Finite Element Method

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Module – 1 Lecture - 3

In the previous lecture, we had looked at the principle of virtual work for the model one dimensional problem that we had introduced. We had also introduced concept of the variation of the function, where 'u' was our displacement function and corresponding to 'u' we defined the variation of 'u' as del of u.

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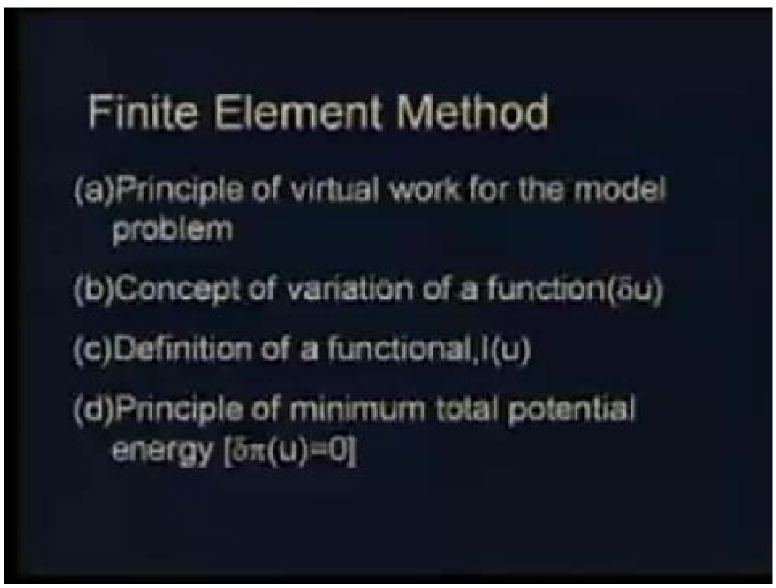

We had also introduced a new concept of a functional, which if you remember was defined as the function of a function; that it would take a function and give us a number. For different functions we get different numbers. From the definition of this functional we reposed our principle of virtual work as minimization of a functional and that functional we got corresponding to the model problem of interest turns out to be the total potential energy corresponding to the system.

We had said that the corresponding equations in the integral form could be obtained by taking a first variation of the functional that is del of pi and setting it to 0. That is, we are looking for the function u which minimizes the functional pi.

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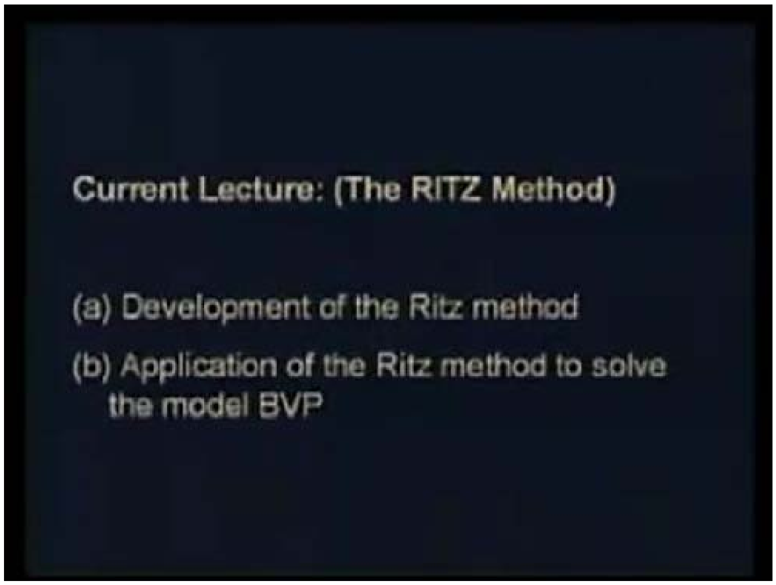

After we have done all these, we had also introduced the RITZ method as one of the techniques to get an approximate solution to the problem of interest. The question is why do we want an approximate solution? As we have shown that most of the boundary value problems that we may be interested in will not have a readily available exact solution. That is getting a close form solution may be almost impossible. In that case we would like to obtain an approximate solution to that problem and in the one dimensional setting we would like to introduce all the concepts that are needed in order to obtain a good approximate solution using the method of our choice. Given this introduction what we will do is develop the RITZ method that we introduced last time in greater detail. And we will apply it to some typical examples corresponding to the model problem that we have introduced and we will show how good or bad the RITZ method does with respect to the solution of these model problems.

Essentially we are creating an artificial situation where we would like to gauge how good the RITZ solution does or how bad it does. Let us see how we can improve the accuracy of the solution, and what can be the causes of the solution being bad.

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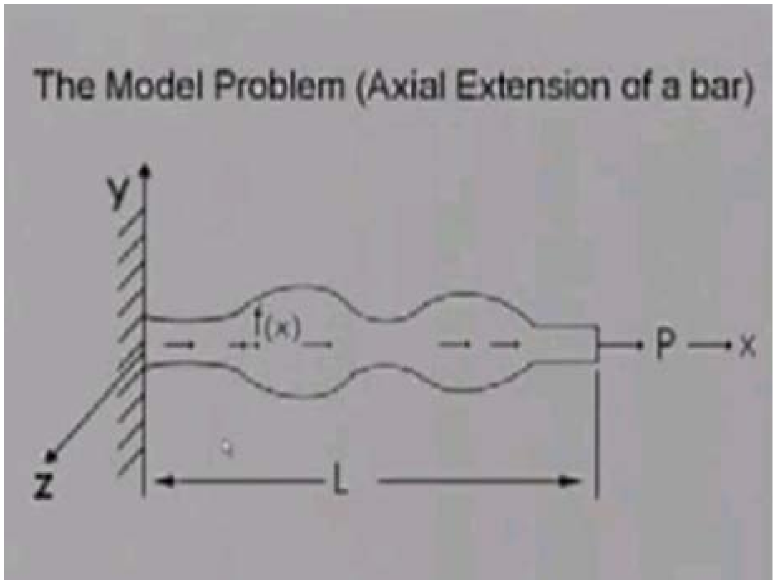

Remember that our model problem was this bar problem where the cross section of the bar was non-uniform and it is subjected to a distributed body force $f_{(x)}$ constrained at the point x=0 that is a displacement is set to 0 at the point x=0 and at the point x=L a force P is applied. Under the action of this force P this distributed load $f_{(x)}$ and the constraint at the point x equal to 0, I would like to obtain the solution to this problem.

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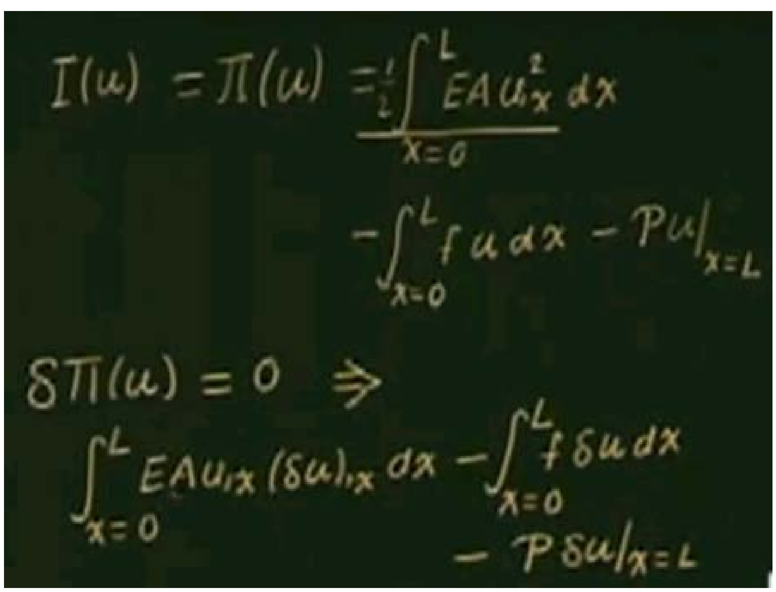

Corresponding to the model problem that we have introduced let us rewrite the functional that we are interested in that is I (u) is equal to pi of u which is equal to integral we will take from x=0 to L, 1/2 in front EA u comma x whole squared dx minus integral x=0 to L f u dx minus P u evaluated at x=L; this is our functional. What we had said as far as the solution to the problem was concerned; the solution u corresponds to a minimum of this potential pi. We said variation of pi of u is equal to 0 equal to as we have defined, the operation of variation for functions and as well as for functionals this will be equal to EAu, x variation of u, x dx minus integral x=0 to L f variation of u dx minus p variation of u at x=L.

Remember that we had said the variation of this quantity 1/2 of EAu, x^2 was as if I have taken the derivative of this expression with respect to u, and so we end up getting - the 1/2 goes and we will get integral of EA u, x the variation of u, x dx.

