 is ϕ_3 and ϕ_4 . Exactly what we have done till now will now be done and we will compute these matrices at the element level, assemble them in the global matrices and that, we have done. But we have to do for the mass matrix also, where the integral will be integral of $\rho \phi_i \phi_j$.

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So we have to now compute the Element Level Stiffness and Mass Matrices. How do I go about doing it? Let us see. So we have our generic element with node $l, l+1$. This is the element I_l . Here the basis function from the figure I have drawn which is non zero. The basis function will be ϕ_l and ϕ_{l+1} . These will correspond to the element level N_1 of the element and this will be N_2 of the element. If I want to write the $U(x)$, the approximation of it in the element I_l will be equal to $\alpha_l \phi_l$ plus $\alpha_{l+1} \phi_{l+1}$. These things will not change. These things remains the same as what we have been doing till now. We are going to essentially find the effect of these in the integrals.

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$$\int_{x_l}^{x_{l+1}} EA N_i' N_j' dx = K_{ij}^l \quad \begin{matrix} i=1,2 \\ j=1,2 \end{matrix}$$

$$[K^l]$$

$$\int_{x_l}^{x_{l+1}} \rho N_i N_j dx = M_{ij}^l$$

$i=1$	\rightarrow	1	,	$j=1$	\rightarrow	J
$i=2$	\rightarrow	l		$j=1$	\rightarrow	l
$i=1$	\rightarrow	$l+1$		$j=2$	\rightarrow	$l+1$

We will have integral at the element level x_1 to x_{l+1} $EA N_i'$ for the element l prime N_j for the element l prime dx , prime means the derivative with respect to x . This is equal to K_{ij} for the element l . i is equal to 1 to 2. In fact, if we use higher order approximation, it goes up to $P+1$ and j is equal to 1 to 2. So this will give me 2 by 2 matrix in the case of the element. This will give me K for the element l . Similarly, I will have integral from x_1 to x_{l+1} ρN_i for the element, N_j for the element, dx . This is equal to M_{ij} for the element l . What is the assembly procedure? Assembly procedure will be this i locally at the element level corresponds to some global I and similarly, the j at the element level corresponds to some global J . So $i = 1$ corresponds to 1, $i = 2$ corresponds to $l+1$ globally. $j = 1$ corresponds to 1 globally and $j = 2$ corresponds to $l+1$ globally. That is I again use the local to global enumeration routine that we had made long ago in the 1 dimensional code to give me this information and then I will go and assemble in the following way.

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$$\begin{aligned}K_{11} &= K_{11} + K_{11}^e \\K_{1,2} &= K_{1,2} + K_{12}^e \\K_{2,1} &= K_{2,1} + K_{21}^e \\K_{2,2} &= K_{2,2} + K_{22}^e \\M_{11} &= M_{11} + M_{11}^e \\M_{1,2} &= M_{1,2} + M_{12}^e \\M_{2,1} &= M_{2,1} + M_{21}^e \\M_{2,2} &= M_{2,2} + M_{22}^e\end{aligned} \quad \left| \begin{aligned}M_{1,2} &= M_{1,2} \\ &+ M_{12}^e\end{aligned}\right.$$

K_{11} will be equal to, these are the global level, K_{11} plus K_{11} from the element. $K_{1,1+1}$ will be equal to $K_{1,1+1}$ plus K_{12} for the element. $K_{1+1,1}$ will be $K_{1+1,1}$ plus K_{21} for the element. $K_{1+1,1+1}$ will be equal to $K_{1+1,1+1}$ plus K_{22} for the element. This is the simple assembly procedure for the K matrix is exactly the same as we what we had done and similarly, we will do for the mass matrix. The assembly that is M_{11} is equal to M_{11} plus M_{11} and $M_{1,1+1}$ is equal to $M_{1,1+1}$ plus M_{12} and $M_{1+1,1}$ is equal to $M_{1+1,1}$ plus M_{21} and finally, $M_{1+1,1+1}$ is equal to $M_{1+1,1+1}$ plus M_{22} for the element. We do this assemble procedure. We loop our all the element and we construct the global K and the M matrices.

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Diagram of a beam with nodes 1, 2, 3, 4, 5 and element lengths l_1, l_2, l_3, l_4 . The beam is fixed at node 1. The displacement at node i is α_i . The stiffness matrix $[K]$ and mass matrix $[M]$ are defined by the following equations:

$$[K]\{\alpha\} = \lambda [M]\{\alpha\}$$
$$K_{ij} = \int_0^L EA \phi_i' \phi_j' dx = \sum_{i=1}^4 \int_{\xi_i}^{\xi_{i+1}} EA \phi_i' \phi_j' dx$$

Now the question is that the K and the M , if I take again our problem that we had considered, the mesh that we have taken, this will be a 5 by 5 system, but from the boundary condition at this end, I have to put α_1 equal to 0 because the $U(x)$ has to be 0 at the point x equal to 0. So how do I enforce α_1 equal to 0?

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The equation $[K]\{\alpha\} = \lambda [M]\{\alpha\}$ is written. The boundary condition is set $\alpha_1 = 0$ (from B.C.). The matrix equation for the unknown displacements $\alpha_2, \alpha_3, \alpha_4, \alpha_5$ is:

$$\begin{bmatrix} K_{22} & K_{23} & K_{24} & K_{25} \\ \vdots & \vdots & \vdots & \vdots \\ K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix} \begin{Bmatrix} \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix} = \lambda \begin{bmatrix} M_{22} & M_{23} & \dots & M_{25} \\ \vdots & \vdots & \vdots & \vdots \\ M_{52} & M_{53} & \dots & M_{55} \end{bmatrix}$$

One way is to actually take the global systems and eliminate α_1 . Set α_1 equal to 0 from boundary condition. We can do that reduce this 5 by 5 system to 4 by 4 system that is that remove the variable α_1 and the first equation. We actually solving for the system which will have $K_{22}, K_{23}, K_{24}, K_{25}$ all the way down to $K_{52}, K_{53}, K_{54}, K_{55}$, this into $\alpha_2, \alpha_3, \alpha_4, \alpha_5$. This will be equal to λ into M_{22}, M_{23} and so on, M_{25}, M_{52}, M_{53} and so on up to M_{55} into α . So this will be the reduce system or we can use the tool that we have developed for doing the elimination.

