Finite Element Method
Prof. C. S. Upadhyay
Department of Mechanical Engineering,
Indian Institute of Technology, Kanpur

Module – 11 Lecture - 02

Till now we have discussed the static problem in 1 and 2 dimensions and with a little bit of introduction to what happens in 3 dimensions. Let us now move on to a different class of problems.

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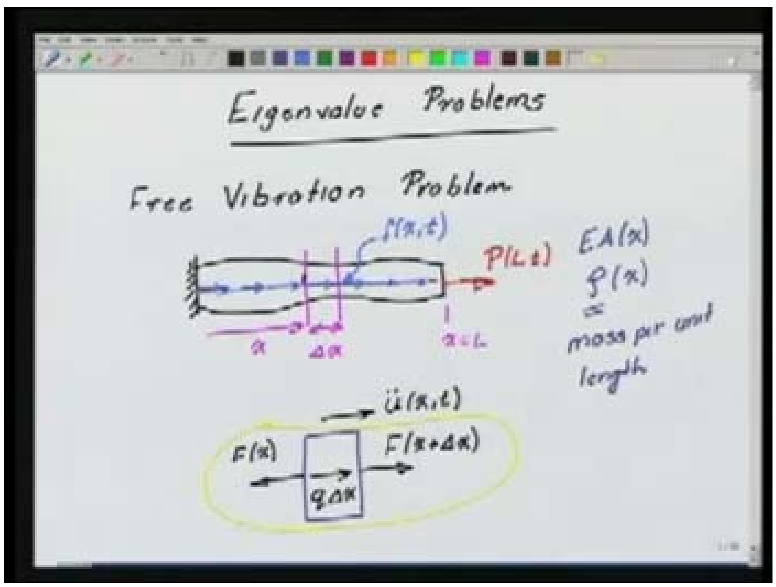

So what we are going to look at need not necessarily be from the same type of analysis. We will look at Eigen value problems and under this big class of Eigen value problems, we are going to look at to start up with specific example of the free vibration problem. What do we mean by the free vibration problem? In order to get to the free vibration problem, let us do a review of what do we mean by it in the case of a deformable structure.

Let us again go back to our 1 dimensional example problem of a bar, possibly of a variable cross section but with the central line of the symmetry with an end load P which is a function of where it is being applied and time and a distributed load in the interior. The loading is changing with time that is the end load which is applied on the bar is changing with time as well as the distributed load on the body. It could be due to any temporal disturbances arising from the environment in which the body exists. That is not the major issue but issue is we now like to set up the equation of motion. Till now we had been only talking about equation of equilibrium or from the conservation of linear momentum. We did not have the inertia part. Now we have to build in the inertia part. Go again to a section at a distance x from the end. This is at the end x equal to l and take the section of size delta x. We will assume that the material is such that the E and the area are functions of x. The density is also a function of x and this density is mass per unit length. This is what we mean by this density mass per unit length.

When we have this thing given to us, these quantities, we would now like to pose, what is the equation of motion corresponding to this small piece that we have cut of? Let us take that small piece and look at the force system acting on it. So the interaction of this piece with its neighbor on this side is this actual force F at x + delta x. Interaction of this piece on this side with its remaining neighbor is the force F at x. The effect of the distributed load is q delta x and we have to remember that this piece has an acceleration given by the second derivative of u with respect to time, u double dot as, when I write dots, it means derivative with respect to time. This is essentially under this force system, this piece is going to move. So how do I write the equation of motion? This all must have done in the standard dynamics classes that I will have in the actual direction.

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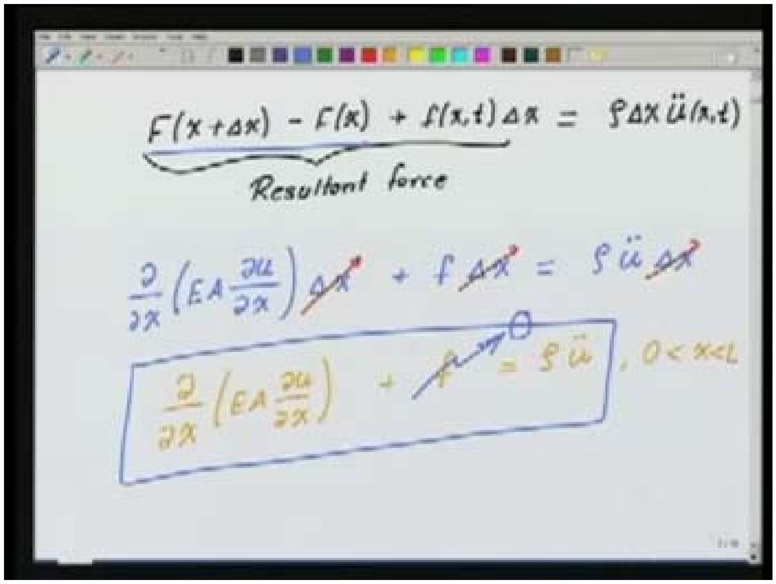

If I look at force balance, F (x + delta x) minus F(x), plus f as the function of x and t, delta x, this will be resultant force acting on this piece in the positive x direction. This will be equal to rho delta x which is the mass of this piece into the acceleration of this piece. The resultant force is equal to the mass into acceleration. So this is very easy for us to do, because we have already done this part in the static case. So this part will give me del del x of EA into del u divided by del x into delta x plus f, I am dropping the x and t part. It is understood now is equal to rho u double dot delta x. In the standard way, we remove this part and we are left with the differential equation del del x of EA del u divided by del x plus f is equal to rho u dot for all x lying from 0 to L. This is for the dynamic case, the equation of motion for this actual bar member. We would like to do the free vibration analysis. What does the free vibration analysis mean? In the case of the free vibration analysis, this f is 0. The equation of motion is del del x of EA del u divided by del x is equal to rho u double dot. How is the vibration happening? It is due to the initial conditions, initial perturbations or disturbances given to the body. We are interested in this analysis. What do we do next in the case of the free vibration analysis?

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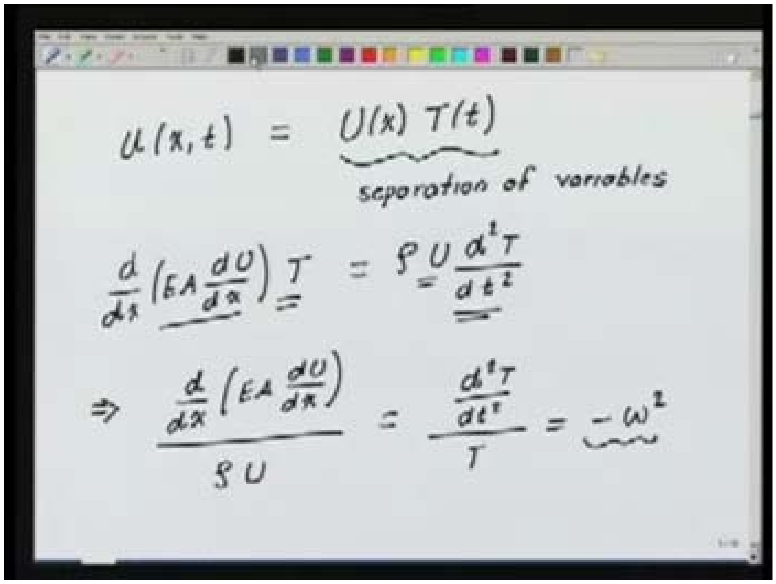

In this case, we can do a partition or a separation of variable approach and write U is a function of x, a pure function of x and pure function of t, of x and t. So this is a partition or separation of variable approach that we have used. So we will use the separation variable approach and once we use it, the differential equation will now become d dx of EA into dU divided by dx. This is the big U into T is equal to rho U into d2T divided by dt squared, so this is what the differential equation will become.