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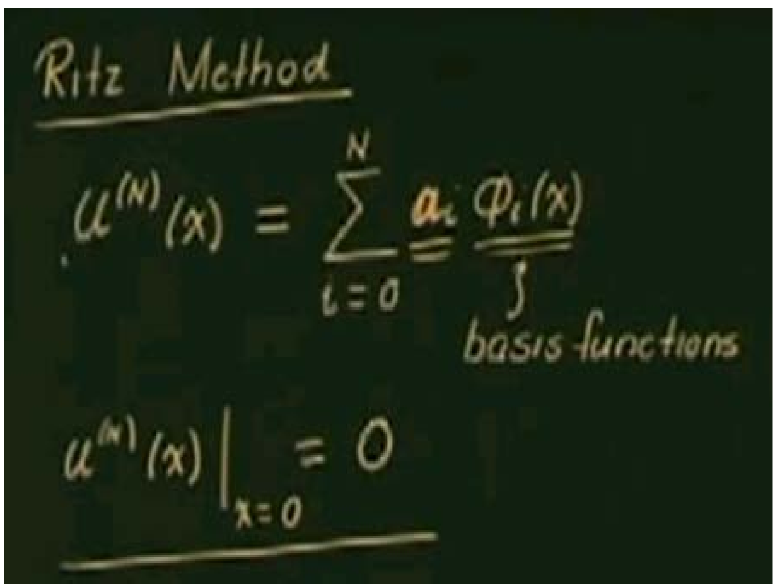

Given this variation what we will do in the RITZ method as we had said earlier. We are going to look for an approximate solution of the following form, I will call it by the name $u^{(N)}$ (x). This will be equal to sigma i=0 to N a_i phi_i of x, where we had said these a_i's are the coefficients which are to be determined and phi_i's are the so called basis functions that are used to define this series solution. What do we want? If we remember from what we had said last time that the $u^{(N)}$ (x) has to satisfy the geometric boundary conditions or the essential or the dirichlet boundary conditions. Which means that we want this one to be at the point x=0 to be =0 that is it is equal to specified displacement at the point x=0. This has to be satisfied by the $u^{(N)}$ of x. Let us choose a particular set of this function phi_i of x the things that we are used to.

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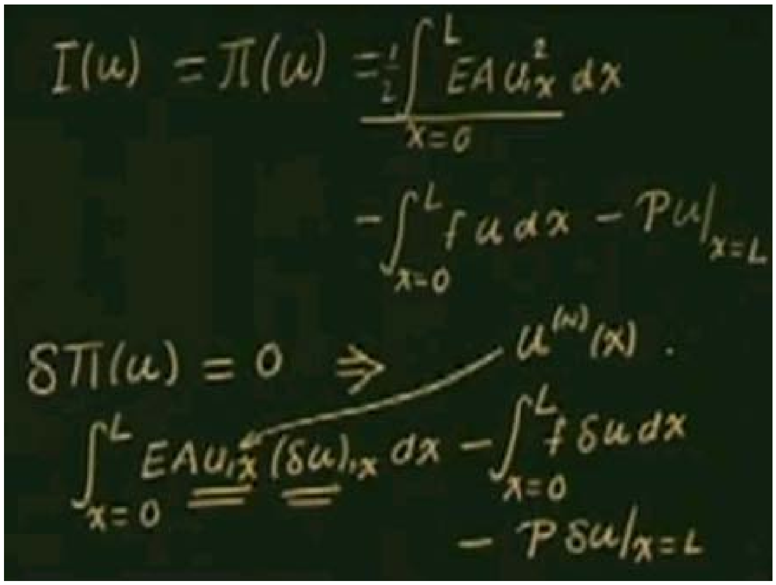

Let us take phi_i of x is equal to x to the power of i for i going from 0 to N. When we take this and substituted in our expression we will end up getting u N of x is equal to sigma i is equal to 0 to N a_i x to the power of i. We want $u^{(N)}$ at 0 to be equal to 0 which is equal to… what do we get by substituting 0 for x to the power of i? We will get a_0; so what we end up getting is a_0=0. This is the constraint that we have enforced. What we end up getting after enforcing the constraint at the point x=0? We will get the series i is equal to 1 to N a_i x to the power of i.

Because a_0 has been determined from the boundary condition, once we get this expression now the question is how do we obtain these coefficients? Let us go back to our variational formulation. Here if you see we have ux and we have this variation of u, x. What we are going to do is instead of u of x, we are going to substitute this with $u^{(N)}$ of x and the variation of u will then become variation of $u^{(N)}$ of x.

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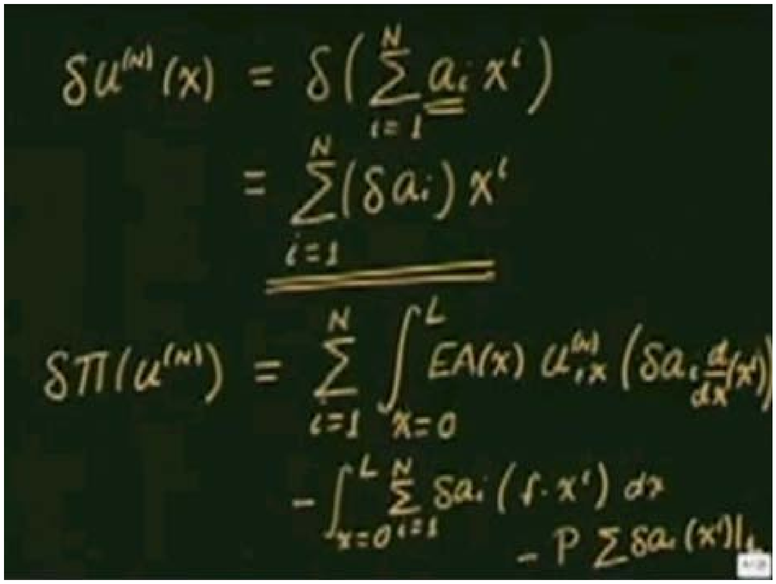

Let us now define what is the variation of $u^{(N)}$ of x. This will be equal to variation of sigma i=1 to n a_i x to the power of i. This function x to the power of i are given to us; what is going to change in order to get the variation is the coefficient a_i. The variation of $u^{(N)}$ of x will become summation 1 to N variation of a_i x to the power of i. Once we have obtained this expression for the variation of $u^{(N)}$, let us go and put it back in our variational formulation. When we put it back in a variational formulation we will get delta of pi of $u^{(N)}$ is equal to summation of i=1 to N EA which is a function of x $u^{(N)}$,x and here we will end up getting variation of a_i and d dx of x to the power of i.

What we have done? We have substituted instead of variation of $u^{(N)}$ in expression this summation. Then will also get minus integral x=0 to L summation, take a summation out any way I will put it here, variation of a_i into f multiplied by x to the power of i the whole thing dx and finally will get the term minus p x to the power of i evaluated at one at the end. This is the whole expression that we will get by substituting for variation of $u^{(N)}$ in our variational formulation.

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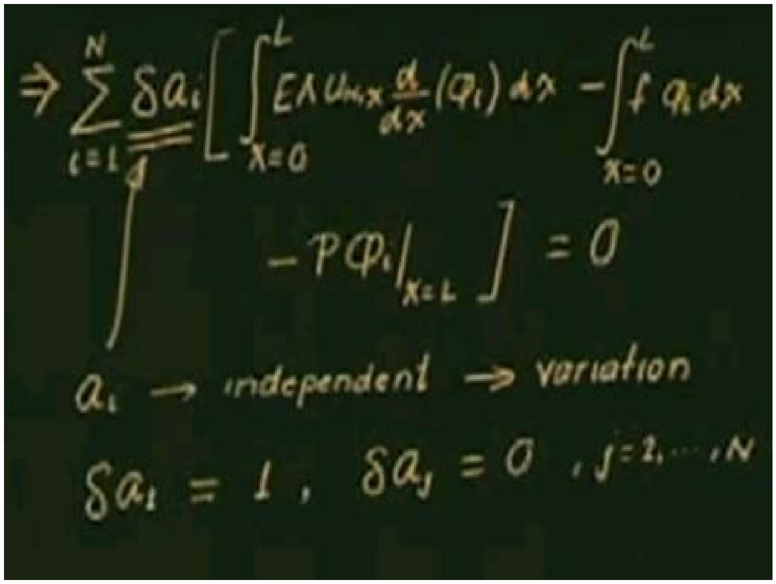

Let us again rewrite the whole thing; we will get - implies summation of i going from 1 to N variation of a_i into, we put the big brackets, integral x=0 to L EA $u^{(N)}$,x d dx I am going to replace x_i by phi_i. One should understand that in our example we have taken x to the power of i is our phi_i dx minus integral x=0 to L f phi_i dx minus p into phi_i evaluated at the point L whole expression is equal to 0, next what? How do we get the equations using which we can determine the unknown coefficients a_i? That is our goal. Then we look at the variation of a_i; this is something that is under our control.

