Finite Element Method

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Module – 9 Lecture – 2

In this lecture, which is our 28th lecture, we are going to continue our discussion on the planar elasticity problem. In this problem, we had taken a domain omega. On this domain I would have some specified boundary conditions about which we are going to talk in detail today. Some parts of the boundary are completely fixed, some parts of the boundary may be on rulers and some parts will have a traction specified. In the interior there will be somebody forces acting on the body given by a distribution of the body force by the vector f as a function of x and y.

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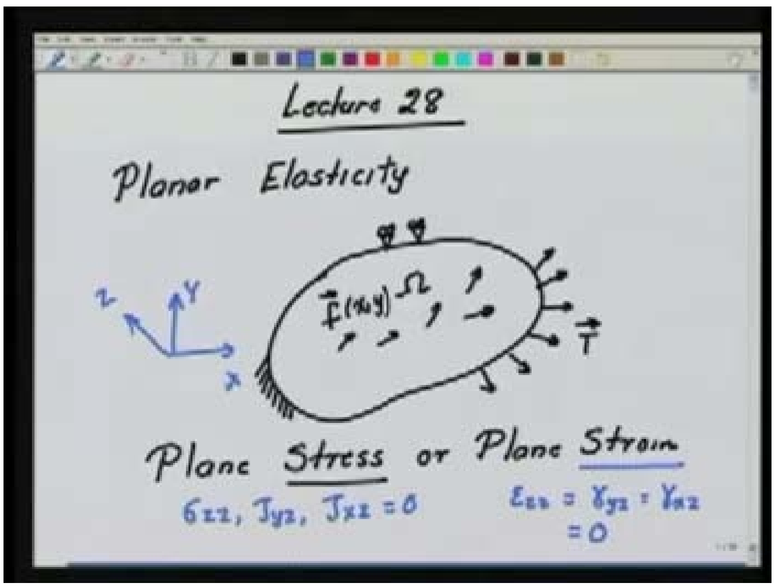

This could be either a plane stress or a plane strain problem. By this we understand that the state of stress and strain do not change with the thickness. That is, in the transverse direction there is no variation of the stress and the strain. Also, if it is a plane stress problem then the state of stress is given by sigma xx, sigma yy. Let me draw the co-ordinate system: this is x, this is y and normal to all this is z. In the z direction there is no variation of the state of stress, that is, displacement strain and stress are assumed to be independent of the z co-ordinate. In this case of plane stress problem, we said that sigma zz, tau yz, tau xz are all zero. In the case of plane strain problem we had E_{zz} is equal to gamma yz, which is equal to gamma xz and this is equal to zero. In general, if you look through a book on mechanics, these problems can be given as a two-dimensional boundary value problem, which is given as follows.

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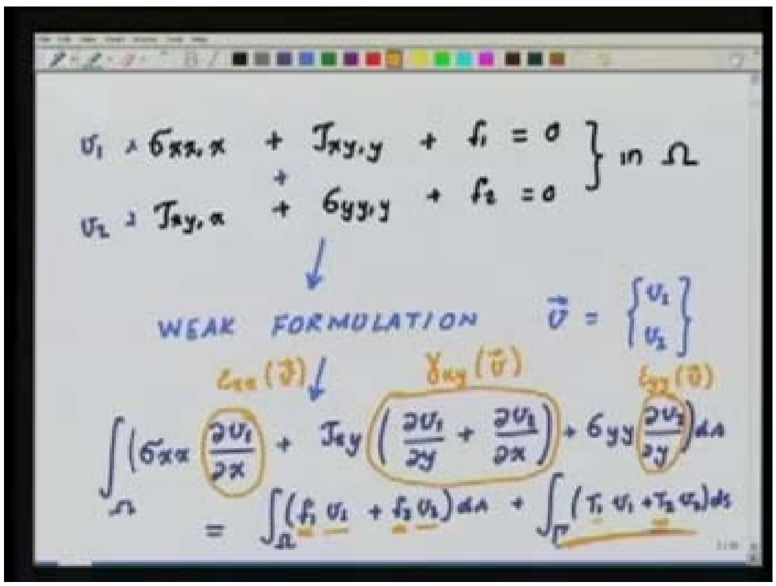

We had sigma xx comma x plus tau xy comma y plus f_1 is equal to zero; tau xy comma x plus sigma yy comma y plus f_2 is equal to zero; in omega. This was the differential equation from which we had obtained, in the last lecture, the weak formulation by taking an admissible virtual displacement vector v, given by components v_1 and v_2. Remember that this is now a vector. That is, it will have a component in the x direction, it will have a component in the y direction and these are both functions of x and y. Given this vector which we had obtained by following the usual procedure of integrating, of multiplying the equations, first one by v_1 and second one by v_2 and then integrating and adding the two equations, we had obtained sigma xx del v_1 divided by del x plus tau xy del v_1 by del y plus del v_2 by del x plus sigma yy into del v_2 by del y whole thing integrated over the area was equal to integral f_1 v_1 plus f_2 v_2 two integral over the area plus on the boundary which is denoted by gamma on the other domain T_1 v_1 plus T_2 v_2 ds. The last part is a boundary integral and if we look at this expression, del v_1 by del x is nothing but the strain E_{xx} for the vector v. The quantity del v_1 by del x corresponds to this strain for the virtual displacement. This quantity corresponds to the shear strain due to the quantity v. This quantity corresponds to the strain E_{yy} for the virtual displacement field v.

What does this expression actually represent? It says that the virtual work done by the given virtual displacement field against the external forces, which is given by either the body force which is f_1 and f_2 and the boundary tractions T_1 and T_2 is balanced by the internal virtual work done, that is, the work done by the strain due to this virtual displacement against the existing stress. In the language of mechanics this is called principle of virtual work.

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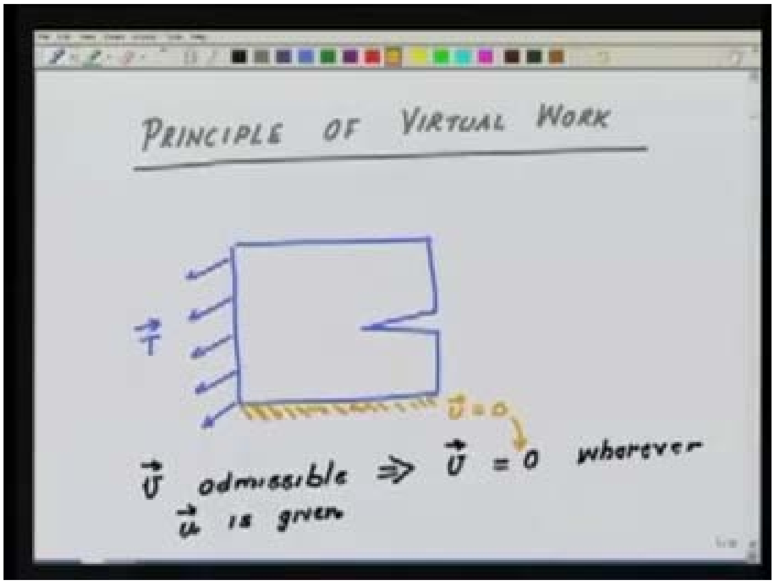

By starting with our weighted residual form, by taking the strong form of the equations or looking at the differential equations and multiplying with virtual displacements and then doing integration and deviation by parts in order to transfer derivatives from the actual stress part to the virtual displacement part, we obtain what is popularly known as the principle of virtual work formulation. Here, there are two questions.