Let us take a simple case first or in turn have to take it. Let us now take this and see that here is a time part, here is the time part, this is the space part, and this is the space part. That is the function of x and t. So let us bring the 2 parts together. So we have implies d dx of EA dU divided by dx divided by rho U. This is equal to d2T divided by dt squared divided by T. Since, this is a pure function of x. This is a pure function of t. The 2 can only be equal, when they both are constant. We cannot have a function of x being equal to function of time for all x and time except when they both equal to constant and that constant, we know from our basic vibration analysis is given in terms of minus omega squared with the argument that really in terms of time, the solution is going to remain bounded and oscillatory in nature. It will not blow up so because of that this function has to be minus omega squared. All that we can learn in our vibrations classes.

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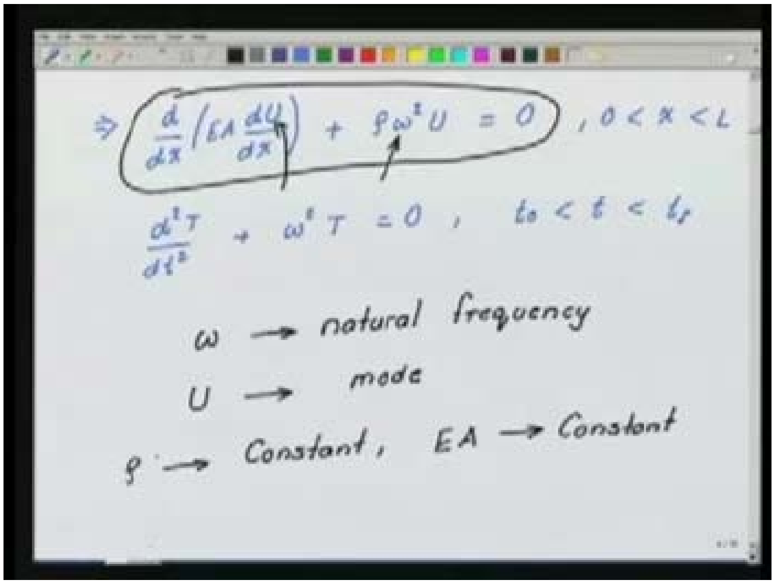

Let us go and look at this problem as the 2 separate problems that implies I will have d dx of EA, dU divided by dx plus rho omega squared U is equal to 0, 0 less than x less than L. And I will have d2T divided by dt squared plus omega squared t is equal to 0, for t_0 is less than t is less than some t final so initial time final time, that is why we have this. We can go on forever also. That is not the issue. We are interested in looking at this problem. Why? Because this problem gives us what is the value of this omega? Corresponding to the omega, what is the value of this U? So where omega forms the natural frequencies and U is the corresponding mode that is corresponding to this omega, what is the corresponding shape deflected shape of the bar as a function of x? How do we solve it?

Let us take a simple case of the bar with rho is constant. It is not a function of x and so is EA that is, I have a uniform cross section and it does not change neither does the material change. If it take that special case, then as an example, we will get implies, if I look at the problem corresponding to the U, I will get, EA d2U divided by dx squared plus rho U is equal to 0 in 0 to L.

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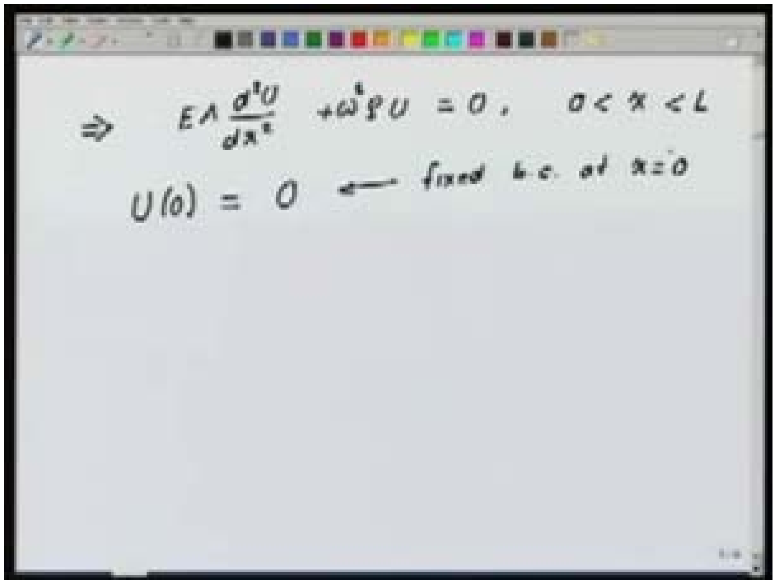

It will be into omega squared, omega squared rho U will be 0. How do I find omega and how do I find the corresponding U's? As for as U is concerned, U at 0 for all times the displacement small u as the function of x and t was equal to 0, at x equal to 0. So U of 0 is equal to 0 from the fixed boundary condition at 0. For this particular problem, it need not be the same for all problems, boundary condition at x equal to 0.

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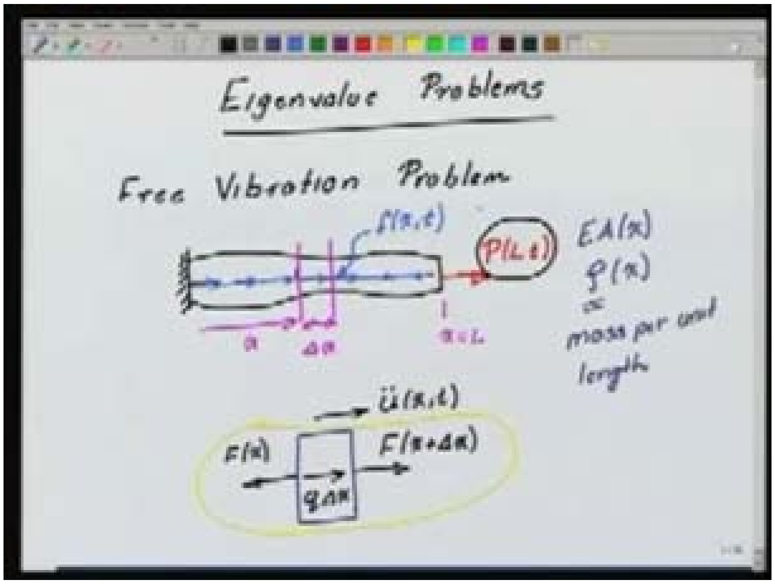

Further, we also have in the case of the free vibration problem, we are also going to force our this P as the function of L and t as the function of time to be equal to 0, because there is no external forcing function acting on the body. So for the free vibration problem, F will also be 0, P will also be 0. So in our case, what we will get? We will show it later on.