The variation is in a way an abstract virtual displacement that I am giving to the structure which is already in equilibrium. I can choose what kind of variation i gives; so in a way what I am saying is that we can vary each of these a_i's independently - i these are independent coefficients. That is variation can also be done independently. Let us take for example that I decide to choose variation of a_1 =1 while variation of all the other a_j=0 for j is equal to 2 to N. Let us put it back in this expression. When we put it back into the expression the summation gives as what?

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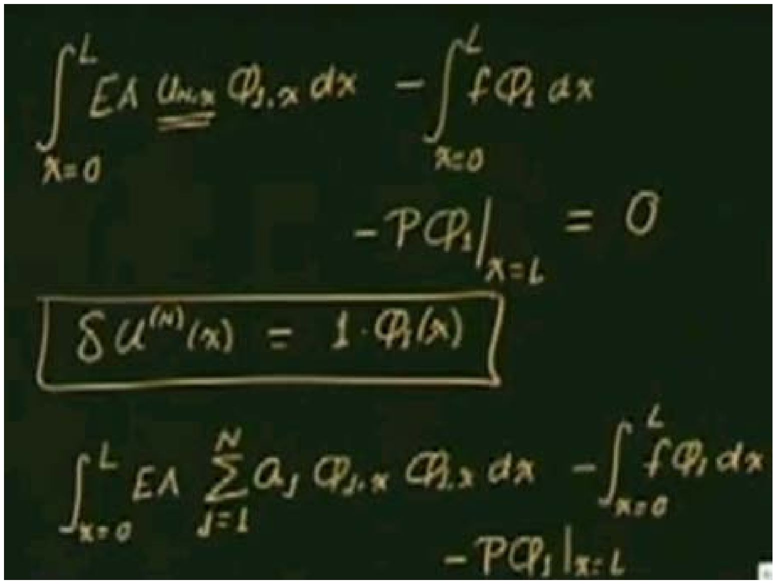

It gives us by putting the value of the variation of a_1 =1 in all the other variation is equal to 0. We will get integral x=0 to L EA $u^{(N)}$,x I will write phi_1,x dx minus integral x=0 to L f phi_1 dx minus P into phi_1 evaluated at x=L =0. Remember what we have done we have taken the variation of a_1 =1 all other variations we have set to 0. That is one choice of the variation that we have taken. That is we have taken del $u^{(N)}$ of x is equal to 1 into phi_1 of x. When we substitute this choice we get this equation. Let us now expand this equation in terms of the coefficient a_i. What will do when we put it there integral x=0 to L EA sigma j is equal to1 to N a_j $p_{j,}$x P 1,x dx minus x=0 to L f P_1 dx minus P into P_1 x at x= L.

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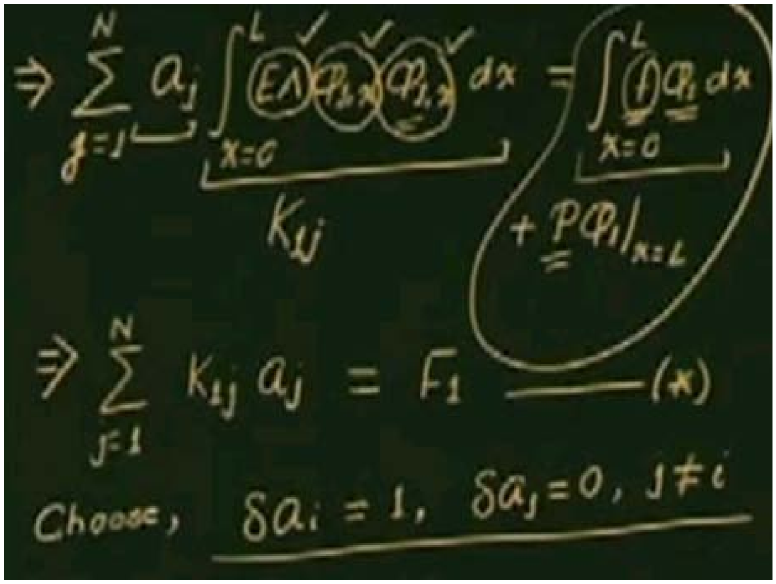

Let us now take the summation out we will write summation of i=1 to N let us put j a_j integral x=0 to L EA what will we get $phi_{j,x}$ $phi_{1,x}$ dx is equal to, by taking the other part to the right hand side, integral x=0 to L f phi_1 dx plus P into phi_1 evaluated at x=L. I have taken the other part to the right hand side simply because you see that f is known, phi_1 is known, P is known and phi_1 at x equal to L is also known, because we know the expression for phi one which is nothing but x. The x evaluated at L we know so all the known's have been brought over to the right hand side. What is the unknown? Unknown is a_j which remains on the left hand side. Now look at this expression; this is a number because EA we know, phi_j we know because we have chosen the phi_j and phi_1 we know we put all of those into the expression of the integral these will turn out to be a number.

This number we are going to give a name; we will call it K_{1j} why do we call it 1 j? Because you see that everything this $phi_{j,x}$ is multiplied by $phi_{1,x}$. So for different $phi_{j's}$ we are always multiplying with $phi_{1, x}$. This is the first index and the second index comes because of this $phi_{j's}$. If I write it now in this index form I will get implies summation of j=1 to N K_{1j} a_j is equal to look at this now on the right hand side. This f is not going to change if you change the phi_i's. This f will remain fixed as the function, f into phi_1 integral over x=0 will be a number corresponding to phi_1. Similarly p into phi_1 at x equal to L will also be a number corresponding to phi_1. We are going to call this as F_1. This is one equation that we get. Then what we can do is similarly choose the other delta a_i's one by one equal to1 and keeping the other ones 0. That is we'll say that we will say delta a_i for a particular i is equal to one and for all others for j not equal to i. This will be a particular choice of these varied parameters for this particular choice.

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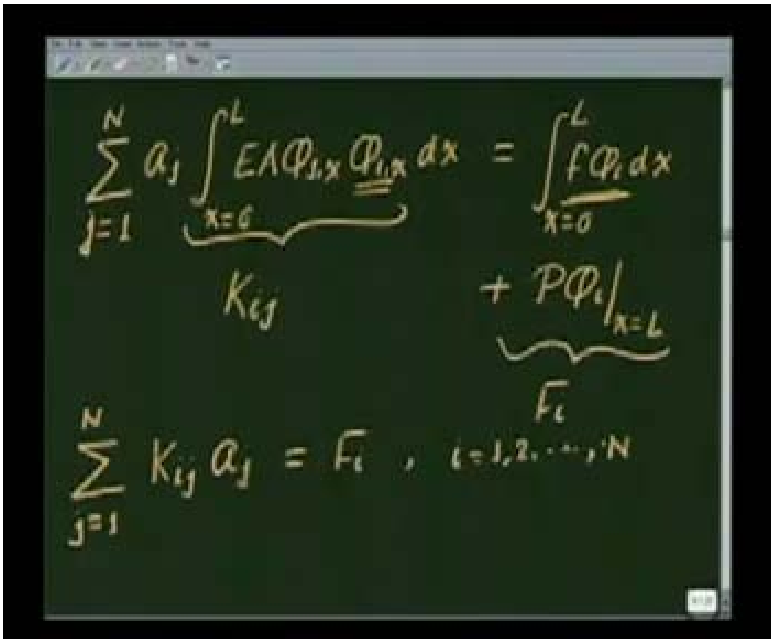

If I go now to the equation, then we can generalize what we have done corresponding to the phi_i or the delta a_i that we have taken; sigma j is equal to 1 to N a_j integral x=0 to L EA $phi_{j,x}$ $phi_{i,x}$ dx this is equal to integral x=0 to L f phi_i dx plus P phi_i evaluated at L. Again you see this expression; here the j is going from 1 to N that is summing over the j's but this i is fixed. What we can write here is we can replace this by coefficient K_{ij}. This integral becomes the coefficient or a number $K_{ij.}$ Similarly this one where corresponding to the choice of delta $u^{(N)}$equal to 1 into phi_i will get this one else f in to phi_i integrated plus P into phi_i at L.