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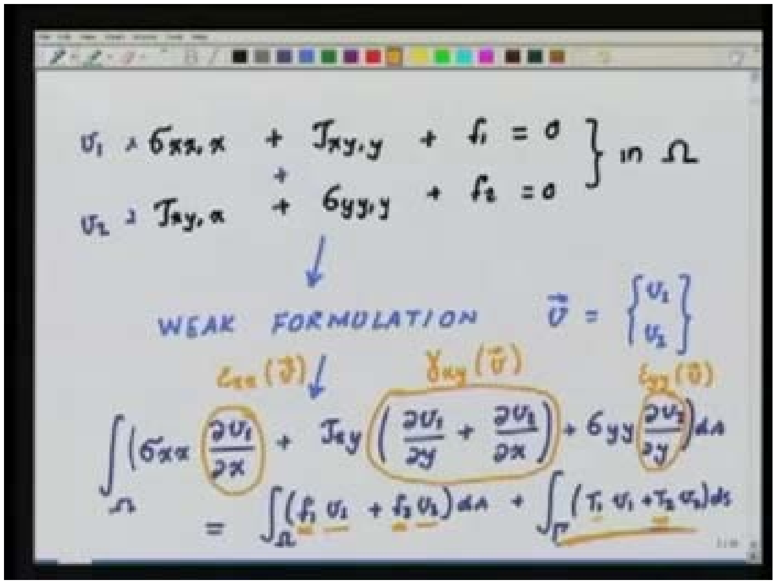

What kind of boundary conditions can we have on the boundary of a domain?

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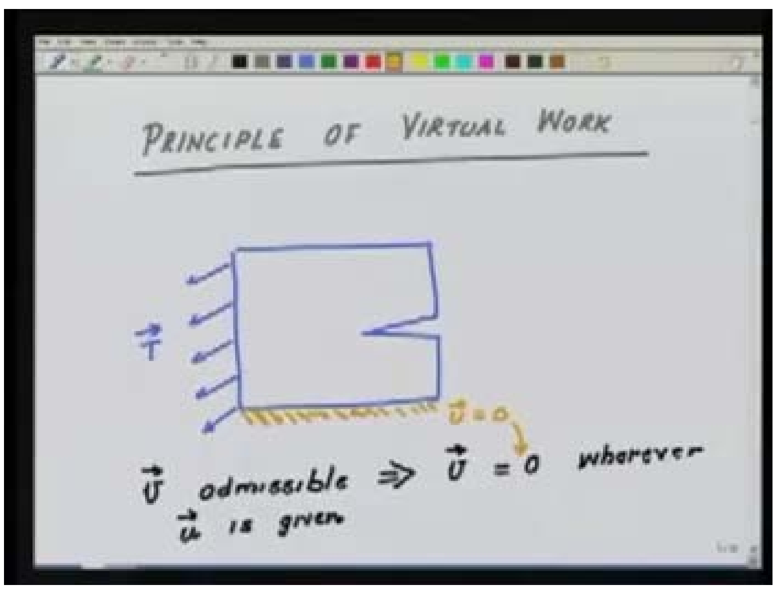

Let us take for simplicity, a simple domain with an edge crack. This domain is loaded by some traction on this face. I could also, at the same time, say that I am going to fix this part. How does this translate to the kind of boundary conditions that are possible for this problem? In the one-dimensional case we had said that our weak formulation or the principle of virtual work formulation is going to give us the effect of the boundary conditions naturally. That is, what kind of boundary conditions can we specify?

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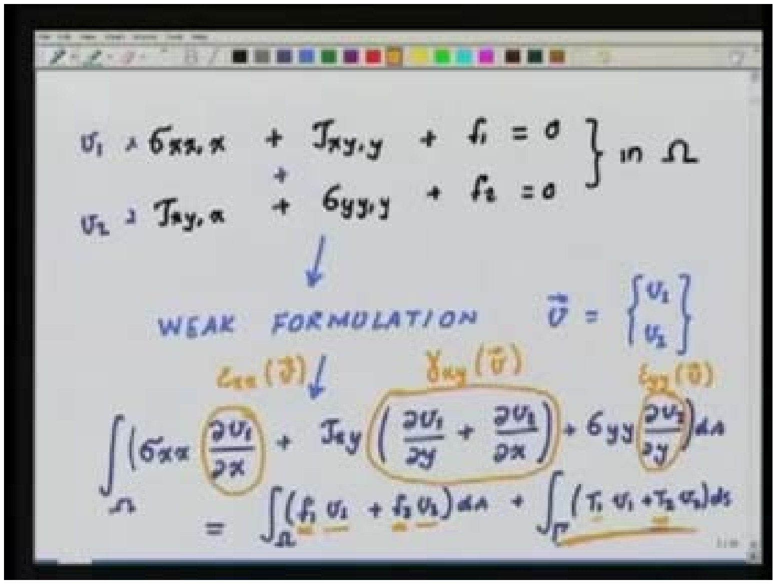

If I look at the second part of the integral on the right hand side, I have T_1 into v_1 plus T_2 into v_2 and v has to be admissible. What does the admissibility of v imply?