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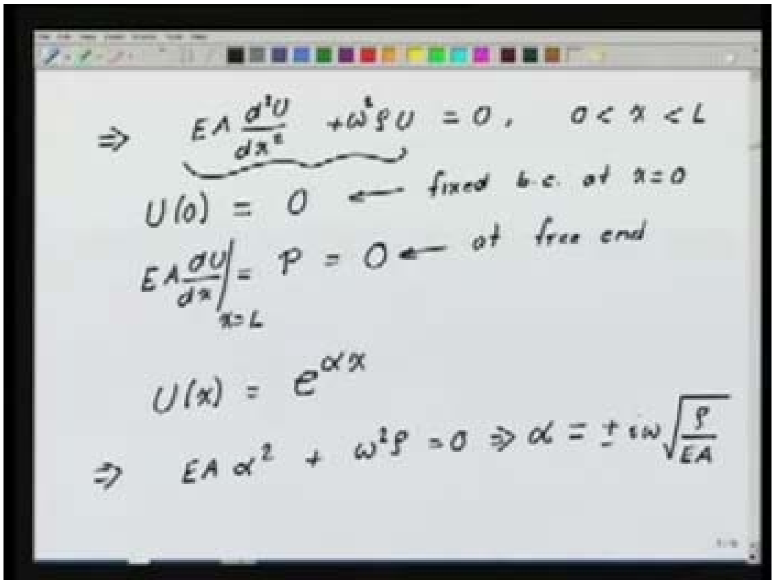

EA dU divided by dx is equal to P is equal to 0. This is at point x equal to L. This is equal to P which is equal to 0. So this is at free end. The force at the free end will be given in terms of EA dU divided by dx because that is the actual force. This is equal to P which is equal to 0.

As far as this is concerned, this representation of the solution, in this case the solution can be represented as e to the power of alpha x implies we will get EA alpha squared plus omega squared rho is equal to 0 implies alpha is equal to plus minus i omega root of rho by EA, if I do this, alpha squared is equal to minus omega squared of rho divided by EA in the root of the minus will be plus minus i omega root of rho by EA.

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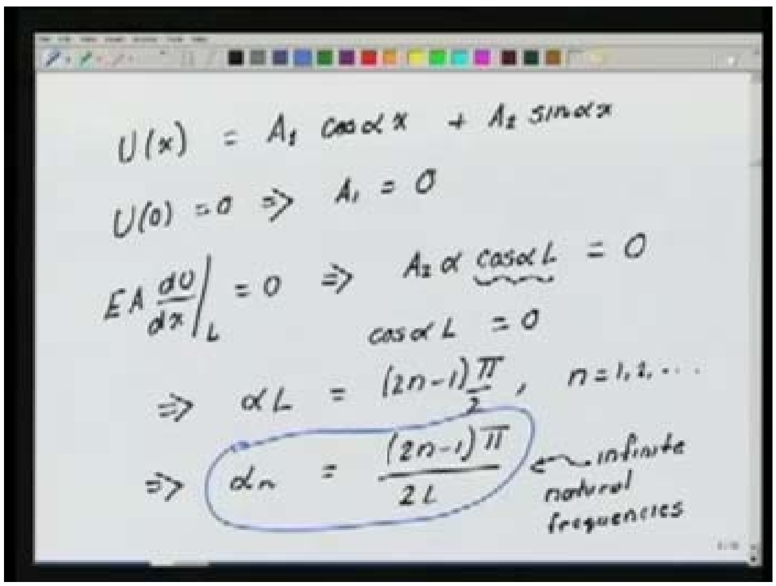

This will give the representation of U of x becomes equal to, we can write it, A_1 cosine alpha x plus A_2 sine alpha x because the alpha came out to be imaginary. So it is in terms of this. Now U at 0 equal to 0 implies the A_1 is equal to 0, because cos alpha x becomes 1. This term becomes 0. So I am left with $A_{1.}$ We will have EA dU divided by dx at L is equal to 0 implies A_2 alpha cos alpha L is equal to 0. We are interested in the free vibration analysis that is we want the non trivial modes and the corresponding frequencies. Now if I say, this is equal to 0 which means A_2 equal to 0. We will end up getting a trivial solution. In order to not get a trivial solution, we want cos alpha L equal to 0 implies alpha L is equal to (2n-1) pi by 2, n is equal to 1, 2 and so on. It implies we will now write alpha_n is equal to (2n-1) pi by 2L, we have so called infinite modes. In the case of a continuous system, we will have infinite natural frequencies. This is the very important point that we have to keep in mind before we progress to how to do the finite element approximation. So we have these infinite natural frequencies, for n going from 1 to infinity, 1, 2, 3 up to infinity and given this alpha$_n$, we have the corresponding mode.

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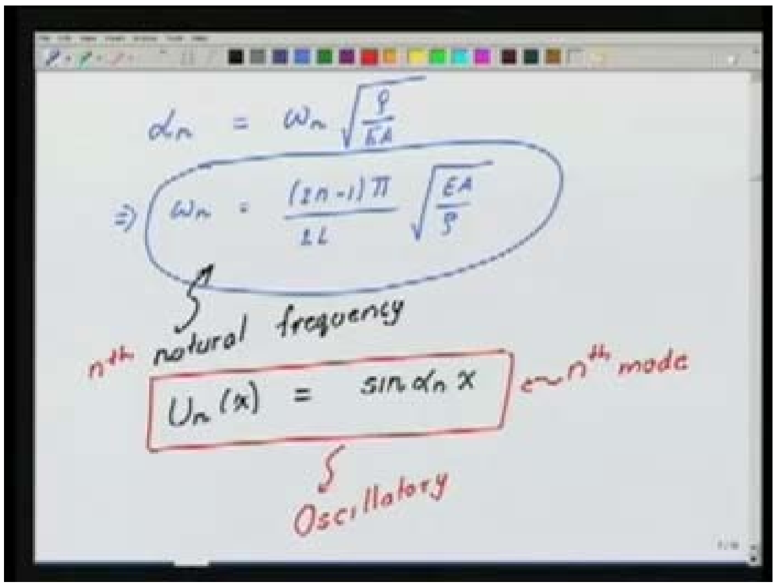

We will have alpha$_n$ is equal to omega$_n$ root of rho over EA implies omega$_n$ is equal to (2n-1) pi by 2L into root of EA by rho. This is what we call as the natural frequency and by putting this value of alpha$_n$, this expression for alpha$_n$ in the expression for the corresponding U see remember , we had this U here, we will get, U what we know as the corresponding mode for the natural frequency, this will be equal to sin alpha$_n$ x, n th natural frequency, nth mode. This is very important thing to keep in mind, before we go for the approximation. Before we do the approximation of anything, we should not do it blindly, we should know what exactly the nature of the functions is, we are trying to approximate. All of the entities that we are going to approximate, nature not that we have to know the exact function. We should be able to say what is the nature? The nature is that these are oscillatory in nature and not only oscillatory as the n rises the oscillatory nature becomes more and more pronounce.