We will call this by a name F_i which is again a number. What will get as the equation j=1 to N K_{ij} a_j = F_i; now for i going from 1 to 2, to…N. That is for each of these I will get an equation. What we have is a system or linear equations or simultaneous equation in terms of the unknown coefficients $a_{j.}$

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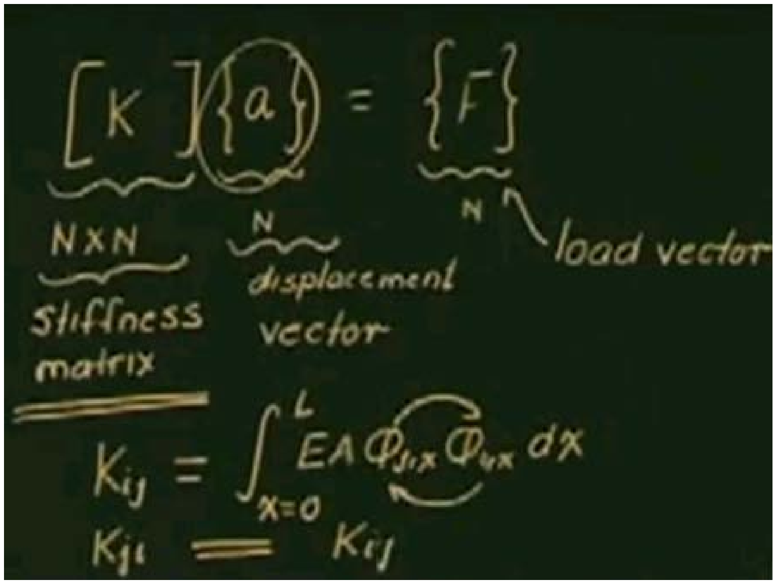

This we can write in matrix form as matrix k operating on the vector a is equal to vector F. We look at this matrix K. This matrix K is of size N by N for the problem that we have taken. This is the vector of size N that is it has N coefficient a_1 to a_n this is the vector of size N. This matrix as we will see can be called the so called stiffness matrix. This vector we are going to give it a name called the displacement vector. And this vector will be called the load vector. We have this matrix problem Ka=F which if it can be solved will give as the unknown coefficient a. Once I obtain the unknown coefficient a then I know the expression for $u^{(N)}$of x.

Lets us look at this matrix; what are the entries of this matrix? So k_{ij} if I look at elements of this matrix that is the element sitting in the ith row and in the jth column will be equal to integral x=0 to L EA $phi_{j,x}$ $phi_{i,x}$ dx. Now I ask you the following question: tell me what is the element sitting in the jth row and the ith column? That will be obtained by bringing this here and bringing that there. The expression is not going to change because the integral the integrant remains the same so what we get for this case is that this thing is equal to K_{ij}.

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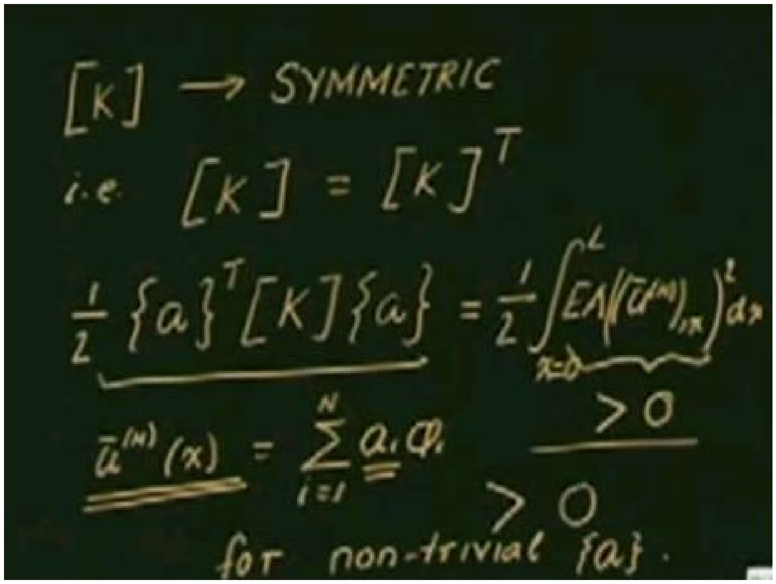

What does that mean in terms of the matrix notation? It means that the matrix K is symmetric that is nk_{ij} is equal to k_{ji} we have K is equal to K transpose. Secondly, if we look at the following expression 1/2 of a transpose K a; let us say I give you 1/2 set of the coefficients a and then I would like to evaluate this expression. What is it equal to? This will be equal to from what we had already defined 1/2 of integral x=0 to L EA in to u bar N, x whole square dx. Where I am going to call u bar N as a function of x is equal to summation i is equal to 1 to N a_i phi $_i$ of x. Tell me this integrant is always greater than 0 when these coefficients a_i are not equal to 0.

This means that when this function $u^{(N)}$ is not the trivial function that is $u^{(N)}$x=0 everywhere then this expression has to be greater than 0 integrant because I am looking at u bar comma x whole squared. When this integrant is greater than 0 what do we have? In that case this expression itself is ½ a transpose KA is greater than 0, for I will say non trivial. What does that mean? That if I give you any choice of this set of coefficients a I can guarantee that if these a's are not all equal to 0 then this expression 1/2 a transpose KA is greater than 0.

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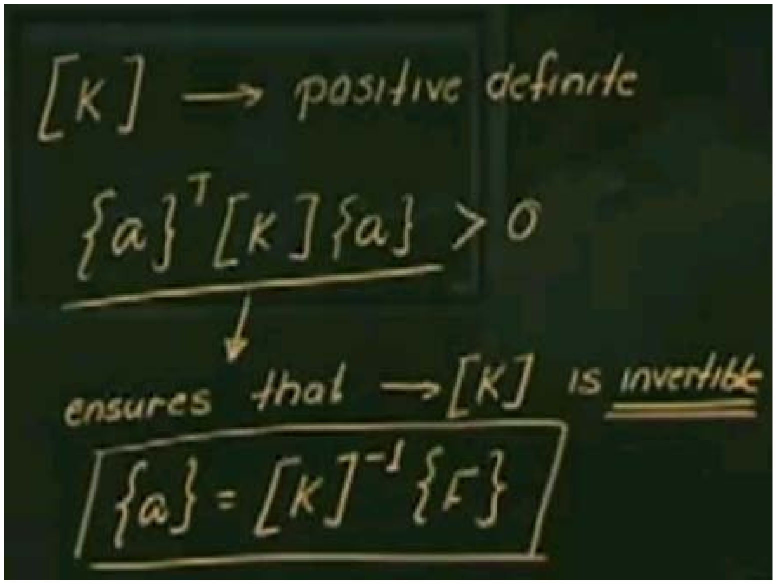

This means that this matrix K is positive definite. The definition for positive definite for matrix is that if I give you any set of these vectors a which are non trivial then a transpose K a has to be greater than 0. And if we look back to the expression that we have taken then the integral that corresponds to this expression a transpose K a is nothing but twice the strain energy of the corresponding to the displacement given by u bar N. And for the structure we know that the strain energy cannot be 0 or negative unless the structure is subjected to a rigid body motion.

Here we don't have any rigid modes so which means that any a, which is non-trivial this is not greater than 0. Why do we need for this positive definitive for the matrix? Positive definiteness ensures that this matrix K is invertible. This was not obvious by looking at symmetry. A matrix can be symmetric but may not be invertible; it could be singular. By this we prove that yes indeed for our Ritz formulation module problem that we have taken this matrix K is a positive definite; which means that whatever be the end we can always invert the matrix K.

Why is it important? Because if we cannot invert K then we cannot get the solution vector a which is given by K inverse F. So once we guarantee invert ability of the matrix then the solution vector a can be obtained.

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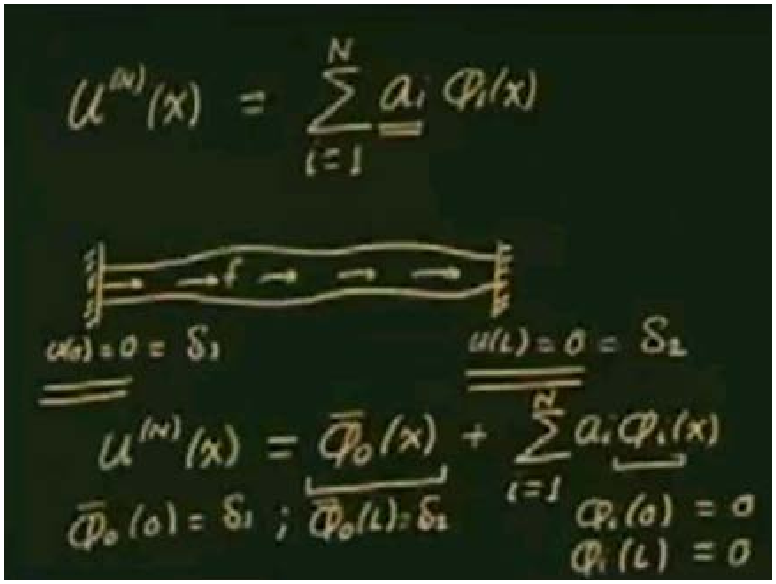

Once we have obtained the solution vector a then now we put it back in our expression for the solution which is at least in our example it is a_i phi_i of x where a_i's are now known. This is how we construct a typical Rayleigh - Ritz or Ritz solution to given boundary value problem. There can be interesting off shoots to this problem; this is not the only module problem that we may be interested in; let us say that I want to solve this problem. Let us say I want to solve this problem. There is the bar subjected to a constraint u at 0 is equal to 0 and a constraint u at L is also equal to 0. This could be a problem of interest. Now in this case how do we go about constructing the Ritz approximations? We see that in the earlier case it was very easy to in force the geometric conditions at the point x=0. We have to in force the geometric conditions both at the points x=0 and x=L. How are we going to do this job? We will take again $u^{(N)}$ of x; we will say it is equal to something called phi_0 bar of x plus summation of i equal to 1 to N a_i phi_i of like this, phi_0 bar has a job of satisfying both these n conditions and the phi_i's will be such that phi_i at 0=0 and phi_i at L=0.