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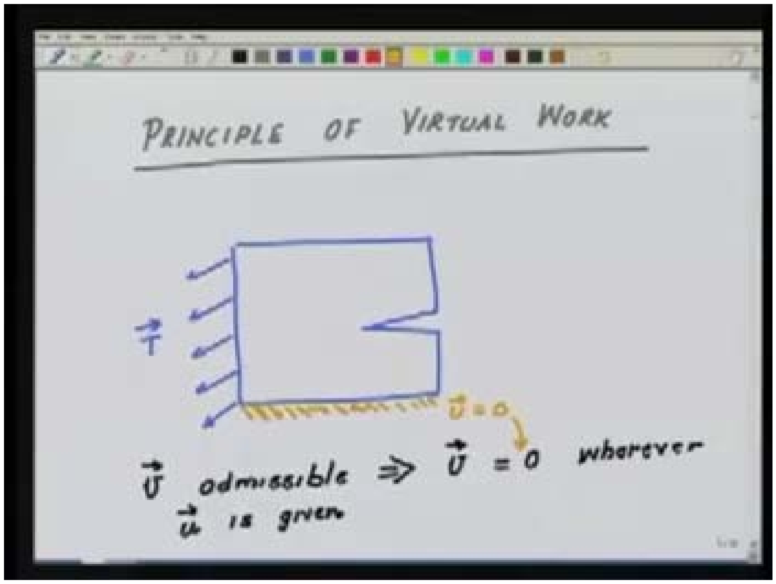

v admissible implies the vector v = 0, wherever the displacement u is given. That is, on the boundary or at points where u is specified, v has to be zero. That is, it is a geometric constraint on v. If I take this part of the boundary, we are saying that the components here, the vector u is equal to zero, that is, both components u_1 and u_2 are zero and this need not be the only boundary condition possible. If that is so, then on this boundary, we want v also zero.

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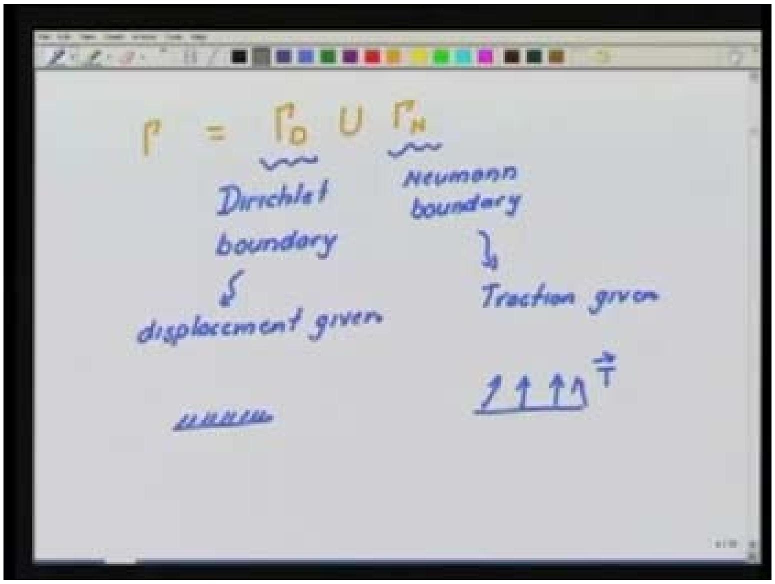

The part of the boundary where displacement is specified is called the dirichlet boundary. This is displacement given. The part of the boundary where the force is given is called a Neumann boundary or the boundary traction is specified. This is traction given. Imagine there is a part of the boundary segment where I have put the displacement equal to zero, this is a dirichlet boundary; and a part of the boundary where I have given the tractions or the distribution of the traction vector is a Neumann boundary.

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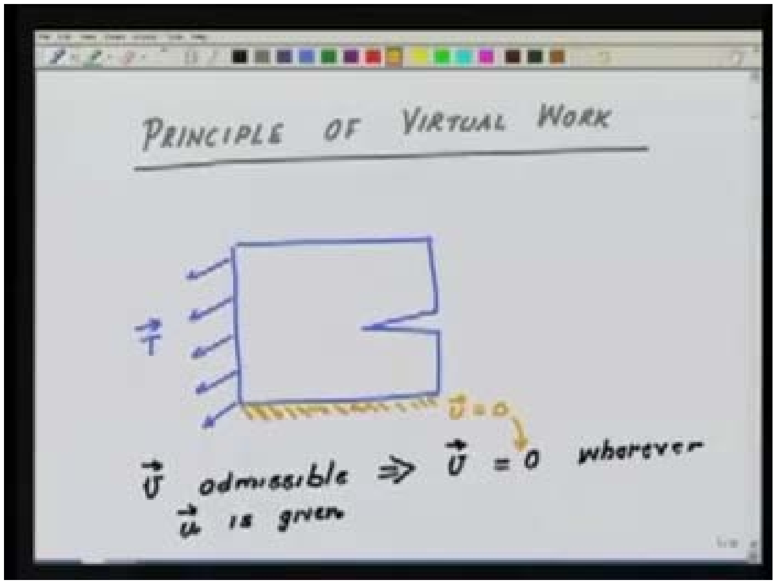

If I go back to a previous figure, we see that, this part of the boundary is a traction boundary. What about this part? If we see this part of the boundary, there is no traction applied. This is a traction free boundary, which means that, in a way here traction is specified and the traction is set to be equal to zero. Neither am I putting geometric constraints, nor am I applying any forces. This is also a Neumann boundary. What about this? This also is a Neumann boundary. These two faces are also Neumann boundaries and this face is also a Neumann boundary. If I look at this figure, this whole part of the boundary is now gamma N and this part of the boundary where I have specified displacement is gamma D. The total boundary can be broken into the part that is a dirichlet boundary and the part that is a Neumann boundary.

We know that at a point on the boundary, we cannot at the same time specify the force in a particular direction as well as the displacement. We specify either the force in a particular direction or we specify the displacement. If we specify the force, then v is unconstrained and if we specify the displacement, v is forced to be zero in that particular direction, at that particular point. There are other possible boundary types.

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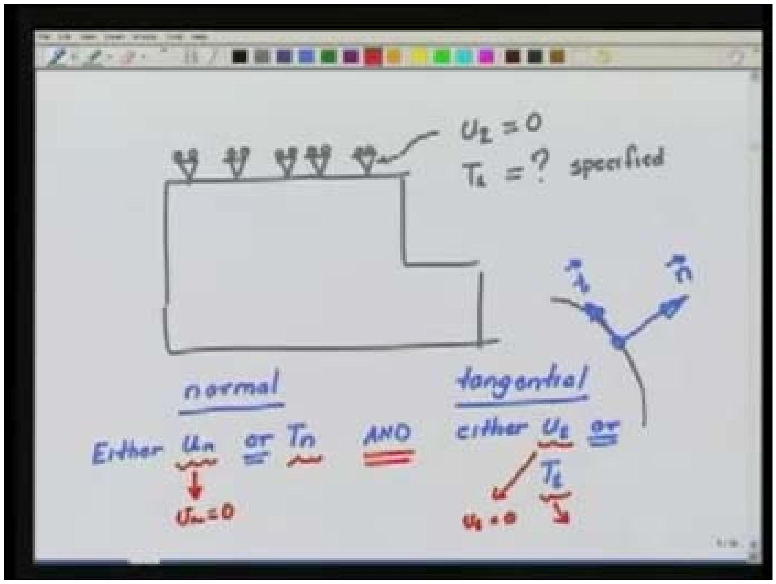

For example, I have roller support on a part of the boundary. As far as this boundary is concerned, (let us say this is a part of a domain, I can make some domain from this edge) we are constraining a displacement in the transverse direction to be equal to zero and as far as the tangential direction is concerned, this member can move in a transverse direction. That is, there is no constraint in the transverse direction. I will have, in the transverse direction, the traction specified. It could be zero or non-zero, depending on what we have. As far as the normal direction is concerned, we have the displacement given and as far as the tangential direction is concerned, we have the traction given. So we can have this kind of a mix of boundary conditions. In general, if we have a curved surface, then on the curved surface we will talk in terms of the normal direction and the tangential direction. This is a tangential direction and this is the normal direction.