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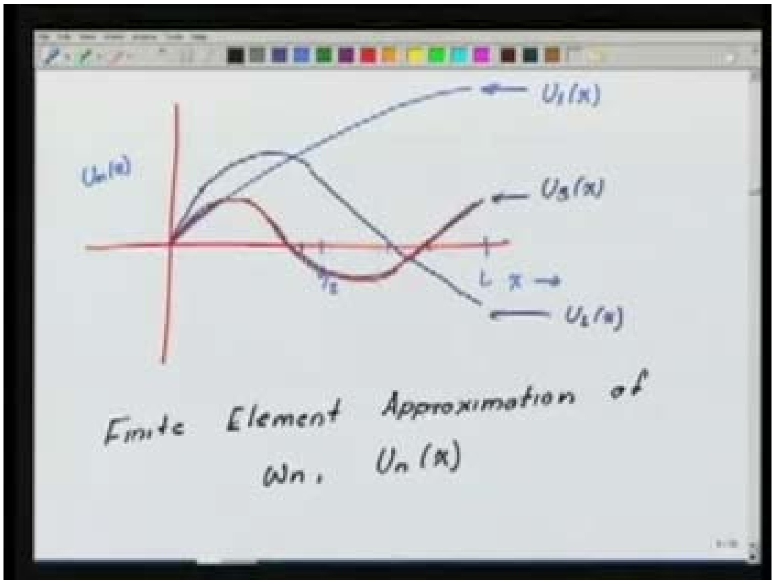

For example for this problem if I plot, this U_n s as the function of x, this is x equal to L, this is L by 2. So here I will plot the $U_n(x)$. U_1 is essentially a sin of $alpha_1$ x which is 0 at x equal to 0, at x equal to l, it becomes 1 and we get something like this as $U_{1,}$ $U_1(x)$. If I look at U_2 what will happen is, U_2 will do the following. It will go up come, this is $U_2(x)$. Similarly, if I look at $U_3(x)$, it will do the following. Let me reiterate that $U_3(x)$ is this function which is given by the corresponding choice. As the alpha is increasing, the n is increasing, this function the mode becomes more and more oscillatory in nature and this way, I can continue. This picture we have to have in mind because we have to see how well is the finite element solution doing as far as this problem is concerned.

We have these solutions for this then how do we go and construct a finite element solution? How to construct a finite element approximation of $omega_n$ and U_n as the function of x? In order to do this, we will go back to the differential equation that we had created as far as the spatial part is concerned, this differential equation. Let us come to that differential equation and write it down.

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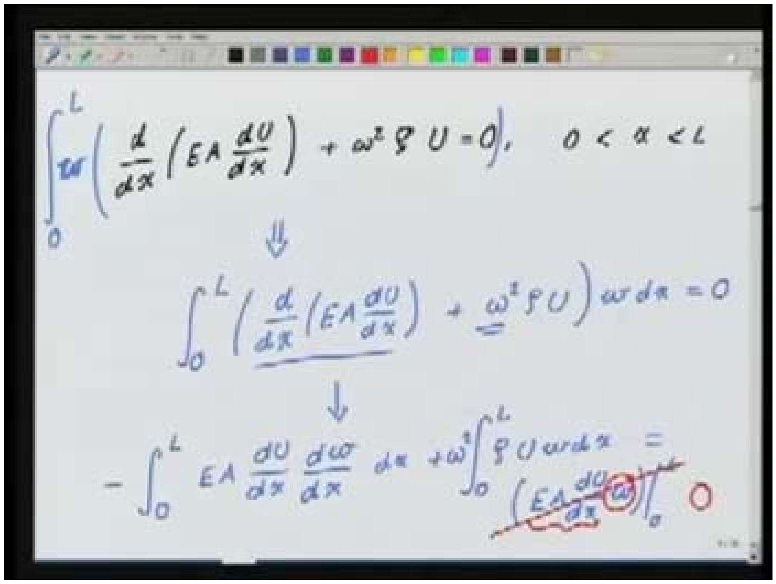

We will have d dx of EA dU divided by dx is equal to rho U, 0 less than x less than L. This is equal to minus omega squared rho U or I will erase it and rewrite it, here for the convenience plus omega squared rho U is equal to 0. In the standard finite element setting that we have been following till now over and over again. Take this differential equation multiply it with a weight function w and let me have w and the omega very clearly written of, integrated from 0 to L, then we will get, this will give me integral 0 to L, d dx of EA dU divided by dx plus omega squared rho U whole thing into w dx is equal to 0. This is the weighted residual form for this given differential equation, where omega is an unknown and U is an unknown. We have to solve for both. First we have to set up that problem. We see here again, we have the same problem that second derivative of u is sitting, w is sitting by itself. So do an integration by parts once for this quantity and this integration by parts will give me, integral of 0 to L, dx minus of this into dw divided by dx into dx plus integral 0 to L, I can take the omega squared out, rho u wdx, this will be equal to EA dU divided by dx into w evaluated at 0 and L.

From what we have done earlier, this part corresponds to the specified force on the boundary. This part has to be made 0 wherever the U is specified to be 0 on the boundary. At x is equal to 0, U is equal to 0, big U is equal to 0. So w also has to be 0, at x equal to l for the free vibration problem EA dU divided by dx is equal to P is equal to 0. So that, this term is 0 at both the boundary. So I can knock this of. So this becomes equal to 0.

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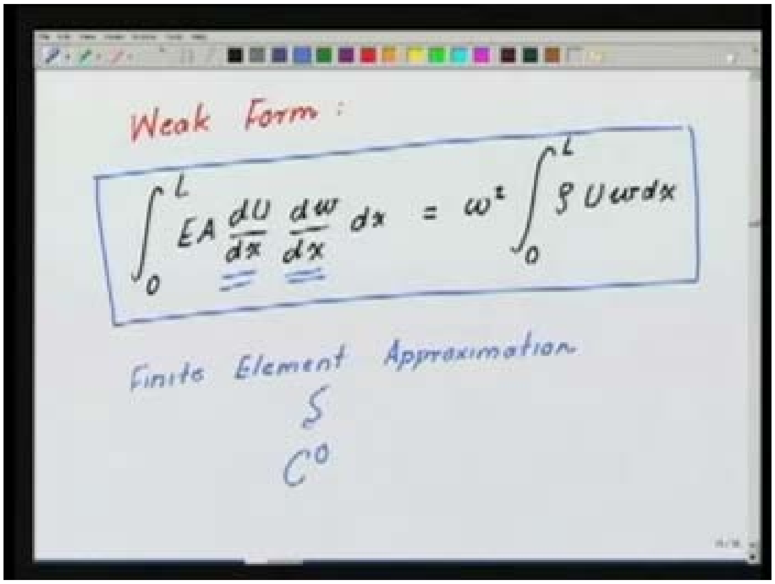

If I now write this weak form, as we have obtained here, this is the weak form, we will write it as integral of 0 to L, EA dU divided by dx into dw divided by dx is equal to omega squared integral 0 to L, rho Uw dx. This is the weak form of the problem that we want to solve. Once we have this weak form then what do we do? We have to now create a finite element approximation. How do we do it? We see that again in this case, we want dU divided by dx to be defined. Similarly, dw divided by dx is to be defined. So we want functions U and w for which the first derivative is defined. That is the functions need to be only continues, so as far as the approximation will go to the C 0 approximation.