What we are going to do is, we are going to choose the phi_i's and such a way that this satisfies the 0 conditions at both ends. While phi_0 bar takes care of the zero displacement conditions at two ends. We see that in all problems you may not have the zero displacements condition, you may have this one as some number delta 1, this one as some number delta 2. In that case phi_0 bar at 0 has to be equal to delta 1 and phi_0 bar at L has to be delta 2 while the phi_i's will satisfy the conditions of 0 value at two ends.

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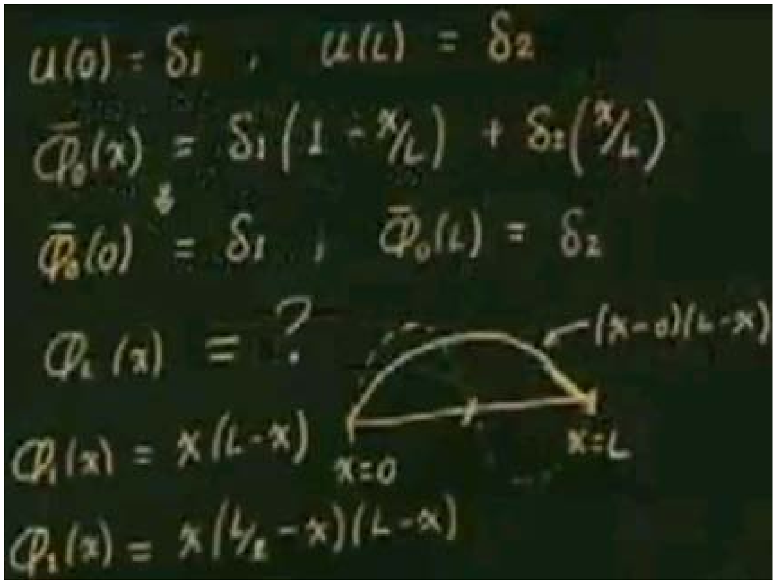

In this case let us say that my u at 0 is equal to delta 1 and u at L is equal to delta 2. Then how do we construct this function phi not of x? Very easy, you take it to be delta 1 into 1 minus x by L plus delta 2 into x by L. If we look at this expression what happens at x=0 will get x by L=0 and this other expression is equal to 0, so at this is going to give me phi_0 at 0 is equal to delta 1 and phi_0 bar at x equal to L. If we substitute then this expression is going to go to 0 what we are going to get here is 1. This will be equal to delta 2. So phi_0 bar satisfies the given geometric constraints exactly at the two points. Then the question is how do I choose the remaining phi_i's? In the expression of the $u^{(N)}$ of x phi_0 bar is completely known; we have to get this phi_i's.

So one option if you remember that phi_i has to be 0 at x=0 and at x=L for this particular case that we have taken. In this case what choice can be made for phi_i? The first function phi_i can be it has to be quadratic. Because quadratic is only function which is going to minimum order polynomial which is going to vanish at the two ends. This can be made x minus 0 into L minus x. So I can make phi_1 of x is equal to x into L minus x; these are all choices that we can make. Similarly phi_2 x will be now cubic. It could be a function going like this. I can take it to be, I am just choosing things according to my wish, L 2 minus x into L minus x.

What I want this function to do is it should vanish at point x=0, at the point x=L by 2 and the point x=L. I get a cubic expression x into L by 2 minus x in to L minus x and so on. I can keep on constructing these phi_i's which are functions of increasing order polynomials by choosing appropriate points where they vanish.

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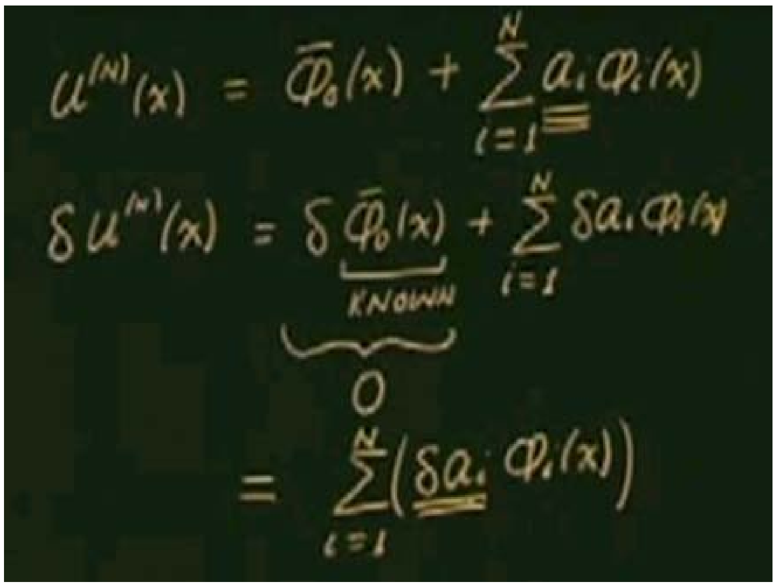

Once I have done this construction then I will put it back in my expression for $u^{(N)}$ of x to get $u^{(N)}$ of x will be equal to phi 0 bar plus 1 to N a_i phi_i of x. Now our job will be again to find this coefficients $a_{i.}$ We will substitute this back in our variational formulation; only thing you should observe is this variation of $u^{(N)}$ of x will be actually equal to variation of phi_0 of x plus, summation i=1 to N variation of a_i phi_i of x. This phi_0 of x is a known fixed function that is we know what it is. The variation of phi_0 of x is 0. What we will end up getting is variation of $u^{(N)}$ is nothing but summation from 1 to N variation of a_i phi_i of x. So by putting it back in the principle of virtual work for variational formulation that we have made and by varying choosing particular values for this delta a_i's we will get now N equations in terms of the n unknown coefficients $a_{i.}$ Exactly the way we have done earlier. The only difference will be the $u^{(N)}$ will carry this expression.

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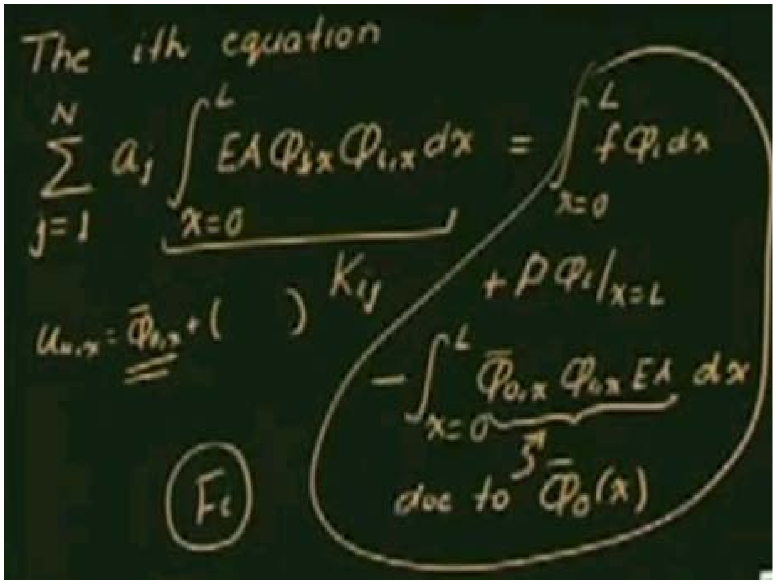

Let me just write it down to make things clear. The ith equation I will just write; that will be summation of j=1 to N a_j integral x=0 to L EA $phi_{j,x}$ $phi_{i,x}$ dx is equal to integral x=0 to L f phi_i dx plus P phi_i evaluated at x=L plus a part which will come (you are $u^{(N)}$ of x which was equal to phi_0 comma x) plus the remainder part this part is a known. This will also go not as a plus but as a minus. We will get minus integral x=0 to L Phi_0 bar comma x in to $phi_{i,x}$ in to EA dx. This part is coming from the known functions phi_0 bar, which satisfies the given boundary condition. This is the additional correction that we have to do to our variation formulation in order to account for this phi_0 bar of x. We have the N equation again in terms of N coefficients this expression will be nothing but K_{ij} and this whole thing on the right hand side will be our F of i.

We go ahead and solve this problem and get the solution that is the coefficients $a_{i.}$ This is how you would construct the Rayleigh - Ritz solution for our rating of our problems with different boundary condition, different load vector and different material coefficients using a polynomial approximation. Nobody tells us that we should use a polynomial approximation.