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The image shows a whiteboard with handwritten mathematical equations. At the top, a 5x5 matrix is written with the first row being $[1 \ 0 \ 0 \ 0 \ 0]$ and the rest of the rows containing terms like $K_{22}, K_{23}, K_{24}, K_{25}$ down to $K_{52}, K_{53}, K_{54}, K_{55}$. This matrix is multiplied by a vector α (represented by a bracketed list of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$) and set equal to λ times another 5x5 matrix. Below this, the equation $\alpha_1 = \lambda \alpha_1$ is written, with an arrow pointing to a circled $\lambda = 1$ that has a large 'X' over it, indicating it is to be discarded. At the bottom, four eigenvalues are listed: $\lambda_2, \lambda_3, \lambda_4, \lambda_5$ with corresponding frequencies $\omega_2, \omega_3, \omega_4, \omega_5$ written below them.

We go to the 5 by 5 system. How did we set α equal to 0 on both sides? We simply eliminate. We will do this, set this term to 1, first entry, then make it 0 then I will have $K_{22}, K_{23}, K_{24}, K_{25}$ and so on, into the α array, α vector is equal to λ into, I will put again here $1 \ 0, 0, 0, 0, 0, 0, 0, 0, 0$ and then M_{22} and so on. First row gets completely decoupled and we get essentially α_1 is equal to $\lambda \alpha_1$. This will give me λ is equal to 1. So we will get 1 Eigen value as 1, throw it out. We will not consider this Eigen value in our system of natural frequencies. We will throw all these Eigen values out and the remaining 4 Eigen values will be the ones, we are going to take. We will take $\lambda_2, \lambda_3, \lambda_4,$ and λ_5 . This will become our modes. This become the first mode, this becomes second mode and so on.

This will actually lead to our ω_1 , this will lead to ω_2 , this will lead to ω_3 and this will be ω_4 . We should check that indeed the Eigen values are the same as what we would have obtained by doing this. These are the next 1 would be equivalent, but throw this Eigen value out on a certain equations. With this, we should be in a position to set up the Eigen value problem and solve it.

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$$\int_{x_k}^{x_{k+1}} EA N_i^t N_j^t dx = K_{ij}^t \quad i,j=1,2$$

$$[K^t]$$

$$\int_{x_k}^{x_{k+1}} S N_i^t N_j^t dx = M_{ij}^t$$

$$\begin{array}{l} i \rightarrow I \\ i=1 \rightarrow I \\ i=2 \rightarrow I-1 \end{array} \quad \begin{array}{l} j \rightarrow J \\ j=1 \rightarrow J \\ j=2 \rightarrow J-1 \end{array}$$

One more thing that we would like to talk of is again while doing this integration, we go to the master element, use all the two set we have developed as for as integration over the master element is concerned. Do the numerical integrations. Create the matrices and we will get a solution.

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Handwritten notes on a whiteboard:

$$d_n = \omega_n \sqrt{\frac{\rho}{EA}}$$
$$\Rightarrow \omega_n = \frac{(2n-1)\pi}{2L} \sqrt{\frac{EA}{\rho}}$$