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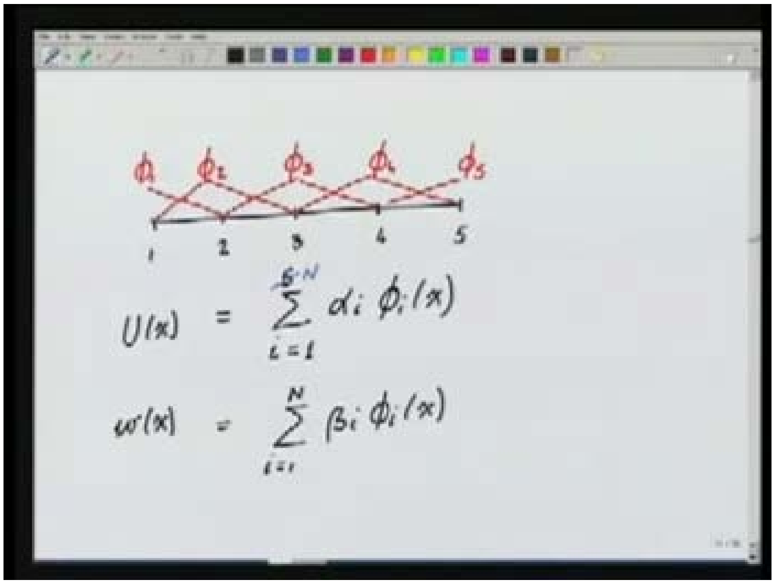

Now we will construct. So let us say that we have a bar, bar is broken into some number of elements. Let us this is 1, 2 as an example 3, 4, 5, I will go for the simplest possible representation of the basis functions which corresponds to the C 0 approximation which we have already done. We have this add functions. Everything else is exactly like what we did in the static problem, at all that we set up. Here we will have our phi_1. This is basis function phi_2. This is basis function phi_3. This is phi_4 and this is phi_5. We will write U as the function of x is equal to sum, i going from 1 to 5, here $alpha_i$ phi_i. If I have, instead of 5, I generalize it. If I have N, then I will replace it by N. As many number of elements that we need, so here we had 4 elements 4 plus 1 is 5 degrees of freedom and if we have N elements and N plus 1 degrees of freedom. That is not a problem. So this is the representation that we do for the U of x. Similarly, the way we have done, the w of x will also be equal to sum i is equal to 1 to N. I will generalize it $beta_i$ phi_i of x.

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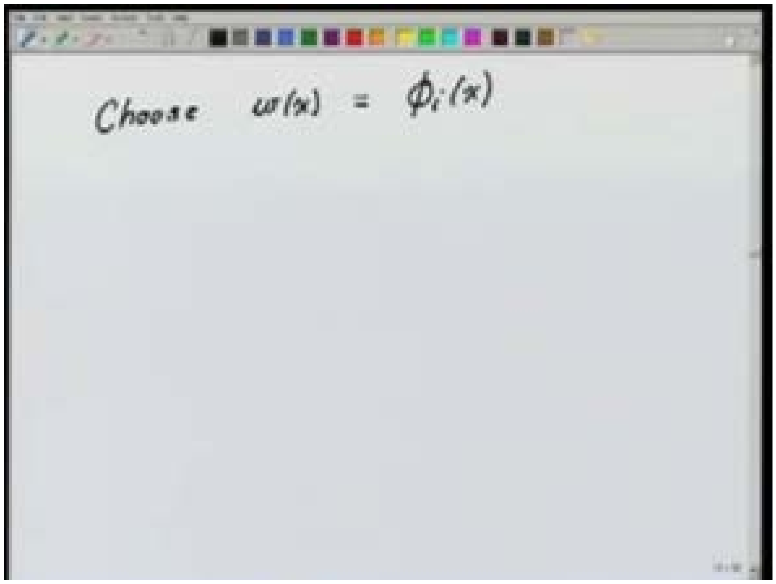

So we substitute this representation, so choose w as the function of x to be equal to phi_i of x because this choice is allowed from what we had argued earlier. Remember that in the charge for the w.

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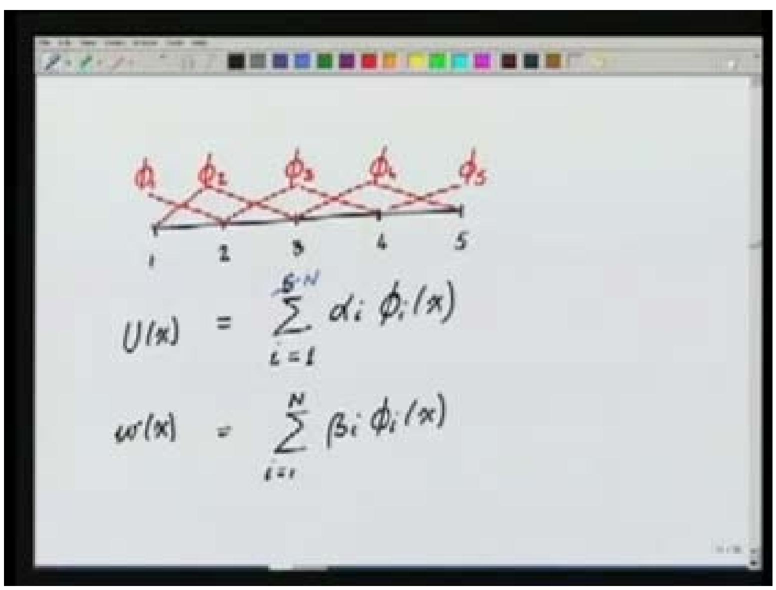

The w has to satisfy the essential conditions wherever they are specified. In this problem for example $beta_1$ has to be equal to 0 that is in the representation of w will actually start from I equal to 2 to N because the part multiplying phi_1 is 0, the w has to be 0 at 0.