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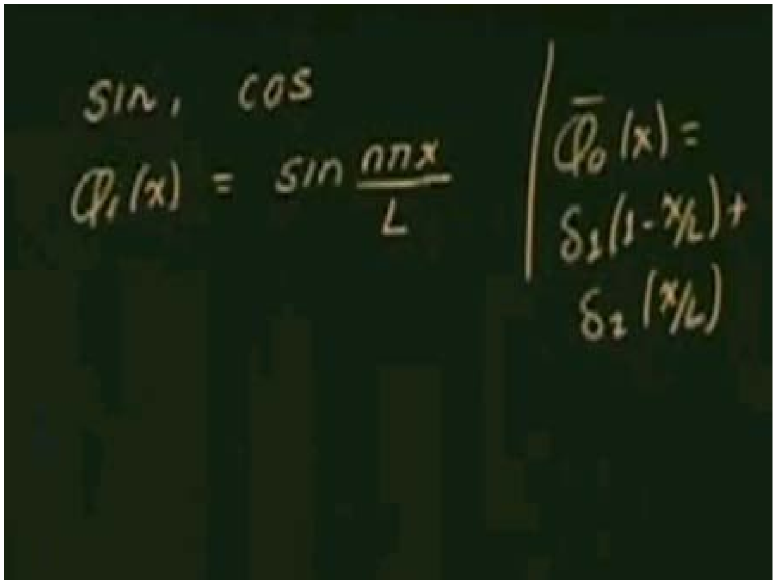

We can also use the sine cosine functions. For example we could have chosen phi_i of x is equal to sin n pi x by L. Very easily we could have done this though our phi_0 of x would have remain equal to delta 1 into one minus x by L plus delta 2 into x by L. But this phi_i's for i=1 to N i could have chosen this in terms of this functions which are trigonometric functions we could have again obtained the series solutions by following the same procedure.

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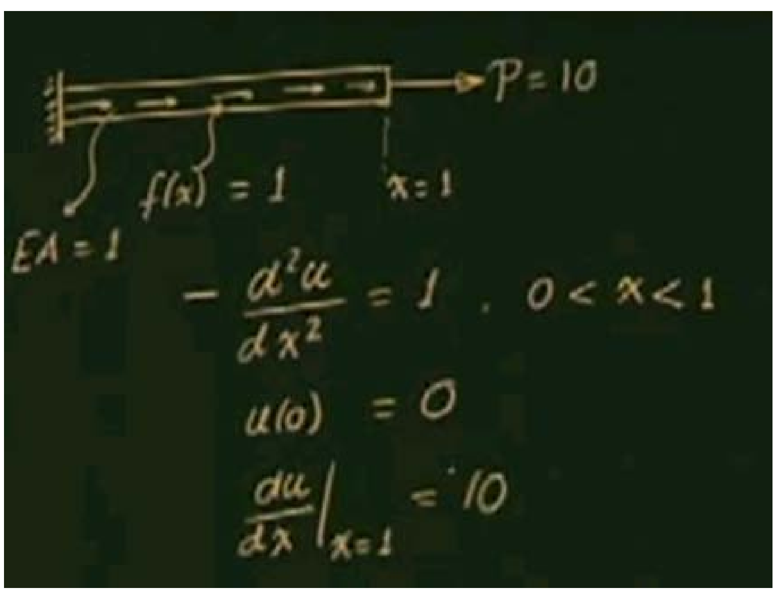

Let us now look at an example with which we will demonstrate how this Rayleigh - Ritz method is used and what is the solution that we will get. Let us take this problem: bar again with an end load P=10 and subjected to a distributed load of uniform intensity that is will say f of x=1 and we will say EA is equal to 1. In this case if we look at differential equation at becomes minus d^2 u dx^2 =1 for x line between 0 and 1. We have taken the bar to be of the length 1. And we will say that u at 0=0 at the end x=1 will have EA du dx where EA now is 1 so you will have du dx at the point 1=10.

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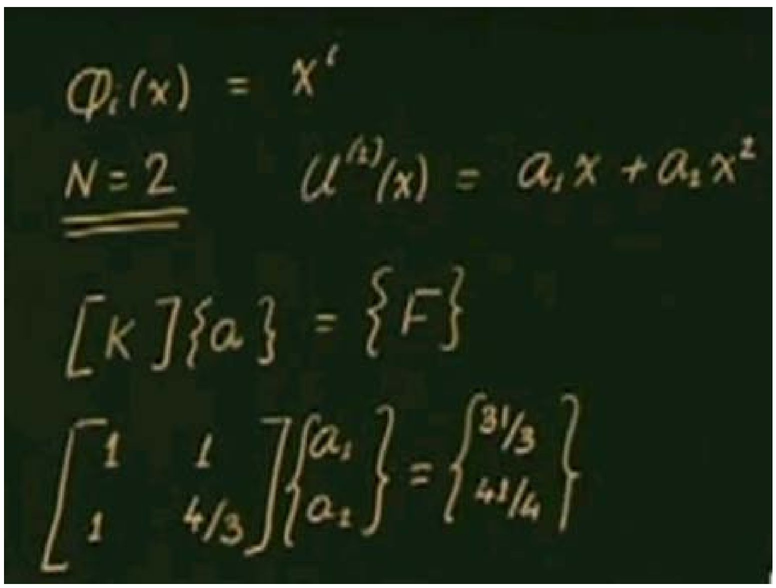

So corresponding to this one if we now go and choose our phi_i's as we have done earlier is equal to x to the power of i and we will take two terms solution let us take N=1.Then our u two of x will be equal to a_1 into x plus a_2 x^2. So going back to our virtual formulation and writing the final matrixes I will get K into a is equal to F where K will have entries 1, 1, 1, 4 by 3 you should all evaluate this entries will convince yourself a_1 a_2 this is going to be equal to the load which is equal to 31 by 3 and 41 by 4.

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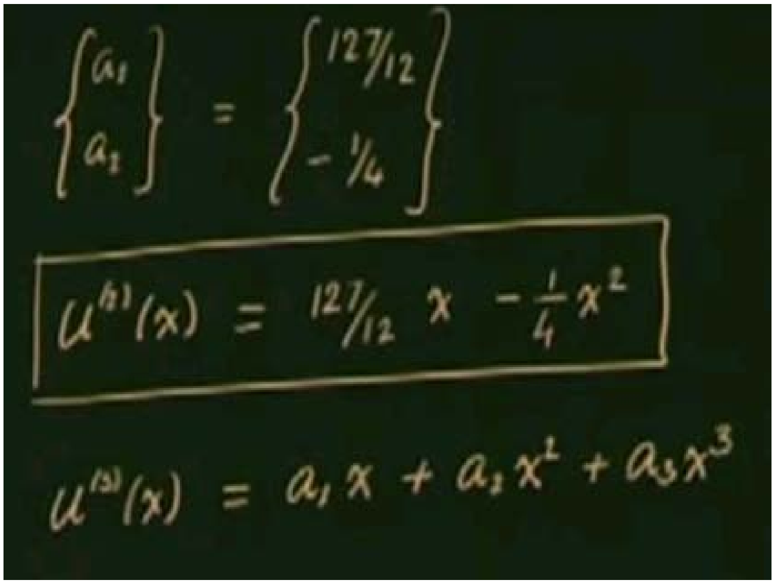

When we solve this we will end up getting a_1 a_2 = 127/12 and -1/4. So our $u^{(2)}$ of x will be equal to 127/12 x-1/4x squared. This will be our (ralaridge) solution taking 2 terms in the polynomial expansion.

Similarly if I now take 3 terms that is I would like to have $u^{(3)}$ of $x=a_1 x+a_2 x^2 + a_3 x^3$.