Talking of a point here, we either specify the normal displacement or we specify the component of traction in the normal direction and either we specify the displacement in the tangential direction or the traction in the tangential direction. To make it more general, we have either u_n or T_n here AND either u_t or the tangential component of the traction. We see that this AND is very important, we have to give two conditions at this point on this boundary. First condition is, we either specify the normal component of the displacement or the normal component of the traction, and we specify either the tangential component of the displacement or tangential component of the traction. If the normal component is specified it implies v_n here is zero. If this is specified, v_n is unconstrained. If u_t is specified, we mean v_t has to be equal to zero. That is, the tangential component of the virtual displacement has to be zero. If this is specified, then the tangential component of the traction is given and tangential component of v is unconstrained. Thus, we have lots of possibilities of having boundary conditions on the edges.

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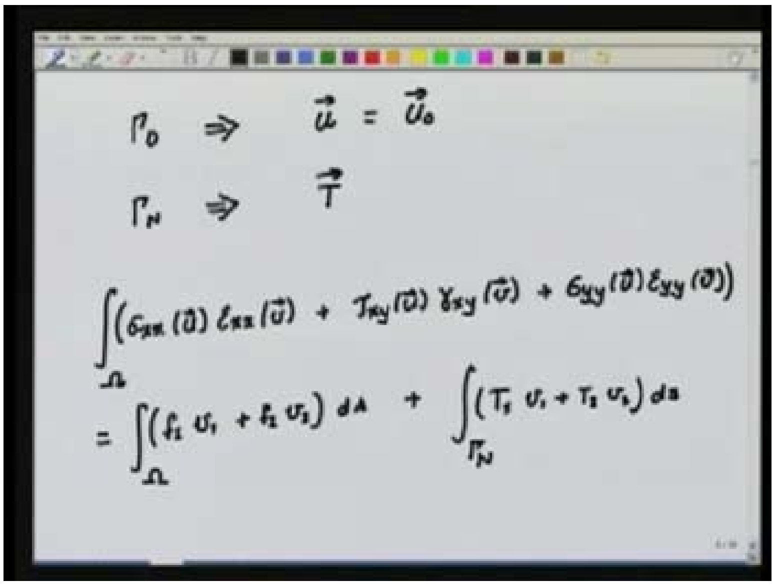

Let us be a little specific and assume gamma d implies u is equal to specified u bar, that is, both components. And gamma n implies both components of traction are given. If I rewrite the weak form, our weak form will now be integral over omega sigma xx with u (due to the actual displacement) into E_{xx} due to v, plus tau xy due to u, into gamma xy due to v plus sigma yy due to u, into E_{yy} due to v, integral over the area. This is equal to integral over omega f_1 v_1 plus f_2 v_2 dA plus integral over gamma n T_1 v_1 plus T_2 v_2 ds. This is because the way we have chosen the specific case, gamma d implies displacement is given. Both components of v are zero and so that part of this integral is knocked off.

We see that carrying this expression is a bit laborious. So what I would like to do is to write it in a more terse form.

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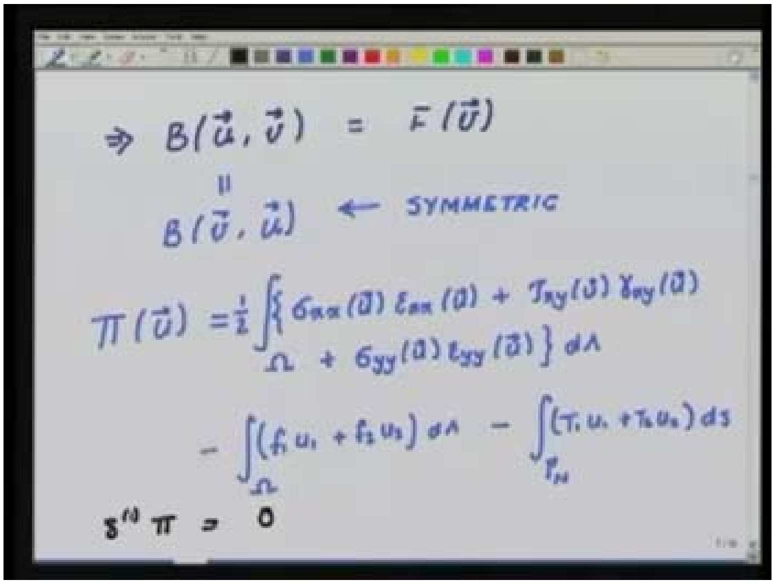

This implies, the bi-linear form for the vector u and v is equal to linear functional due to the vector v.

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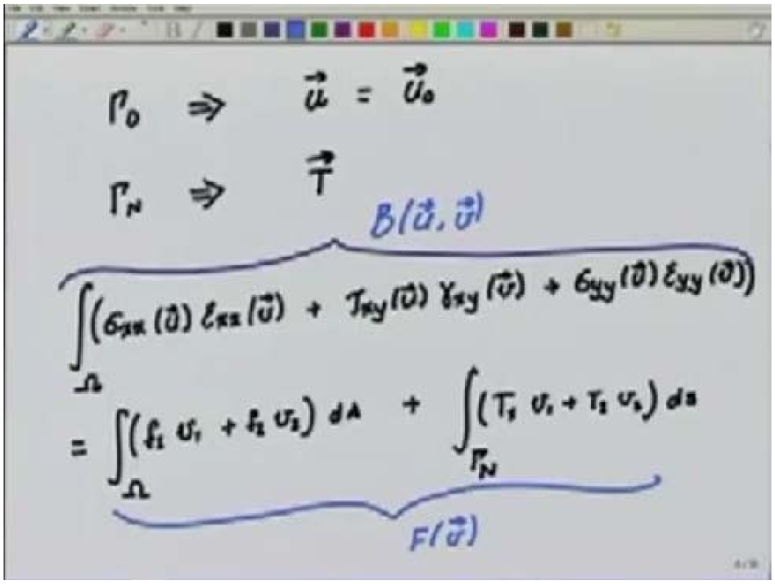

The bi-linear form here is this. This is B due to the vector u, vector v and this is the linear functional F due to v. It is quite easy to show that it is bi-linear because it is linear in each of u and v.