$\omega_1 < \omega_2 < \omega_3$
distinct eigenvalues
 n^{th} mode

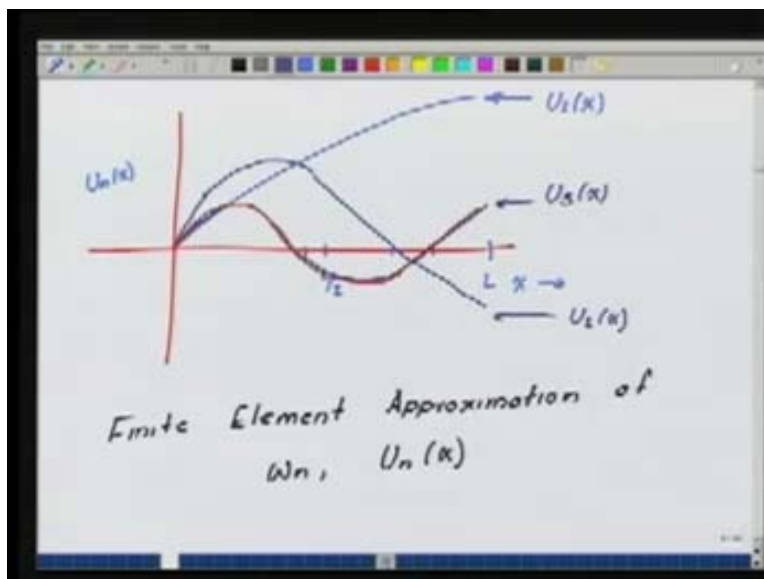
n^{th} natural frequency

$$U_n(x) = \sin d_n x$$

Oscillatory

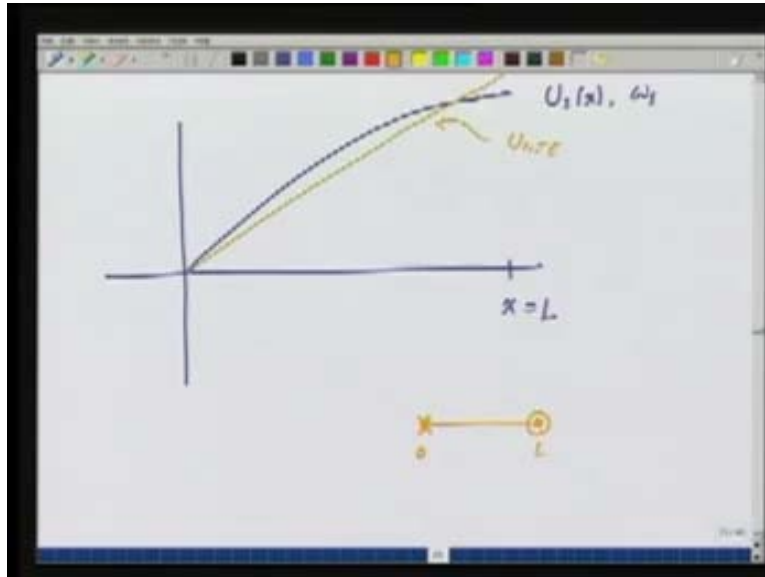
Let us go back and see what we had done as far as the exact solution is concerned. We had these functions as our modes corresponding to the natural frequencies ω_1 , ω_2 , ω_3 and ω_4 and we set these modes are oscillatory in nature and we set keep a note of that because this was needed in what we do.

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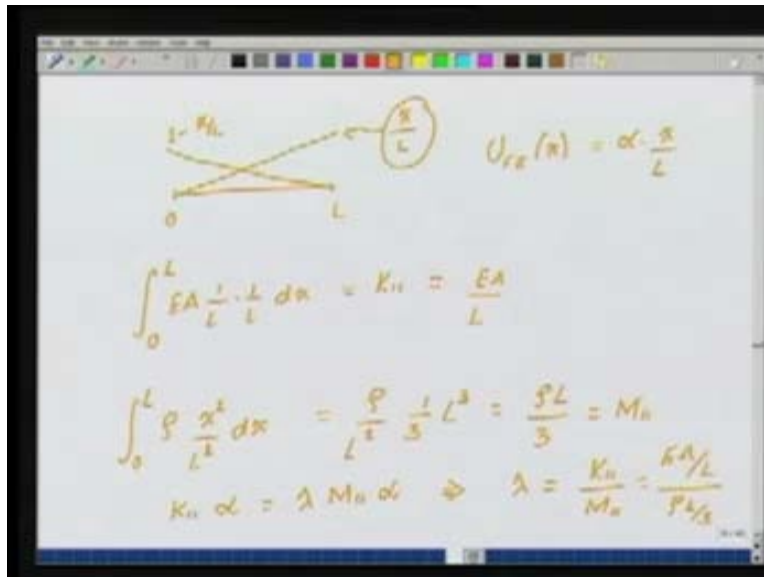
The first mode shape is like this. Second one is like this. Third one is like this. Imagine, let us go back and recreate this picture in what we are doing.

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Let us come here. This is x is equal to L and here is the exact U_1 as the function of x corresponding to ω_1 . What will be the finite element approximation do if I took the one element solution? That is if I took a mesh with only one element. This is L . This is 0 . I simply put one element. In that case, we will see that we will get a 1 by 1 system because one element will have the two nodes. This node will be knocked off will be left only in terms of this one and the solution may look some thing like this. This will be we get as U_{1FE} from the finite element solution. It will be a straight line. This curve is quite a nice curve. So this will not be such a bad approximation and we can see that if we solve the one element problem, that is a single degree of problem, we will get an Eigen value which is a very close to the exact point.

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$$u_{FE}(x) = \alpha \cdot \frac{x}{L}$$

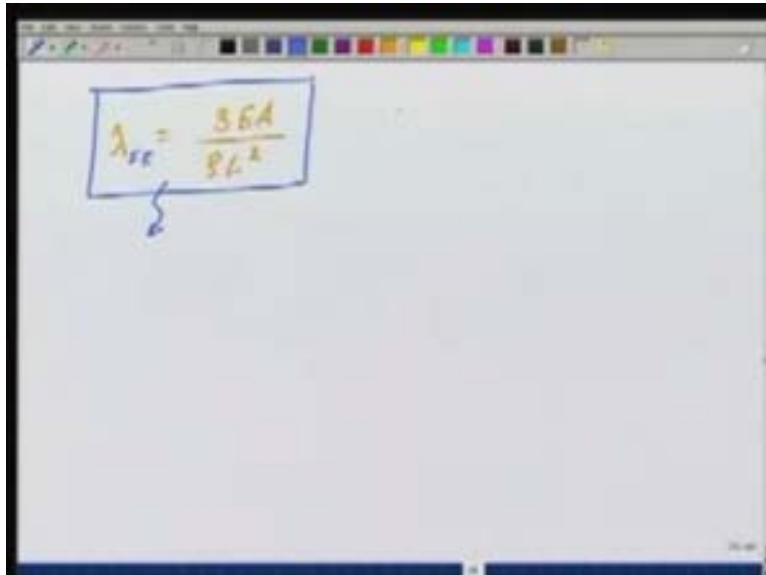
$$\int_0^L EA \frac{1}{L} \cdot \frac{1}{L} dx = K_{11} = \frac{EA}{L}$$

$$\int_0^L \rho \frac{x^2}{L^3} dx = \frac{\rho}{L^3} \cdot \frac{1}{3} L^3 = \frac{\rho L}{3} = M_{11}$$

$$K_{11} \alpha = \lambda M_{11} \alpha \Rightarrow \lambda = \frac{K_{11}}{M_{11}} = \frac{EA/L}{\rho L^3/3}$$