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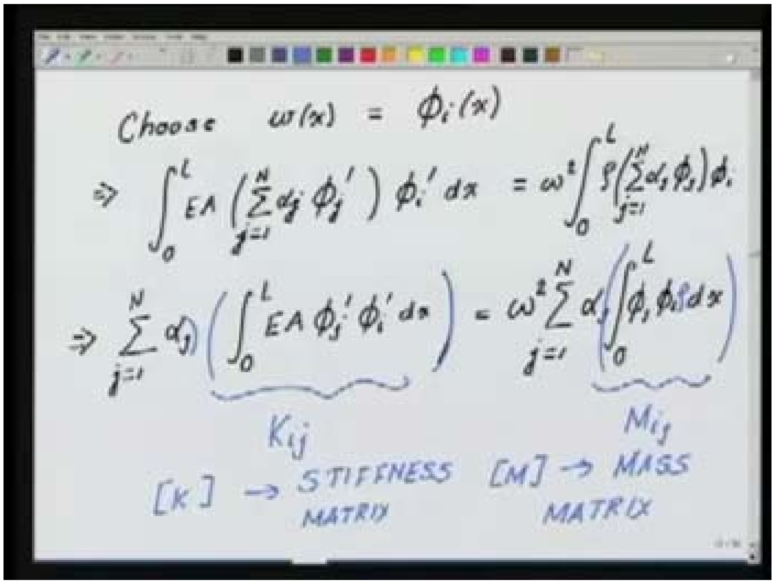

So given all those things, we choose w(x) is equal to phi_i(x) put it in the weak form that we have obtained, we will get implies integral 0 to L EA into sigma i is equal to 1 to N $alpha_i$. Instead of i, I will put it j, just again not to confuse $alpha_j$ phi_j prime into the dw divided by dx which is phi_i prime. Prime means derivative with respect to x is equal to omega squared integral 0 to L rho into $sigma_j$ is equal to 1 to n $alpha_j$ phi_j phi_i dx. This will imply that we will have sum j is equal to 1 to N, $alpha_j$ into integral 0 to L EA phi_j prime phi_i prime dx. This is equal to omega squared sum 0 to L phi_j phi_i dx. This I will put again in the brackets. When w is equal to phi i that gives me the i th equation as we have done earlier. For the ith rho this corresponds to the ijth entry corresponding to $alpha_j$ here also corresponds to the ijth entry corresponding to $alpha_j$. From basic dynamics we will know that this part which comes due to the EA is called the stiffness part, the part coming due to the rho. I will have the rho setting here, the part coming due to the rho is called the mass part so this is essentially component K_{ij} of the so called stiffness matrix this is the component M_{ij} where M, K is called the stiffness matrix and M is called the mass matrix.

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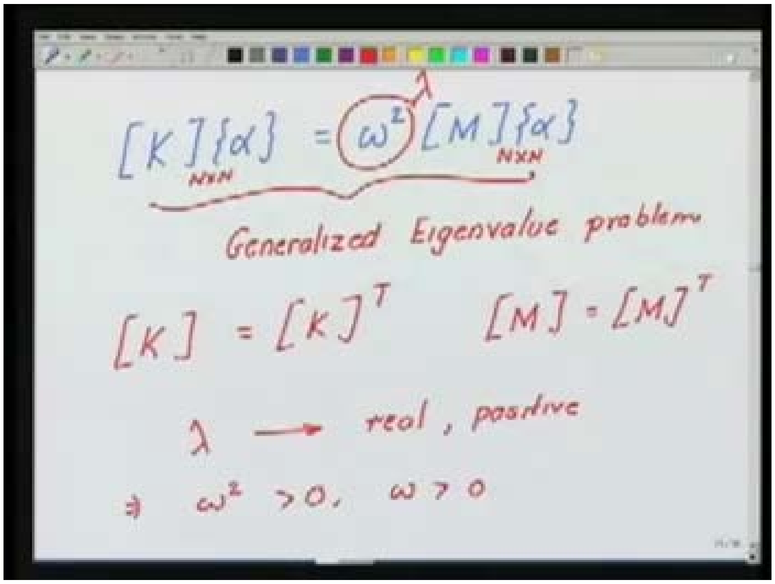

We have here the problem K alpha is equal to omega squared M alpha where we have given what are the entries of the matrix K entries of the matrix M. This problem that we have written is called a generalized Eigen value problem because in general the standard Eigen value problem is in the form K alpha is equal to lambda alpha. This I can write as quantity lambda, omega squared is replaced with lambda look whether things are consistent or not. If I look at how we have done things? K is equal to K transpose that is K is symmetric. Similarly, M will be equal to M transpose that is M is also symmetric, the stiffness and the mass matrix in this case are symmetric. The mass matrix is diagonal not necessarily, but all we can say is mass matrix is symmetric. If these two things are symmetric then we say that the Eigen value lambda is going to be real for this problem which means that our omega squared is going to be a real number if it is a complex then we are in trouble because it makes no sense so this tells success looking at it that yes it is going to give us a real Eigen value. This matrix K will turn out to be K and M or going to turn out to be symmetric positive definite. They will turn out after imposing the boundary conditions to be symmetric positive definite so in this case your lambda will not only be real but it will also be positive. Everything is consistent implies omega squared is greater than 0, implies omega is greater is a real number which is greater than 0. It is a real positive number so this is the crux of what we have done and we see from the actual solution that exact solution that we had done omega was a number. It was a positive number so what are seen that here in the parallel problem that we have created omega or the omega or the lambda turns out to be positive numbers. Here the problem is an N by N system. M and N or N by N matrices where n is a number of unknowns in the problem. In the unknowns in the problem means the number of basis function which have been used to represent the solution u as the function of x.

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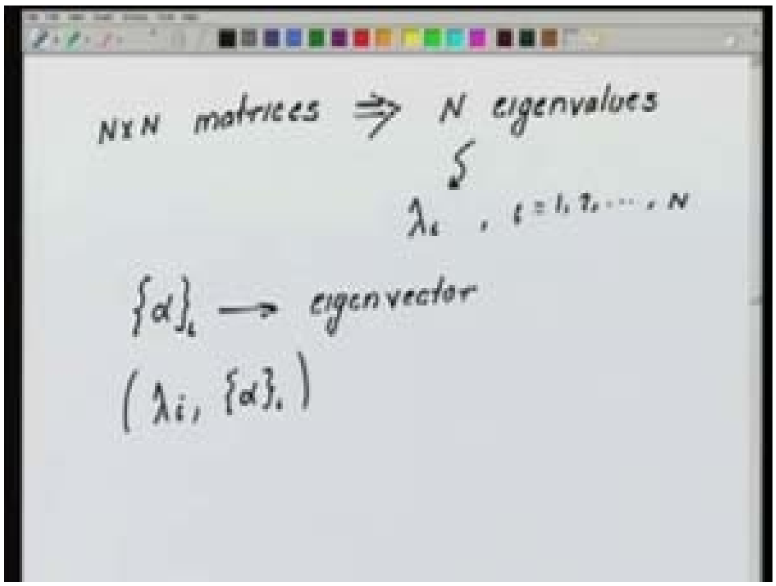

We will have N by N problem which will imply we have N Eigen values which are given by lambda$_i$, i going from 1 to N and corresponding to lambda$_i$ we will get a vector alpha$_i$. This in the matrix problem sense is the corresponding Eigen vector so corresponding to the i th Eigen value we have the corresponding ith Eigen vector alpha$_i$ or we can talk of the Eigen pair lambda $_i$ and alpha$_i$ from this Eigen value problem.