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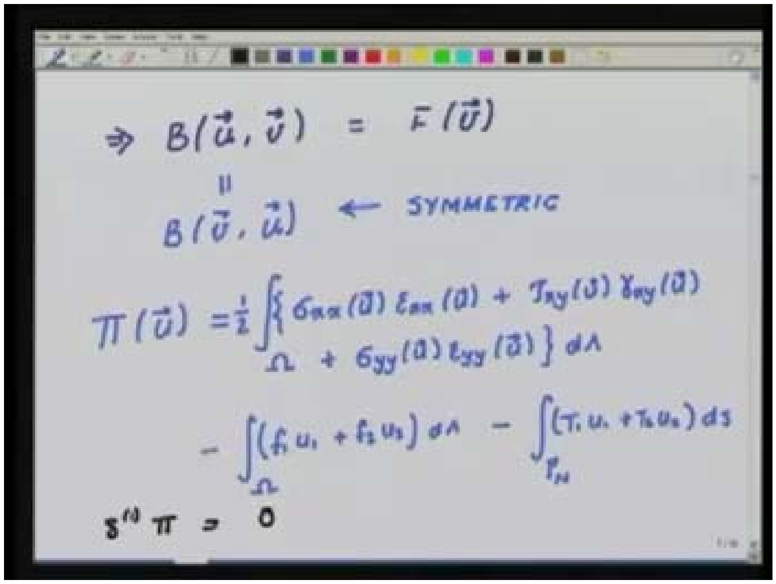

For the model problem that we have taken, it is also easy to show that B (u, v) is equal to B (v, u). That is, this is symmetric. The bi-linear form in this case is symmetric. We will carry on with this notation as far as our construction of various things are concerned. We have our weak formulation. Obviously, one would say that we are used to looking at the total potential energy.

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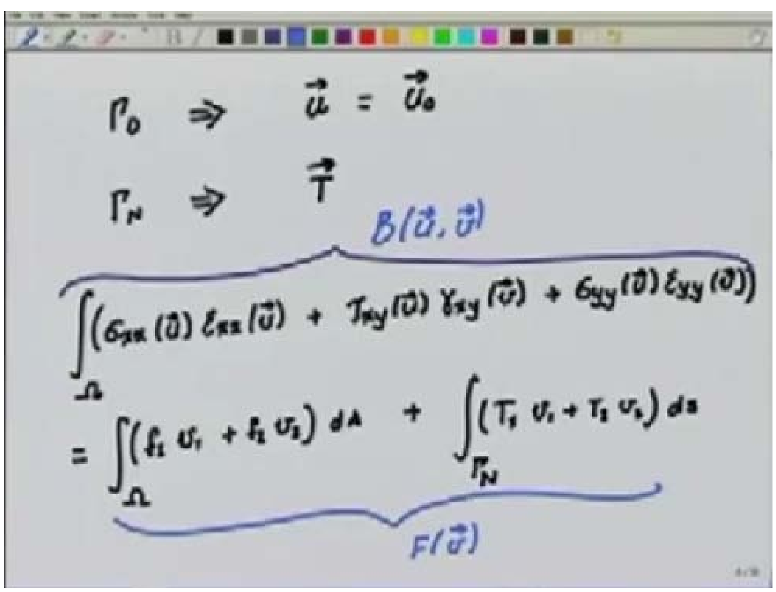

If I have the total potential energy for this problem, what will the total potential energy be?

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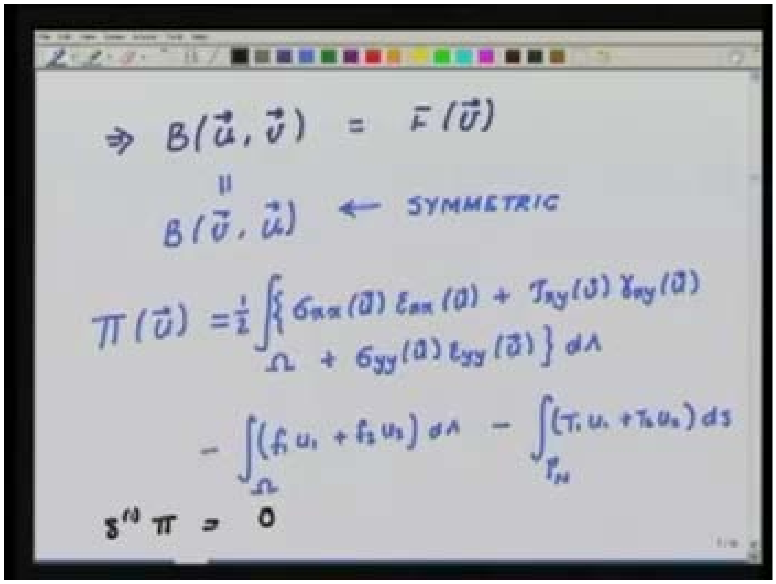

Total potential energy for the problem would be a function of the displacement, this would be equal to the integral over omega, half of: sigma xx due to u E_{xx} due to u plus tau xy due to u into gamma xy due to u plus sigma yy due to u into E_{yy} due to u dA minus integral over omega f_1 u_1 plus f_2 u_2 dA minus integral over gamma N, T_1 u_1 plus T_2 u_2 ds. (And there is a half in front). This is a standard representation of the total potential energy for the system. Is our weak form that we have derived on the principle of virtual work derivable for these problems from the total potential energy? The answer is yes, provided we talk of the linear elasticity problem. It is linear elasticity problem in which the strain displacement relationship is linear and stress strain relationship is also linear. In this case, if I took the first variation of pi as we said for the minimizer of the total potential energy, this is equal to zero.