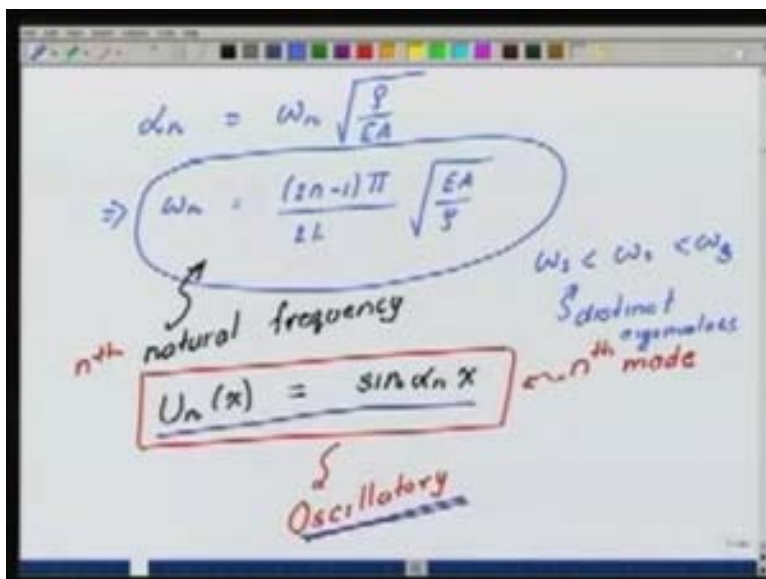
In fact, we can do it here. Let us do it here. Here if this is 0, this is 1 the first shape function will be essentially this one and this one. This shape function will be x by L and this one will be $1-(x$ by $L)$. So we are only interested in this. u_{FE} as the function of x will be equal to αx over L . Integral 0 to L , EA , the derivative of x by L . So 1 by L into 1 by L , dx is equal to K_{11} . This will be equal to EA by L . Similarly, the mass matrix will be 0 to L , ρ into x squared by L squared dx . This will be equal to ρ by L squared into x cubed by 3. So it will be $(1$ by $3) L^3$. So this will be equal to ρL by 3. We get this is M_{11} . If I do this problem, I will get K_{11} into α is equal to λM_{11} into α . So λ will be equal to K_{11} by M_{11} which will be equal to EA by L divided by ρL by 3. This will be equal to $3 EA$ by ρL squared.

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$$\lambda_{FE} = \frac{3EA}{\rho L^2}$$

Lambda will be $3EA$ by ρL squared, lambda from the finite element solution. So this is what we will get for that problem. Once we have this, now what about the alpha? The alpha can now be computed quite easily but let us look at this lambda. What was the lambda exact?

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$$\alpha_n = \omega_n \sqrt{\frac{\rho}{EA}}$$
$$\Rightarrow \omega_n = \frac{(2n-1)\pi}{2L} \sqrt{\frac{EA}{\rho}}$$

$\omega_1 < \omega_2 < \dots$
 \int distinct eigenvalues

$$U_n(x) = \sin \alpha_n x$$

Oscillatory

n^{th} natural frequency

n^{th} mode

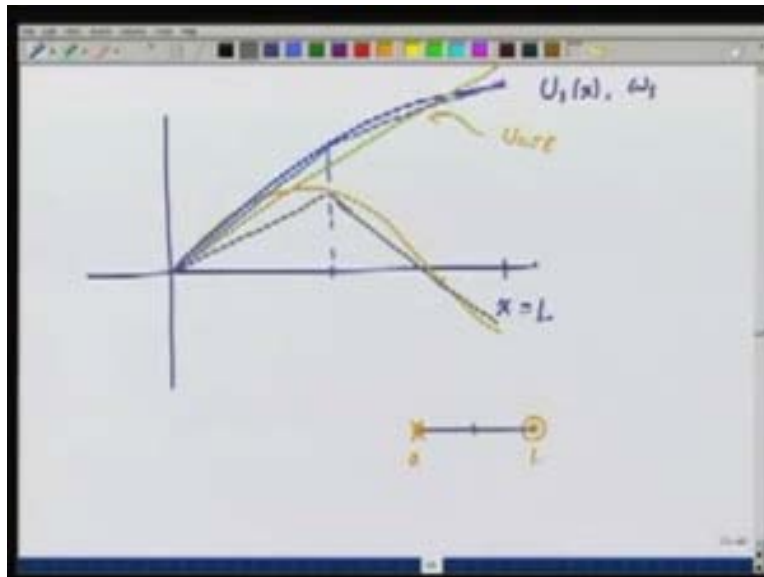
If I go back to the earlier result, lambda exact will be equal to, it will be essentially the square of this, because the way we have done omega squared was our lambda. So let us go to omega and so it will be 2 into n minus 1, into pi by 2L into EA by rho, whole squared pi by 2L into EA by rho whole squared.

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The image shows a whiteboard with handwritten mathematical equations. On the left, a box contains the equation $\lambda_{\text{ex}} = \frac{36A}{9L^2}$. To the right, the equation $\lambda_{\omega} = \left(\frac{\pi}{2L} \left(\frac{EA}{\rho} \right) \right)^2$ is written. Below this, a series of approximations are shown: $\lambda_{\omega} = \left(\frac{\pi^2}{4} \right) \frac{EA}{9L^2} \approx 2.5 \frac{EA}{9L^2}$. The number 2.5 is written above the fraction $\frac{EA}{9L^2}$. At the bottom left of the whiteboard, the letter 'h' is written.

That will be pi by 2L into EA by rho whole thing squared. This will be equal to pi squared by four into EA by rho L squared. If we see that pi squared will be approximately 10, so 10 by 4 and this will be approximately 2.5 and this is approximately 2.5 EA by rho L². This is the lambda exact. The lambda exact is less than the lambda, we have obtained and the natural frequency will be square root of this. This is quite close as an approximation. Not bad with one element, we could create a solution which is quite close to the one which we actually needed.