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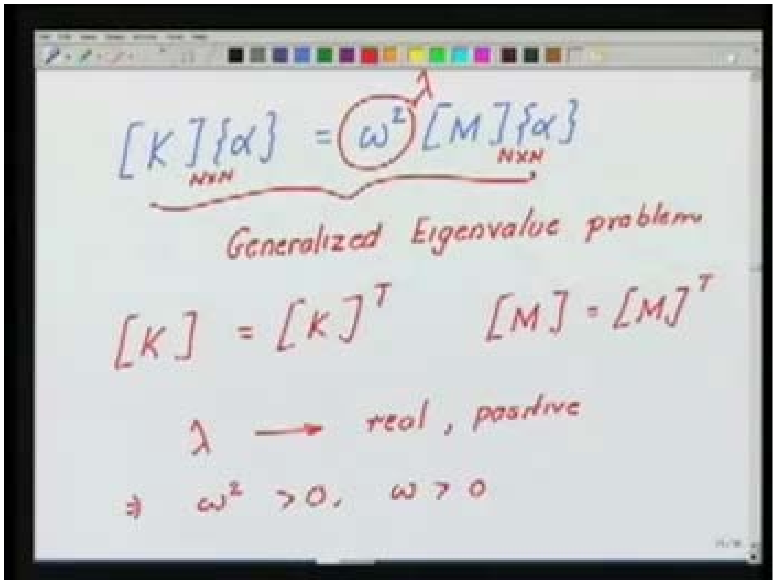

Solve for the Eigen values of this, I get lambda$_i$ put the lambda$_i$ back into this expression and we will get the corresponding alpha$_i$.

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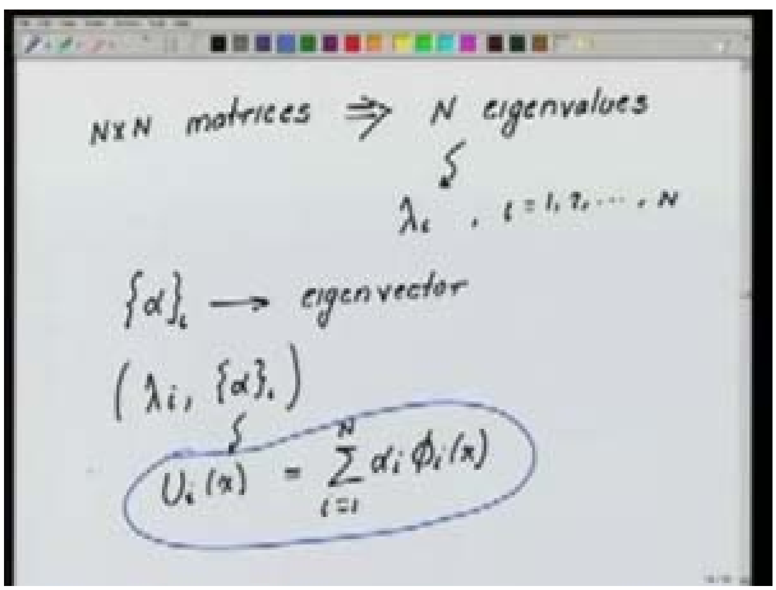

Given this alpha$_i$, a corresponding mode the i th mode $U_i(x)$ will be equal to sigma i is equal to 1 to N alpha$_i$ phi$_i$ so given this Eigen vector alpha$_i$, we can construct the Eigen function or the mode shape which is N by $U_i(x)$. So this essentially is the crux of what we are doing with the finite element method here, some more features of what we get is that here if I look at the exact solution, the exact Eigen values or the Eigen vectors corresponding to what we have done here.

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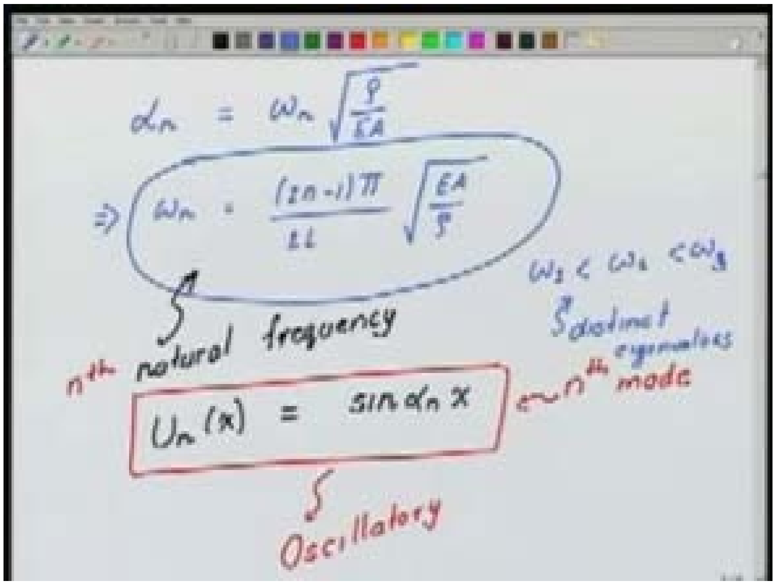

We can arrange this omega's in the sequence such that omega$_1$ is less than omega$_2$ is less than omega$_3$. We have in this problem, we have so called distinct Eigen values in fact these are the Eigen values of the continuous problem or the distinct natural frequencies.

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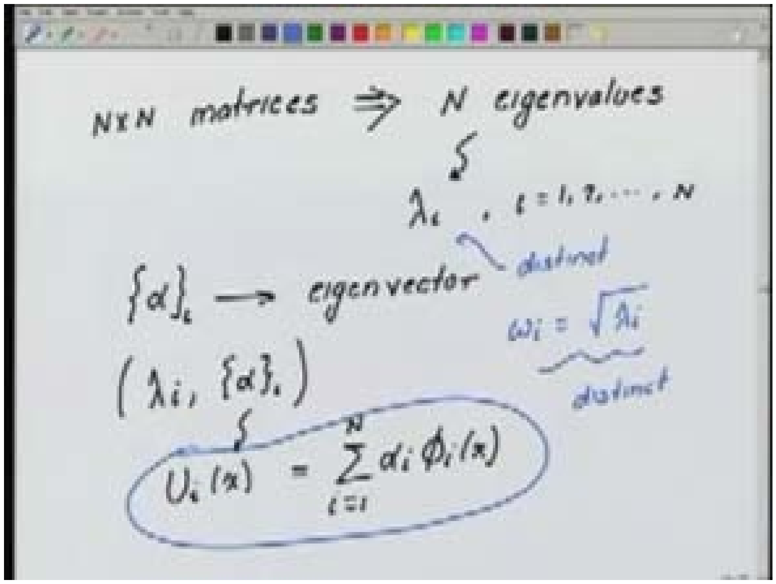

In the case of the problem that we have posed the matrix problem, we will get the lambda_is are also distinct that is you have a different value of lambda_i. Given this lambda_i your omega_i is equal to root of lambda_i which is also distinct. These are distinct, in this case these are some things that we have to keep in mind that some features of the solution which may not always be true. Here we are talking of distinct Eigen values that is, an Eigen value is not repeated if the Eigen value is repeated which means corresponding to that Eigen value we have multiple Eigen vectors. That is the completely different situation which has to be handled in a different way, I mean we have to talk about it differently we do not handled it differently. Here we are only concentrating on problems where the Eigen values are distinct so I have this representation of $u_i(x)$ and now pretty much we have the solution.