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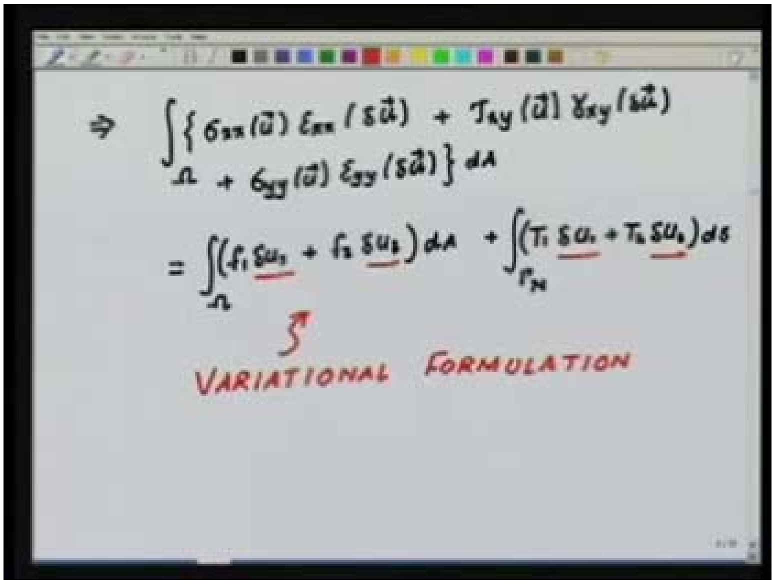

We will see that this implies integral over omega of sigma xx due to u into E_{xx} due to this vector delta u (this is the perturbation or the variation of u), plus tau xy due to v into gamma xy due to delta u plus sigma yy due to u into E_{yy} due to delta u dA. This is equal to integral over omega f_1 delta u_1 plus f_2 delta u_2 dA plus integral over gamma N T_1 delta u_1 plus delta u_2 ds.

If I put delta u is equal to the virtual displacement v, because it could be that delta u is a candidate virtual displacement, because delta u satisfies all the conditions that the v has to satisfy. That is, on the geometric boundary, the boundary on which u is specified, the delta u has to be zero. I see that if I replace this with v, I will get exactly the same formulation that we got using a principle of virtual displacement. This one is called the variational formulation. The variational and the weak form are the same. We have shown that either I come from minimization of the total potential energy or I come from the weighted residual formulation and get to the principle of virtual work. Both ways we end up getting the same weak formulation. We are interested in the weak formulation in order to get our finite element solution for this problem. As we had already talked, this problem is in terms of the two unknown functions u_1 and u_2, the two components of the displacement vector.

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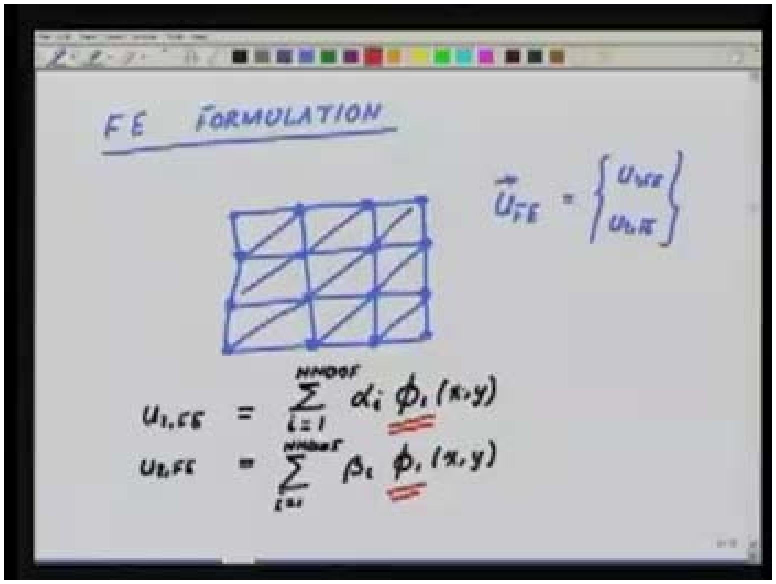

Let us look at the finite element formulation. Till now we did not really bother about the finite element formulation, we were only looking at getting the weak form, which is also true for the exact solution or for the continuation of this. How do we go ahead and do the finite element formulation? As far as the finite element formulation is concerned, we are going to make a mesh of triangles or quadrilaterals in this domain. Let us say I have linear approximations, that is, I put these nodes, then in terms of these nodes and the basis functions corresponding to these nodes, I will define the u_{FE} vector which will be equal to u_{1FE} u_{2FE} where u_{1FE} and u_{2FE} are both equal to sum over one to number of degrees of freedom as many nodes, of $alpha_i$ phi_I, which are a function of x y.

This will be equal to sum over i going from one to NNDOF the same number of degrees of freedom, $beta_i$ phi_i as a function of x and y. I could take higher order approximations also. That is, the basis functions, the definition of which are independent of whether I am talking of a single variable problem or of the two variable problems, that is, either the heat conduction or the elasticity problem. These basis functions are going to remain the same. Why?

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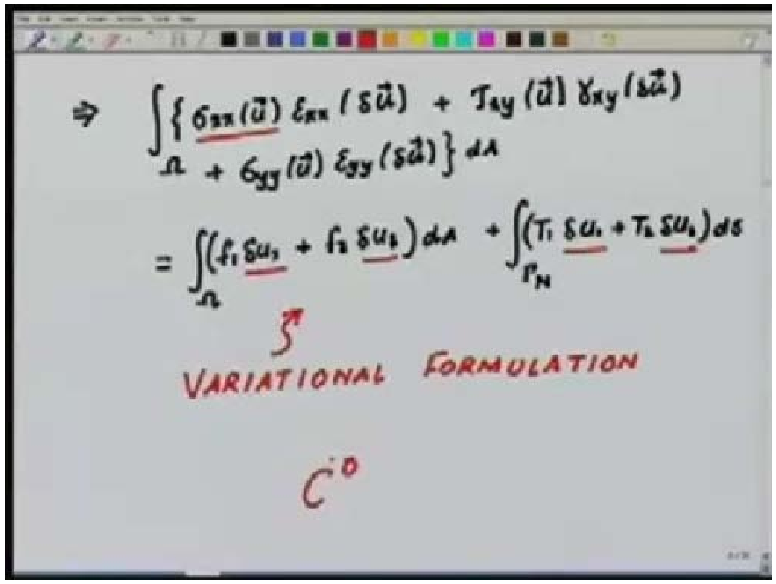

Because, from our weak formulation, all I need is the first derivative of components of u to be defined, that is, del u_1 del x del u_1 del y del u_2 del x del u_2 del y and also for v. For those to be defined, I need C_0 continuity for the basis functions. C_0 continuous basis functions are what we had already created for the heat conduction problem. We are going to use the same basis functions here.