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Now let us go back and look at the two term solution, if I did. What will happen, if I did the two term solution? Put this, what will happen if I do the two term solution that is when I put two elements? This approximation of the mode shape will become better. As this becomes better, I will see that the λ_{1FE} will come much closer to λ_1 exact and further, if I took this 1 term solution, I just cannot expect to do a good job for the second mode. There is no second mode which I can capture. With the two term solution now, I will have two Eigen values. Two frequencies can seemingly be approximated with the two term solution. What was the second mode? The second mode would do this. In the two term solution, for the one mode, very good, for the second mode, I will do most probably, I will do this. This is certainly not a very good approximation of the second mode, but nevertheless, it is an approximation and we will see that when I took two element mesh, the λ_{1FE} will be very close to the λ_1 that I exact but the λ_{2FE} will not be as close as to the λ_2 exact but it will again be higher, not so bad, but badness will be certainly a mode as compared to the λ_1 . As we refined the mesh further and further, the quality of the approximation of the natural frequency or the Eigen value corresponding to this problem becomes better and better.

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$$\lambda_{FE} = \frac{3EA}{9L^2}$$

$$\lambda_{ex} = \left(\frac{\pi}{2L} \left(\frac{EA}{9} \right) \right)^2 \approx 2.5 \frac{EA}{9L^2}$$

$\lambda_{ex}, \omega_{FE} \rightarrow \omega_{ex}$

$$|\lambda_i - \lambda_{iFE}| \leq C U(U_{i,ex} - U_{i,FE})$$

$$U = \int_0^L EA (U_{i,ex}' - U_{i,FE}')^2 dx$$

With mesh size h , the λ_{FE} or in that sense ω_{FE} tends to the ω_{ex} . There is a very interesting way of measuring it and that is given by if we say in this case, where we have distinct Eigen values. λ_i minus λ_{iFE} due to the finite element solution for a given mesh, I could have hundred elements in the mesh or I could have five elements in the mesh but as the mesh is refined that is has the h changes, I am using the h method of the finite element method, this will be less than equal to some constant into the strain energy of U_i exact minus U_{iFE} . This is the very important result. This U is the strain energy which means that is given by integral 0 to L , EA into U_i exact prime minus U_{iFE} prime whole squared dx . So this depends on the strain energy of the error, this is called the strain energy of the error in the Eigen mode or in the Eigen function. How well can finite element method approximate the corresponding mode also reflects on how nicely can we obtain the corresponding frequency?

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$$\underline{|\lambda_{i,ex} - \lambda_{i,FE}|} \leq \underline{C_i h^{2p}}$$
$$|\lambda_{i,ex} - \lambda_{i,FE}| \leq C_i h^{p \min(2, p)}$$

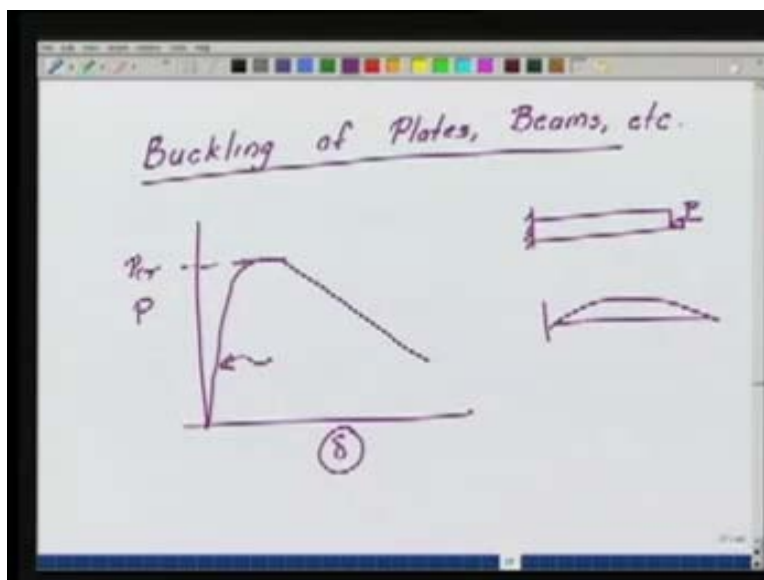
This number is dependent on this squared and here, in the case of Eigen value problem, essentially for such a case, $\lambda_{i,ex}$ minus $\lambda_{i,FE}$, actually I should have it other way now but since I have taken the absolute value. It is something $C h$ to the power of $2p$. C will change for the corresponding mode that is as we go to the higher and higher frequencies, this constant will degenerate further. We will get bigger and bigger as the weight of the convergence will be this. We get twice the rate of convergence of the normal finite element solution. We had talked about the energy norm that is square root of this strain energy. Here, the error in the Eigen value converges as the strain energy itself. Eigen value problems are much more forgiving as compared to getting information about the point by stresses point by strains. We can obtain our Eigen modes with the much coarse mesh, pretty good values of the Eigen values or the natural frequency can come with the much coarse mesh as compared to what will be needed to get point wise estimation of the stresses and so on and other issue.

This is one thing that one should keep in mind that depending on the particular problem, we have to design the approximation in such a way that we get accurate results and we should know something more about how the solution behaves, in order to be confident of what we are computing and what we are giving to the particular user. Incidentally, this same feature what we

had discussed here also works in the two dimensional problem. Let us say, I am talking of a domain with the crack, a crack domain. Here also I may be interested in solving the elasticity problem. I may be interested in getting the natural frequencies. In that case, what we will get is essentially the nature of solution and how would we do the approximation in the vicinity of the crack tip is going to effect the quality of the natural frequencies that we compute for the crack rate. So the meshing in the vicinity of the crack tip becomes very important. We can not ignore it and in fact, in this case, $\lambda_{i \text{ exact}} - \lambda_{i \text{ FE}}$ will go essentially as some constant h to the power of minimum of $2\alpha, P$.

It depends on whether α is smaller P is small and in general, the crack leads to where α is the exponent of the leading singular part of the solution. In the case of the crack, it is essentially half. Any P take will be certainly greater than half. So α will dominate that is the nature of the singularity at the crack tip is going to dominate the rate of convergence and then we have to do all the things that we did for the stress computation to get the good solution but nevertheless, the rate of convergence has the two in the power. It is certainly better than the point wise data. Here, we are talking λ as a global data.