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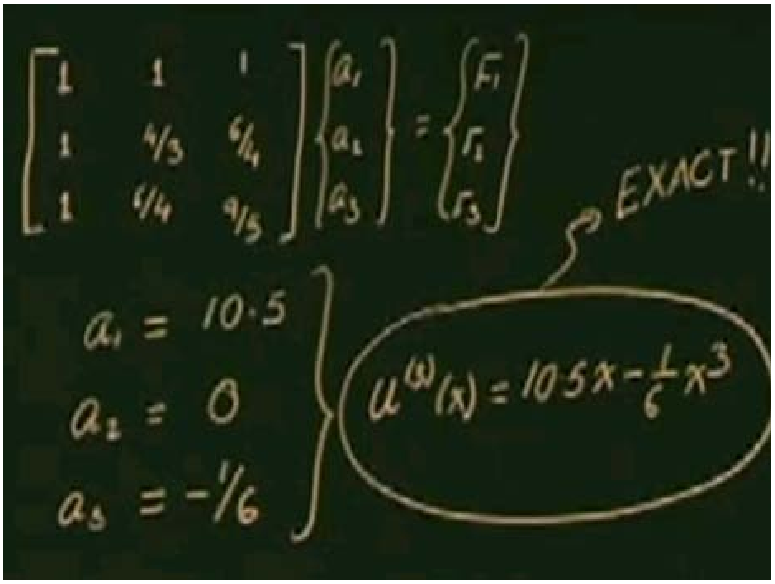

And again we go through the same exercise we will get the matrix a K as 1, 1, 1, 4/ 3 1 6/ 4 9/5 6/4 also has to be here because it is symmetric. This in to a_1 a_2 a_3=F_1 F_2 F_3 here F_1 F_2 F_3 can be evaluated from the integrals. We will end up getting coefficients a_1=10.5 a_2=0 and a_3= -1/6. So our $u^{(3)}$ of x will be equal to 10.5x -1/6 x^3 and you will see that this is exact solution of this problem. The exact solution of this problem turns out to be a cubic polynomial and by taking three terms in the series expansion we have exactly captured that which had to happen. There is no other thing we could have obtained because we have said that this phi_i's should be able to completely represent the highest order polynomial. That is in this case, the linear combination of the three phi_i's should be able to exactly capture cubic polynomial and here exact solution was the cubic polynomial which we have captured.

If we get 10.4x minus something you answer is wrong then you should go back and check your calculation. This answer should be correct to the last decimal digit.

Let us look at the plots of this expression both the two terms solution and the three term solution that we have obtained. Whenever we obtain an approximate solution we should again look back at what is the goal of this whole computation. The goal is to obtain the response quantities of interest to sufficient level of accuracy.

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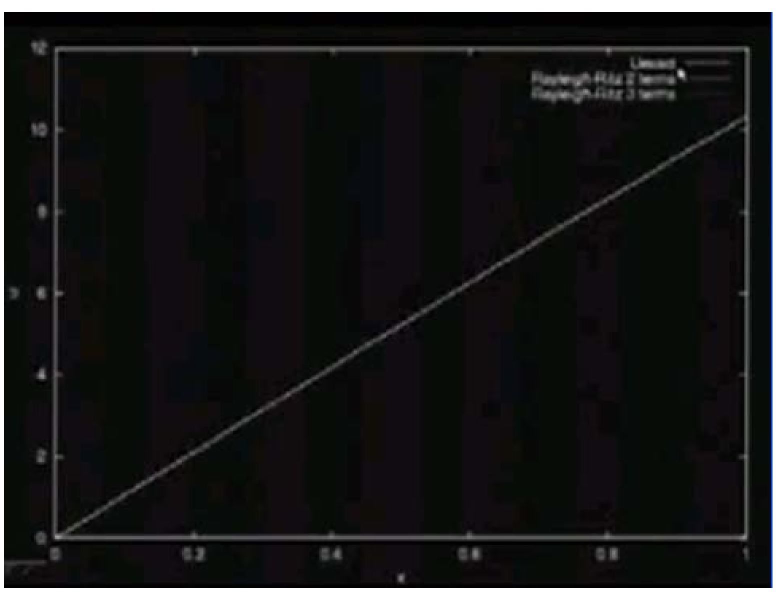

Let us look at this graph that we have plotted of the displacement of a two terms solution and for the three term solution as a function of x.

You see that here you can hardly see a difference the two graphs almost overlap with the exact one.

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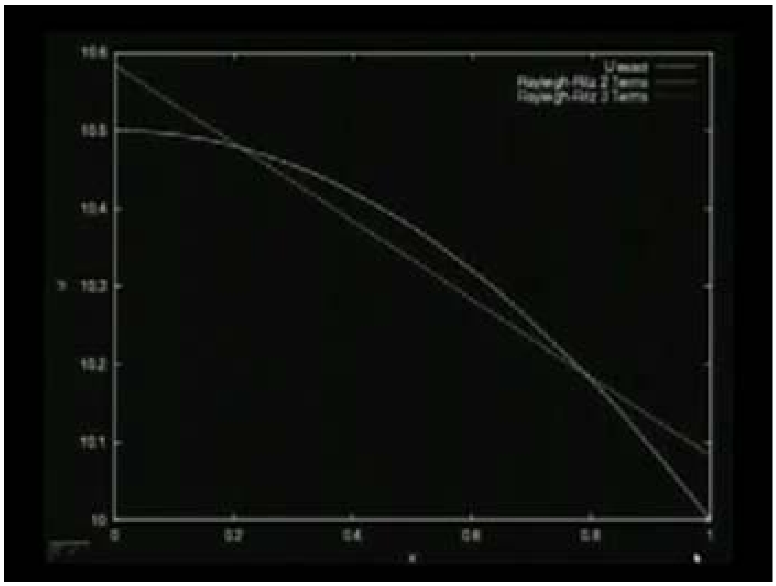

Let us now see what happens when we look at the derivative information. When we look at the derivative information obviously with the three term solution will over lap the exact one because it is the exact solution while the two term solution does not do a great job with respect to the derivative. Though it is not very far if we look at the end value you will get an error if we look at this point, error in the range of 1% 2% in the value of the derivative. But still if we look at the derivative information that it is inferior to the accuracy of the value itself. This is going to be a feature of all the approximation methods. That is if we see good accuracy with respect to function itself, if we go and taking higher and higher derivatives, the accuracy is going to decrease. That is we are going to pay the penalty by taking the derivative of the less accurate function.

Once we have seen that this Rayleigh - Ritz method seems to be doing a great job for the problem of interest then why go to anything else. We can stick to the Rayleigh - Ritz method and solve all the boundary value problems using the Rayleigh - Ritz method. Let us see an example of a boundary value problem where the Rayleigh - Ritz method may not do well.

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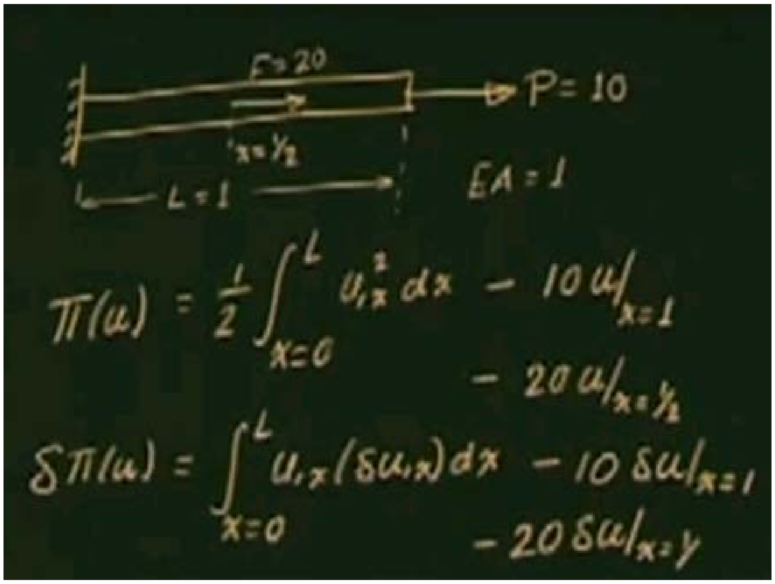

Let us take a very simple problem; again our actual bar. This is our fictitious example that we have created with an end load 10; this is our bar of length 1. At the point x=1/2 I am going to apply a concentrated load f =20 units. This is at the point x=1/2.

If I write total potential energy corresponding to this problem what will we get, again putting EA=1? We will get pi of u=1/2 integral x going from 0 to L u, x whole square dx minus 10 u evaluated at x=1-20 into u evaluated at x=1/2. So work done by the external forces is nothing but the work done by these point loads at applied at the point 1/2 and 1. So if I do the variation of pi u will again get integral x=0 to L u, x variation of u, x dx-10 variation of u evaluated at the point x=1-20 into variation of u evaluated at the point x=1/2.