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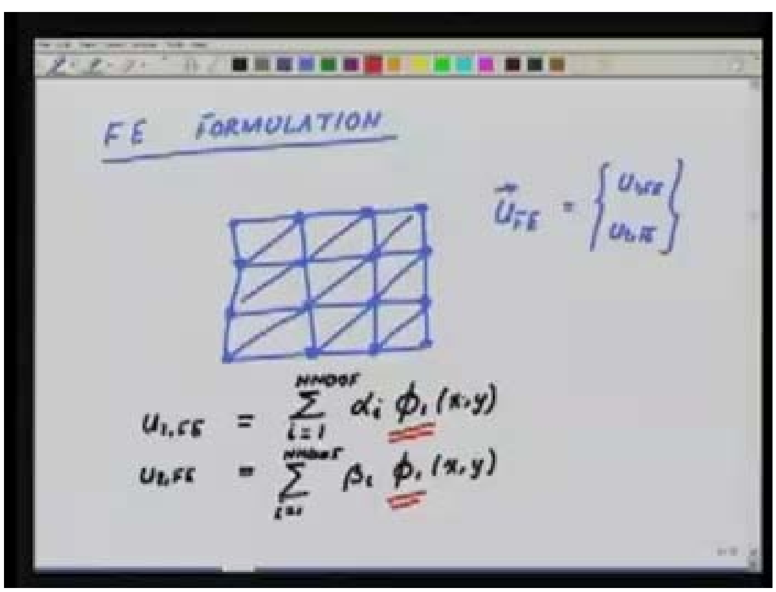

We have these basis functions and likewise, their element wise representation. That is, the element shape functions for each individual element in terms of these basis functions. We write each component of the displacement vector. If we have written it like this, then what can we do next?

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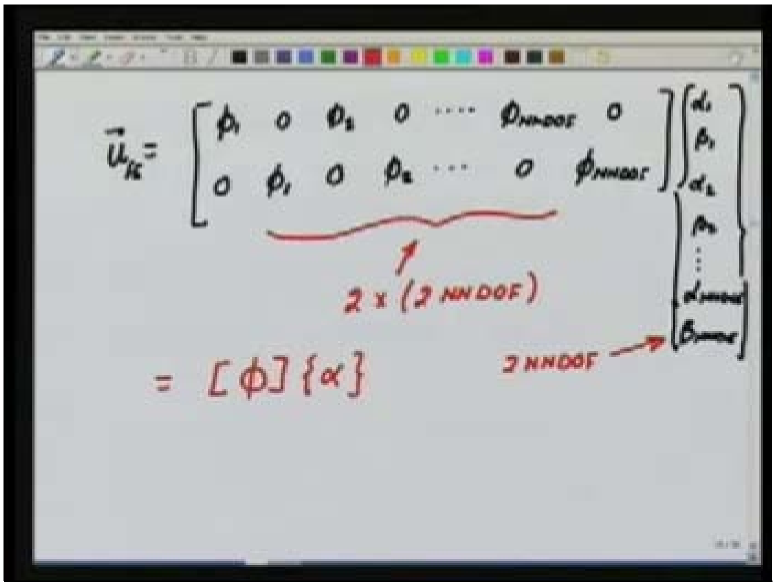

We can write this as vector u is equal to phi_1, zero, phi_2, phi_{NNDOF}, zero, zero, phi_1, zero, phi_{NNDOF}, zero, this into the vector $alpha_1$, $beta_1$, $alpha_2$, $beta_2$ and so on $alpha_{NNDOF}$, $beta_{NNDOF}$. This is how I am going to rewrite this vector and we will see it will be exactly equal to $u_{1FE,}$ being equal to some $alpha_i$ phi_i, u_{2FE} equal to $beta_i$ phi_i. If there is NNDOF number of nodes, we see that the size of this matrix will be two rows into twice NNDOF columns. Number of columns is two into NNDOF and number of rows is 2. This vector will be of size two NNDOF.

This one we will say, is equal to the array phi, by which we understand, each component given by this into vector alpha, where we understand the components of $alpha_1$ $beta_1$ $alpha_2$ $beta_2$ up to $alpha_{NNDOF}$ $beta_{NNDOF}$. If I have this, then what do we have? We can then write the string.

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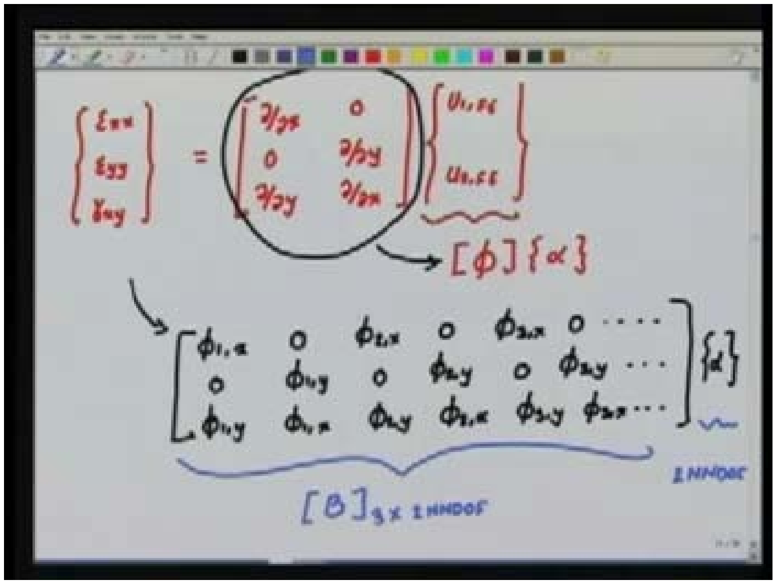

String is going to be equal to the strain vector E_{xx} E_{yy} gamma xy, which is equal to (I am going to write it in a form which is going to be useful to me) del by del x, zero, zero, del by del y, del by del y, del by del x of u_{1FE} and u_{2FE}. This strain, for the finite element solution, can be written as this. This will be equal to del u_1 del x, del u_2 del y, del u_1 del y, plus del u_2 del x - that is exactly the definition that we want.

We are going to write this quantity as phi operating on alpha. Then our strains will be equal to this thing operating on phi into the vector alpha. This operating on phi will be a very interesting form. I will get $phi_{1,x}$, zero, $phi_{2,x}$, zero, $phi_{3,x}$, zero and so on. For E_{yy} I will get zero, $phi_{1,y}$, zero, $phi_{2,y}$, zero, $phi_{3,y}$ and so on. For gamma xy I will get $phi_{1,y}$, $phi_{1,x}$, $phi_{2,y}$, $phi_{2,x}$, $phi_{3,y}$, $phi_{3,x}$, into the vector alpha. The strain can now be written in terms of the derivatives of the basis functions operating on the vector alpha. This is called the matrix B and we see that this is going to be three rows into two NNDOFs columns. This vector is two NNDOFs columns. So this will give me a vector, which is three by one.