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Similarly, Eigen value problems can also be there in the case of buckling, buckling of plates, beams etc. In this case, as we take the P against deflection for a beam, let us say the tip deflection then where I have how is the load applied. I have a compressive load being applied to the structure this center. This compressive load should lead to only an actual deformation but they comes a value if I keep on increasing the load, that comes the value of the load at which the beam or this bar does not deflect inclined. It wants to do this. It tries to have a transverse deflection. How do you take care of that? For that we will look at equilibrium in the deform configuration and there is a whole theory behind how to do the analysis. Nevertheless, as the P is increases, this is the transverse deflection. Let us say it is of the tip. This increases that comes the critical value then it falls. This is essentially what we call as a buckling load of the critical and we try to approximate it through this is solving a non-linear problem. We try to approximate it using a linearized problem. In the non-linear problem, we use concepts from finite deformation elasticity and all these things, finite elasticity but we are not going to use it here.

The idealization is based on assuming that up to the critical load, the member essentially behaves in a linear elastic way that is it is only going to give actual deformations, negative deformation because of the compression.

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The image shows a whiteboard with handwritten mathematical derivations for buckling analysis. At the top left, the critical load is given as $P = P_{cr} = E$. To the right is a simple diagram of a horizontal beam of length L with a compressive load P applied at the right end. Below this, the equilibrium equation is written as $\int_0^L EA \frac{d^2 u}{dx^2} dx = -P u|_L$. The next line shows the strain definition $\epsilon_{xx} = \frac{du}{dx}$ leading to the nonlinear strain expression $\epsilon_{xx}^{NL} = \frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2$, with a note "Van Karman approach" under the second term. The final equation is the energy balance: $\int_0^L EJ \frac{d^4 w}{dx^4} dx = P \int_0^L \frac{dw}{dx} \frac{d^2 w}{dx^2} dx$.

At the critical load, it develops, transverse deflections also and without going into the details of it, we will get that let us say that I am applying a load P is just less than P is equal to P critical minus a very small number, in that case equilibrium under the action of the load for this member is governed by standard linear elasticity that is I will get integral 0 to L in the energy sense $EA \frac{du}{dx}$ divided by dx . I am just writing the weak form of the barred problem. $\frac{dv}{dx}$ into dx is equal to P into v at L or minus P into v at L . This is what drives the problem.

At the critical load, at this point, increases the load little bit so that I get P critical. At that point, this strain we had assumed is given by $\frac{du}{dx}$, not any more, at this point, the strain will be governed by the non-linear or the finite strain or the green strain. It will be given as essentially $\frac{du}{dx}$ plus as an approximation half of $\frac{dw}{dx}$ whole squared where w is the transverse deflection. This is essentially by throwing out some higher order terms. This is called Von Karman theory approach. Cut anywhere the actual forces P and then we can show that we will have when we increase the load little bit. This part is assumed that I am in equilibrium corresponding to this load. From here, I am applying the extra bit that is I am applying the extra load to go to the new equilibrium position. The new equilibrium position given this will have an additional displacement in the plane and the actual direction and the transverse direction. This is generally given in terms of what we call as perturbation displacements. From the current equilibrium configuration corresponding to this and then it can be shown that this whole thing can be written in terms of the following weak form. This will be equal to P integral 0 to L into $\frac{dw}{dx}$ P divided by dx into \bar{w} P dx .

This essentially gives if I take now w P as the incremental or the perturbational transverse deflection, write it using the finite element representation using the c 1 elements, this one will be approximated using the finite element way and this will be corresponding test function or the rate function and this is again on the right hand side, I have the unknown w P and here is the test function. This will lead again to an Eigen value problem of the following type.

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$$[K]\{\alpha\} = \overset{\lambda}{P}[K^G]\{\alpha\}$$

$$K_{ij} = \int_0^L EI \phi_i'' \phi_j'' dx$$

$$K_{ij}^G = \int_0^L \phi_i' \phi_j' dx$$

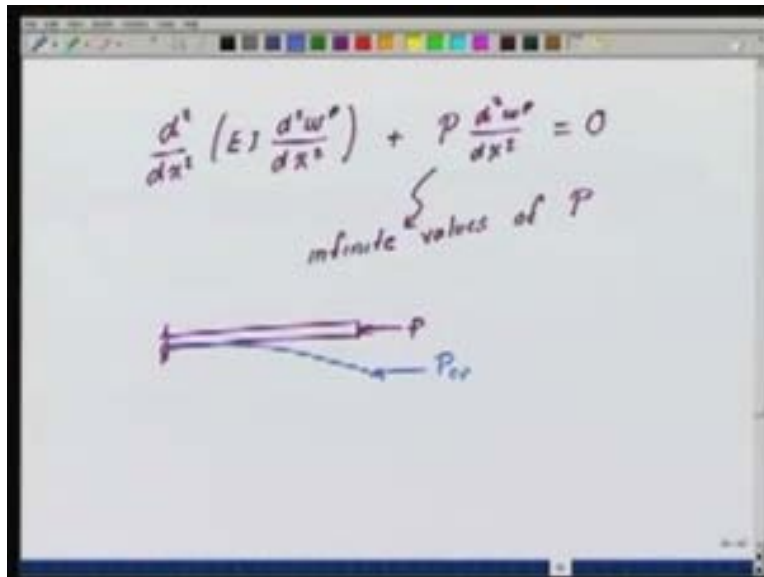
$$w^p(x) = \sum_{i=1}^N \alpha_i \phi_i$$

$P_{cr} = \lambda_{min}$

The K, this is for the bending into alpha is equal to P into what we call as K G into alpha, where is this coming from, this is called geometric stiffness matrix or geometric part, in the sense, that it is coming from this part of the strain that is by representing the large strain components. The entries of K, K_{ij} is equal to integral 0 to L, EI $\phi_i'' \phi_j''$ and K_{ij} geometric is equal to integral of 0 to L. I will go back and see what do I have ϕ_i' , ϕ_j' dx. I am not writing these things in detail. I am really interested in finding this load P actually this happen at which the transverse deflection is present. There again I setup this. Now ϕ_i or c 1 functions, this is very important that these will be c 1, because we want $\phi_i'' \phi_j''$ to be defined.