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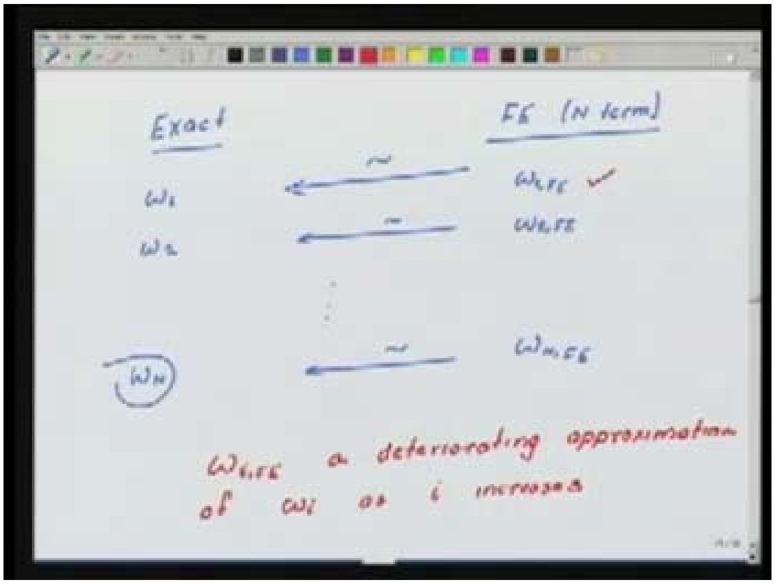

Here is the exact and here is the FE where from N term. The exact omega$_1$ is this, omega$_1$ FE is this. The first Eigen value corresponding to the finite element solution this is an approximation of the exact one. Here is the exact one omega$_{2,FE}$ is the approximation of the second Eigen frequency and so on we continue till omega$_n$ so the nth omega that we get from the finite element solution or n th Eigen values the square root of that is an approximation of the nth natural frequency because they are distinct so this is an approximation of the nth natural frequency. We will get here, this value will be very good depending on the mesh and as we keep on going further down that is go to the higher frequencies. The omega, as I go higher and higher omega$_{i,FE}$ will be a more and more inferior approximation of the actual omega$_i$. This will be a deteriorating approximation of omega$_i$. As i increases, this is very important that as the i increases the omega$_i$ that we get from the finite element solution becomes worse and worse approximation of the actual omega$_i$. So in most of the applications we are only interested in the first few natural frequencies, so in order to get those first few natural frequencies properly we have to use a mesh which is sufficiently fine and we see that here if I look at again I go back to what we had drawn, the Eigen functions have particular shapes.

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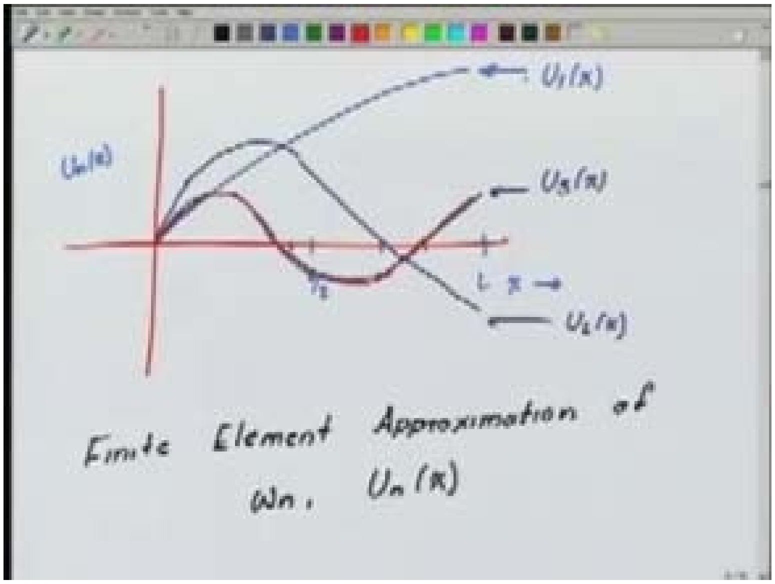

The whole business of how well is the approximation doing depends on what is the corresponding Eigen function and how well can the finite element solution approximate the Eigen function?

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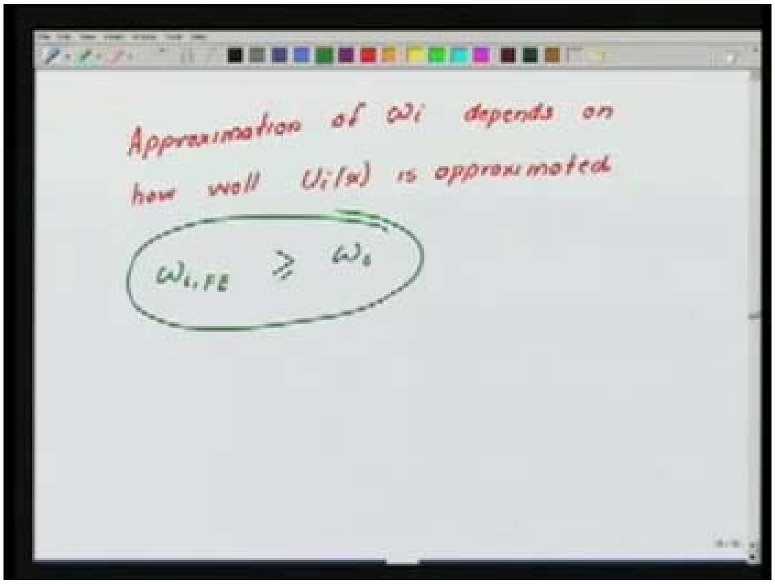

There is a generalized result here which says that the approximation of $omega_i$, the quality of it depends on how well $u_i(x)$ is approximated by our finite element representation. Further we will see another feature of it. I will stop this lecture with this particular feature that we will see that $omega_i$ finite element is greater than equal to the actual $omega_i$. This is the very important result, the frequency obtained from the finite element solution in this case is greater than equal to the frequency that we get from the exact solution as compared to the exact frequencies.

With this we are going to stop this lecture. In the next lecture we will continue a little bit further with these Eigen value problems and look at some more features of it.