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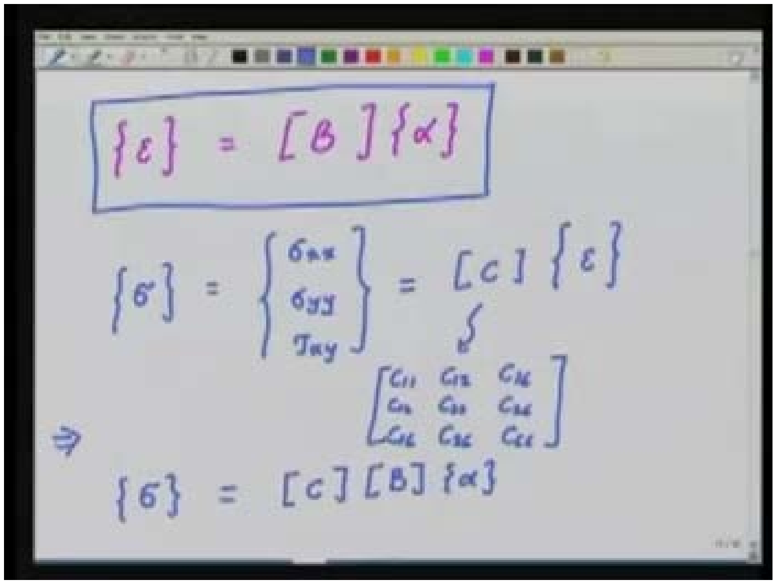

We can, in a simplified way, write the strain vector is equal to the B array operating on the vector alpha. I am not trying to break it up into the element wise contributions and so on. I am simply trying to look at it in terms of what happens with respect to the global basis functions. We have to keep in mind that what we are approximating is always in terms of this globally defined function. How we do it in terms of implementation on the computer comes later and we are in the process of building up the case for making it easily implementable in a computer program. I have this definition of the strain in terms of the derivatives of the basis functions operating on this vector alpha.

If I have this, then how can I write the stress vector in the engineering sense? This is sigma_{xx}, sigma_{yy}, tau_{xy}. This will be equal to the C matrix operating on their size and vector. So we have the stress which we have written as a stress vector, given in terms of the material matrix C, operating on the strain vector. We have already given the components of C. That is, this will be having components C_{11} C_{12} C_{16} C_{12} C_{22} C_{26} C_{16} C_{26} $C_{66.}$ It will have these components. In specific cases like for isotropic elasticity, this will be much simpler. That is, it will have only C_{11} C_{12} C_{22} $C_{66.}$ C_{16} and C_{26} will be zero. We see that C is also a symmetric matrix.

Once we have this, then I can write this stress for the finite element solution, in terms of the representation of the finite element strain, in terms of the basis functions, which is easy. This implies sigma is equal to [C] [B] operating on alpha. The finite element stress at a point can be written in terms of C operating on B, which operates on alpha.

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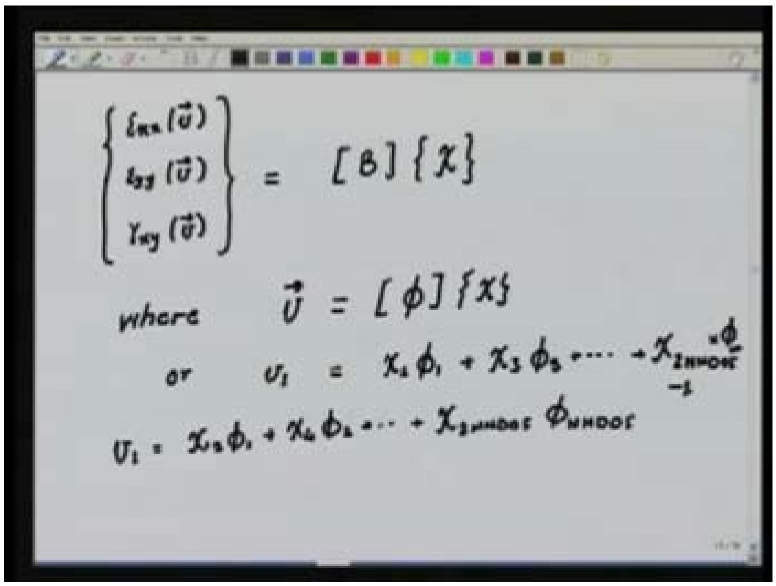

Similarly, we can write the strain due to the weak. We have the E_{xx} due to the virtual displacement v, E_{yy} due to the virtual displacement v, $gamma_{xy}$ due to the virtual displacement v. How can we write this one? This can by written in many ways, let us do it in a in one particular way, that this is going to be equal to B operating on some other vector chi, where (remember that v has the same form as u), the vector v is equal to phi operating on chi or v_1 is equal to chi_1 phi_1 plus chi_3 phi_3 and so on plus chi_2 NNDOF or minus one into phi_{NNDOF}. V_2 is equal to chi_2 phi_1 plus chi_4 phi_2 plus up to chi_{2NNDOF} phi_{NNDOF}. V_1 and V_2 could be chosen as this. How do we choose this virtual displacement vector v, such that, we get two NNDOF linearly independent equations in terms of the two NNDOF unknowns alpha.

Our question is, how we choose the v such that we get two NNDOF numbers of equations in terms of the two NNDOF number of unknown coefficients alpha, given by the vector we have written here. How do we do that?

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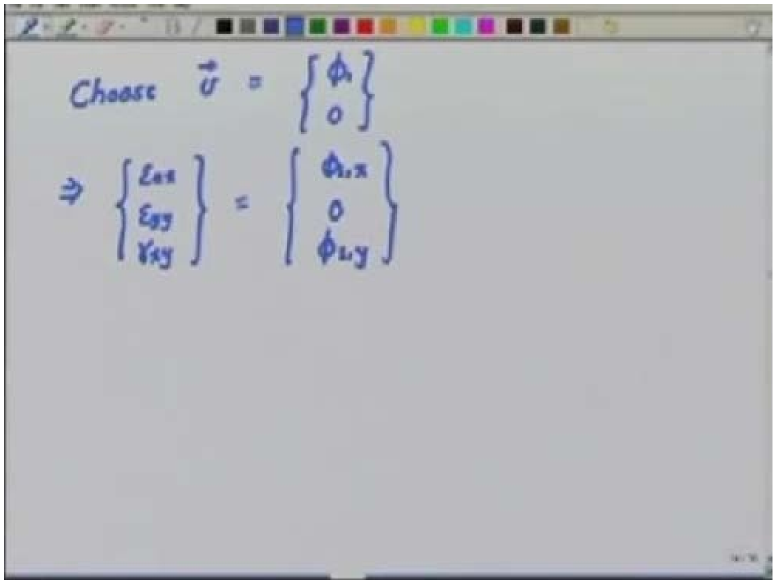

We will say that choose v equal to phi_1 zero. If I choose v equal to phi_1 zero, this implies strain due to v will be equal to $phi_{1,x}$, zero, $phi_{1,y}$. I choose this strain due to the v in the virtual work formulation or the weak formulation that we have obtained to get our first equation corresponding to the basis function phi_1.