We do the same thing. Again this is the where the w P as the function of x is given as sum $\alpha_i \phi_i$, i is equal to 1 to some number of unknowns N. This gives me the Eigen value problem where lambda is equal to P that is the buckling and that is the load unknown load and then the critical load out of this is essentially equal to the minimum Eigen value that we have. That is what I am looking at. I would like to obtain the minimum Eigen value corresponding to this problem. Here again I get P critical is equal to the minimum Eigen value and how many Eigen values will get for this system will get N by N depending on the number of unknowns.

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The image shows a handwritten differential equation on a whiteboard:
$$\frac{d^2}{dx^2} (EJ \frac{d^2 w^p}{dx^2}) + P \frac{d^2 w^p}{dx^2} = 0$$
 Below the equation, it is noted that there are "infinite values of P". To the right, a diagram illustrates a rod of length L fixed at the left end (x=0) and free at the right end (x=L). A horizontal force P is applied at the free end. The diagram shows the rod in its undeformed state (top) and its buckled state (bottom), where the rod has curved downwards. The critical load is labeled as P_{cr}.

Similarly, if I went and solve this in the continual form that is we will get, if we do the formulation of this in an exact form P. This would be the exact differential equation that we have and again if we solve using what we had already used for the free vibration problem, I will actually get infinite values of P because this is an Eigen value problem at the continual level, infinite values of P. They will be distinct value of P and corresponding to each P, I will get a buckled mode shape. That is exactly what we do in our buckling analysis and if I will take a rod like this and apply to this, a compressive load, it deforms. It will deform like this. So the rod will become this at P critical. When the load becomes P critical, the rod will become this.

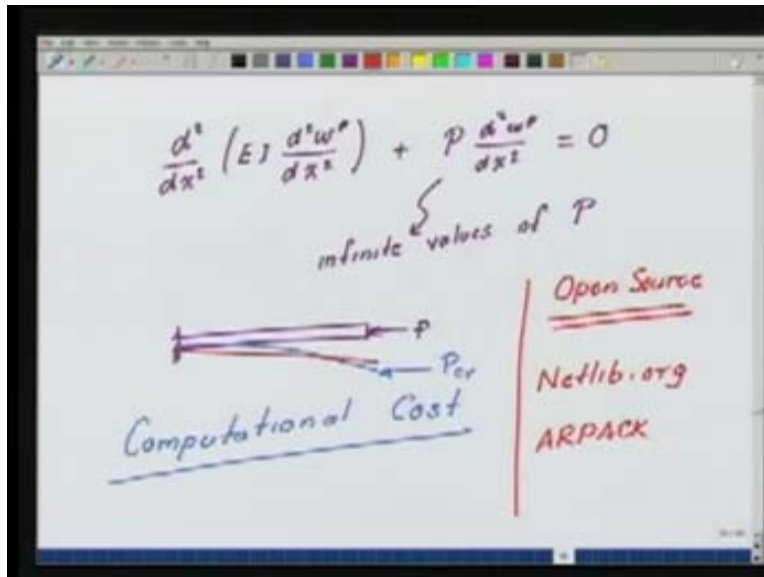
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$$[K]\{\alpha\} = P^\lambda [K^c]\{\alpha\}$$
$$K_{ij} = \int_0^L EI \phi_i'' \phi_j'' dx$$
$$K_{ij}^c = \int_0^L \phi_i' \phi_j' dx$$
$$w^c(x) = \sum_{i=1}^N \alpha_i \phi_i$$

$P_{cr} = \lambda_{min}$
 $\lambda_1, \lambda_2, \dots, \lambda_N$
 $\lambda_1 < \lambda_2 < \lambda_3 < \dots$

As we had talked about earlier here, we have λ_1, λ_2 up to λ_N such that λ_1 is less than equal to λ_2 is less than equal to λ_3 and so on. The first Eigen value is the Eigen value I am interesting in. This is very important also from an implementation point of view because in this analysis, we are only interested most of only interest in the critical load. I am interested only the first Eigen value. If I am interested in the first Eigen value, I really do not need to compute for the remaining Eigen values. If I do not need to compute for the remaining Eigen values, I can use faster solvers which give me the first Eigen value very nicely. The cost of computation comes down tremendously, if I am interested in only the first few Eigen value and I am not demanding all the Eigen values.

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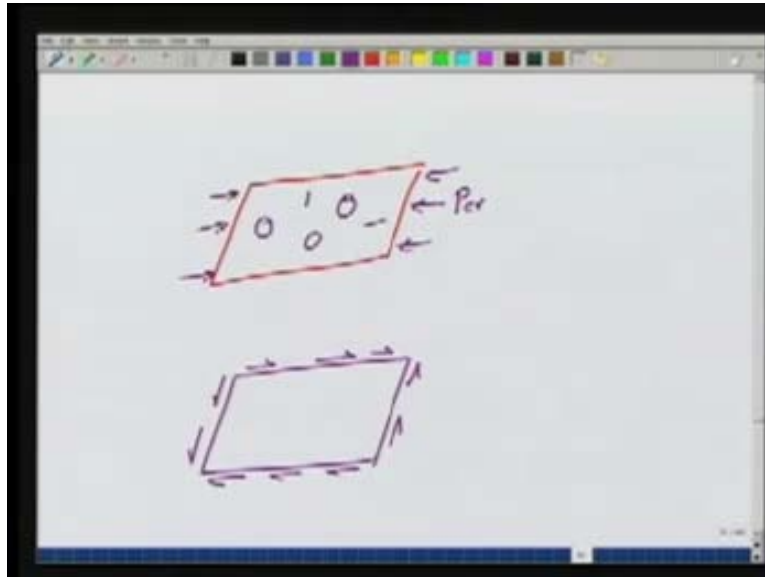


So as an implementational issue computation cost, for the Eigen value problems, this is the big issue. When I use I want to get information about the first few Eigen values or one Eigen value and the cost comes down tremendously. What will be the accuracy of the P critical that we obtained? Again, it will be determined by essentially, this bending strain energy of the perturbation transverse displacement w P. How well are we approximating w P that is how well is the finite element solution going to represent this mode? With one element, I will get this, so exactly the same theory. Certainly, there are certain differences but the same ideas can be extended to the Eigen value problem under computation.