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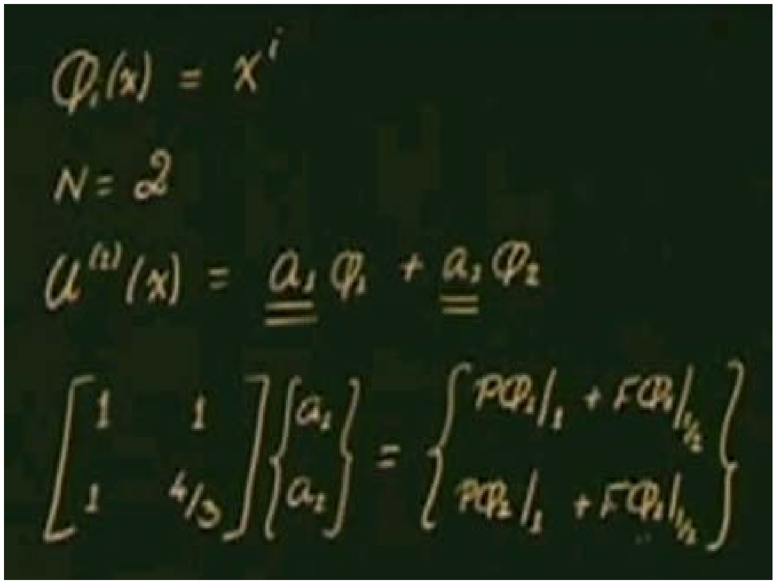

Let us now use the same approach that we had taken earlier that is will take phi_i of x to be equal to x to the power of i. We can again take two terms solution N=2. What will get $u^{(2)}$ of x=a_1 phi_1+ a_2 phi_2. Substitute back in our variation formulation and get the equation corresponding to a_1 a_2 by what we have done. What we will end up getting is 1, 1, 1; that is the same stiffness matrix that we had obtained for the previous example because the material has remain the same and phi's are also the same. There is no reason why the stiffness matrix should change. This will be equal to the load vector. Load vector is what? If you remember it will be P into phi_1 evaluated at 1 plus F into phi_1 evaluated at half, and here it will be p into phi_2 evaluated at 1 plus F in to phi_2 evaluated at 1/2.

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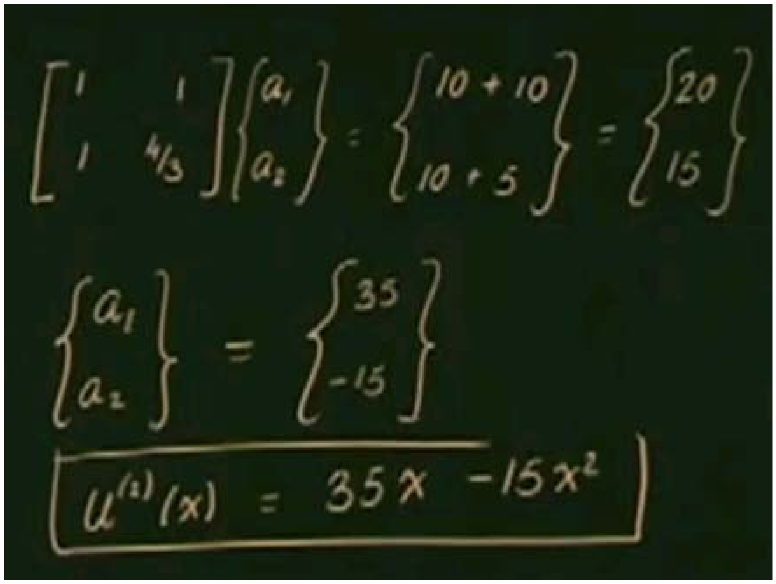

We will get again if I rewrite 1, 1, 1 4/3 intoa_1 a_2 =P into phi_1 evaluated at one is ten plus F in to phi_1 evaluated at 1/2. So x evaluated at 1/2 is 1/2 so 10+10. Second one, P into phi_1 phi_2 evaluated at 1 so phi_2 at 1=1. I will get again 10+F into phi_2 evaluated at ½.

phi_2 is x square evaluated at 1/2 is 1 4. So ¼ into 20 will give us 5. This will be 20 and 15. Out of this, the coefficients a_1 and a_2 will come out to be equal to will be equal to 35 and -15. That is $u^{(2)}$ of x=35x-15x squared. Can you tell me what is the exact solution to this problem? The exact solution to this problem will be obtained in two parts.

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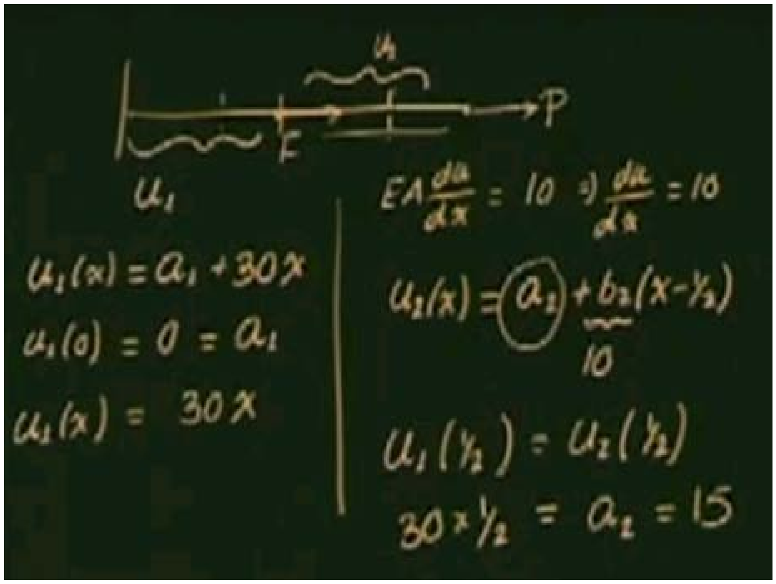

I am just drawing the bar here; this is F and this is P. What will you get if you cut any where here from our standard mechanic solids? We will get EA du dx in this region up to this point x=1/2=10 implies du dx here is equal to 10 which is the constant. I will call this part as part with the solution 2. So $u^{(2)}$ of x will be equal to I will write it as a_2+b_2 into x-1/2 I could always write it as linear because the slope is the constant where b_2 will be equal to 10. Similarly if I look here in this part I will call the solution as u_1. So u_1 of x will be such that I will get the slope equal to what? Slope will be equal F+P which is 10+20 which is 30 30x. If I impose at condition at 0=0= a_1.So u_1 will be equal to the function of x - 30x.

We have to obtain a_2. How do we obtain a_2? By enforcing the continuity of the displacement at the point x=1/2 that is u_1 at ½ = u_2 at ½. So this is going to give us 30 into1/2=a_2 which is equal to 15. So a_2 becomes 15+10 into x-1/ and u_1 becomes 13x.

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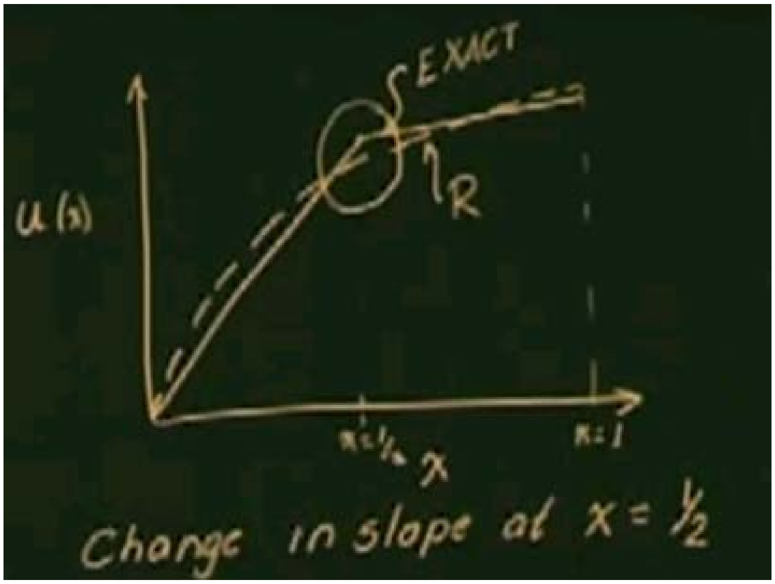

Now if you plot this function as a function of x; in the first part the slope will be 30 up to the point x=1/2 and beyond the point x=1/2 the slope will be 10. So up to the point x=1. If you now take the solution that you obtained using Rayleigh - Ritz method it will do something like this. This is the EXACT and this is Ritz two terms. If you take three terms it was still do this. That is what we are going to get as a solution is a polynomial. At this point x=1/2 you will not be able to capture the change in the slope using the Rayleigh - Ritz solution whatever we do. Your function will make them close in terms of the values at the point but the derivative will be completely 1/2 at the point x=1/2 because this function is going to get change in slope at the point x=1/2.

So as an engineer that is a cause of worry because this will be one of the points where I would like to know what is the derivative that is I would like to know what are the stresses such that I can decide whether this point is going to be safe or not. But our Rayleigh - Ritz solution is not going to give us the derivatives very accurately here; we will end up getting bad information regarding the state of stress at the point x=1/2. The question is why Rayleigh did - Ritz method go wrong here? The answer is quite obvious that in the Rayleigh - Ritz method we are trying to fit the polynomial over the full domain.

While the solution as you have seen is a piecewise polynomial that is in the region x=0 to ½ it is the polynomial, which in this case as a region and it is the region ½ to 1 it is also the polynomial with of the different slope which is also linear. So somehow in our approximation, in the choice of phi_i's we have to build in this kind of an information that we should choose pi which can reproduce the function which are piecewise define, that is the motivation of using a finite element method. I would say one of the motivations. In the next class I am going to highlight this point little further, elaborate on it, and make a case for using the finite element method where we will say what we do in order to rectify these kind of problems.