There are lots of Eigen values solvers available in the open source. We can download it from the net or numerical recipes in Fortran and C then so many books on those they also have Eigen value solvers which are generally a little bit more difficult to make program and finally make it useful for computation as compared to the solvers, we had for the inversion of the matrices. From the open source, we can have many such algorithms which are available. We can go to a site called netlib.org that is an open source site, we can download lots of bookings from there. There is something called ARPACK which is very powerful, generalize Eigen value solvers

package available again as a free open source package. It can be used to compute Eigen values of large systems. Here we did not really say what the size of the system is.

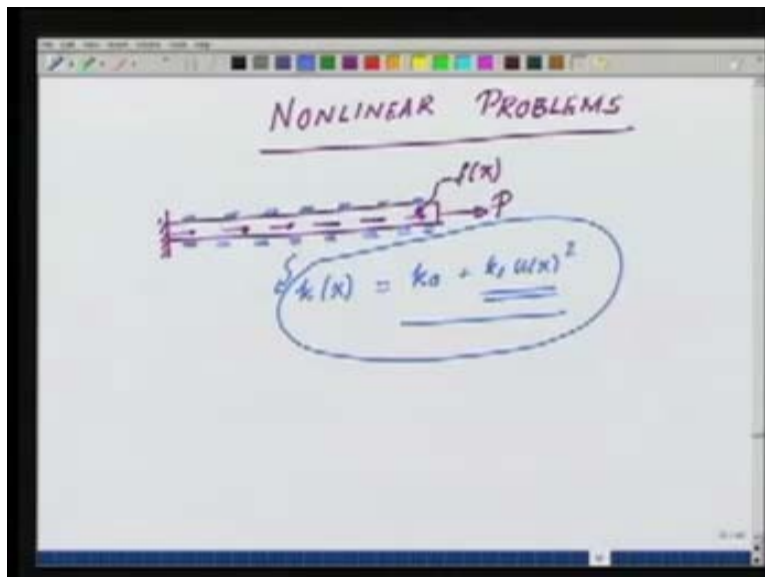
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Depending on the problem, 1-D problem, meshes are small. Approximation is small. When I want to do this Eigen value problem in 2-D or 3-D for example, in the 2D, very important thing is when I am interested in plate buckling, in this case, the plate is subjected to a compressive load or even to a shear load. Here is the plate subjected to some shear loads. In that case, I am interested in the critical value of this load. In that case, if the plate has cut outs everywhere, crack sitting some where at various sites. I have to resolve these geometrical details properly so that the finite element solution corresponding to the buckled mode shape is not bad. The finite element solution has to be good in order to have a good value of the P critical. We have to do the meshing which is detailed enough to take care of these details. This leads to a very large sparse Eigen value problem. The mass matrix and the K and the stiffness matrix could be 20000, 30000 or 40,000 by 40,000 size. Then getting the Eigen values becomes the tough job and some of these algorithms have to be used. So there is the lot of literature available on how to simply solve a large sparse Eigen value problem. We can use straight away from the literature.

In this lecture, we are trying to cover some classes of Eigen value problems that arise in mechanics. Some typical example that we have taken is the free vibration problem or the buckling problem, both of which give rise to Eigen value problems which have to be solve first and formulated in the finite element sense and then they have to be solved and then check whether those numbers are reliable or not. Benchmark problems have to be solved and for the students it will not be very difficult to program this in the existing static analysis codes because we already have the structure with which we have the shape function routines, we have our element calculation routines, where all we have to do is add this extra parts which may be for computation of the mass matrix or computation of the geometric stiffness matrix add them. Do follow the same procedure that we are following for the assembly. Assemble these matrices. Obtain, download or write whichever way we like to do. Get an Eigen value solver. Put it and check. This way, we can now expand our codes that we have developed the 1D and the 2D code, to also take care of Eigen value problems and so we can create much more information which is useful to a designer or to an analyst.

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In the next lecture, we are going to look at some problems which fall in the category of non-linear problems. Till now we have only dealt with linear problems. We will add non linearity in

to the system which is not forcibly done but this is where the real situation is actually, most of the problems in real life are non-linear in nature. Those nonlinearities, if we add to the system then what happens to the response of the system in terms of the finite element solutions, that is how do I go ahead and do a finite element computation, which will give me the information that I need correctly? So we are going to look at finite element formulation for nonlinear problems and from there we will start. A very simple example is this. I take a bar again a stick to a bar that we had. Here is P . Here is $f(x)$ and I will now make it lie on elastic supports on either side such that this can be represented by distributed actual spring which can be given with the spring constant $k(x)$. We had taken it earlier as k_0 . I will add to it to a part which is this.

The spring now has the nonlinear part in the definition of the spring constant. How do I take care of the response of the bar with this kind of a support spring? Example, this could be the effect of the rubbery support, where the rubber behaves in the nonlinear way. We should try to find its information will be respect to the force which is applied. This could be a model of rubbery spring, something like that. We could add more complexities but primarily, we would take this problem as our problem for which we will develop the non linear analysis tools. We will solve the problem and we see, how well or how badly do we do with respect to, what is the exact solution of this problem.