Finite Element Method
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Module – 09 Lecture-1

Introduction

In this lecture, we are going to talk about other problems in two dimensions where we will have either a single variable problem or a multi variable problem. Specifically, here we are going to talk of something called the axisymmetric problem.

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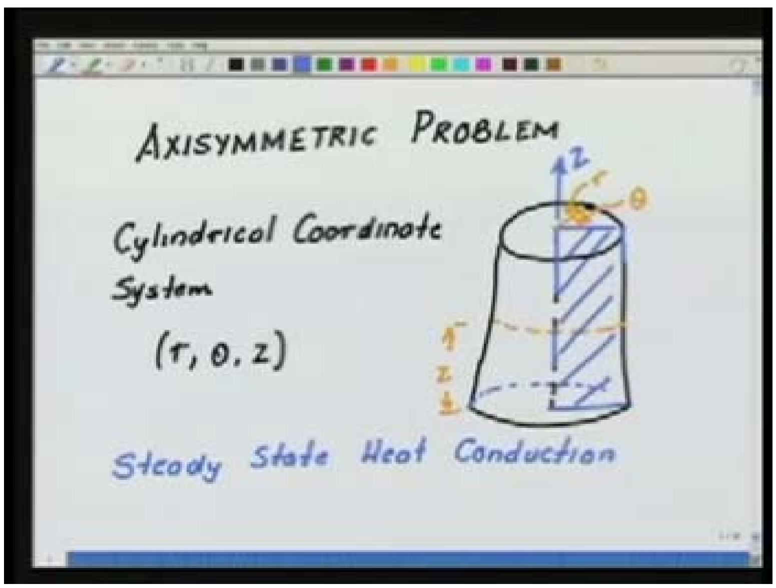

This problem is essentially concerned with a domain of a very special type; a domain which is obtained by revolving a plane about a particular axis. You have a domain of revolution; so we can think of this as formed by revolving this plane about the z-axis.

For such a domain, we are going to use a cylindrical coordinate system, which is a departure from the coordinate systems that we have been using till now, which was the Cartesian coordinate system. In the cylindrical coordinate system, we are going to write the variables in terms of the radius vector r, the angle theta and z; where radius vector r, this is my r for any point and this is the angle theta. (Refer Slide Time: 02:40 min).

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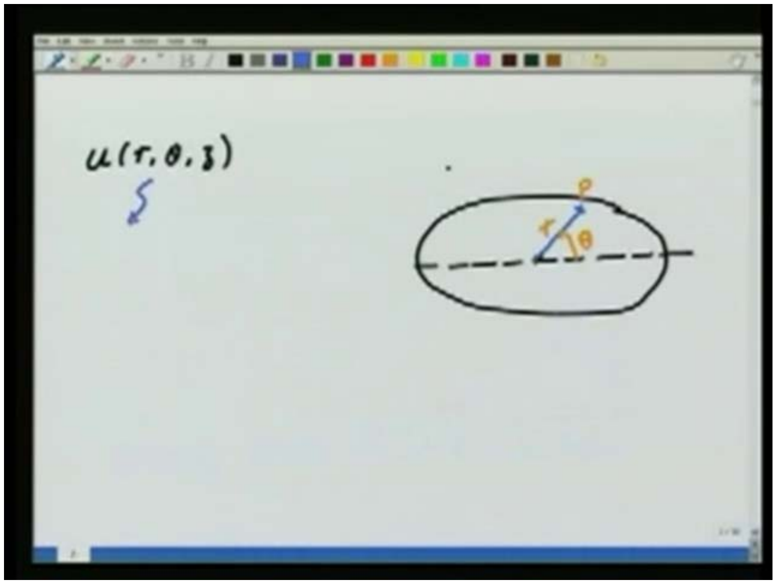

In a more expanded form, if I take a section out, take a point here, let us say this is my theta equal to 0 line. Then with respect to this line, this is the angle theta and for this point p this is the radius r. Depending on where I am with respect to the z, that is where I decide to cut, the height z from the bottom, that gives me the z coordinate. (Refer Slide Time: 03:35 min) This is the coordinate system that we are going to use and we will have a variable let us say a function u, which is now a function of r, theta and z.

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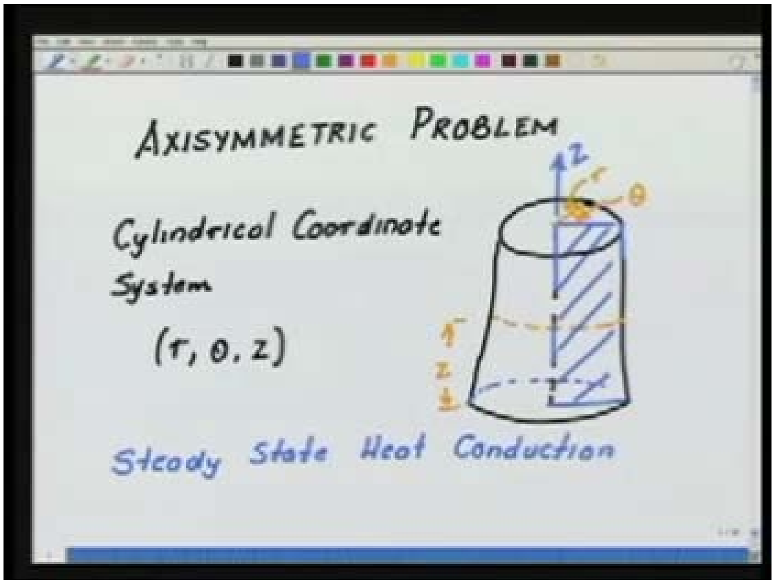

A particular problem which can be posed over this domain is a problem of steady state heat conduction.

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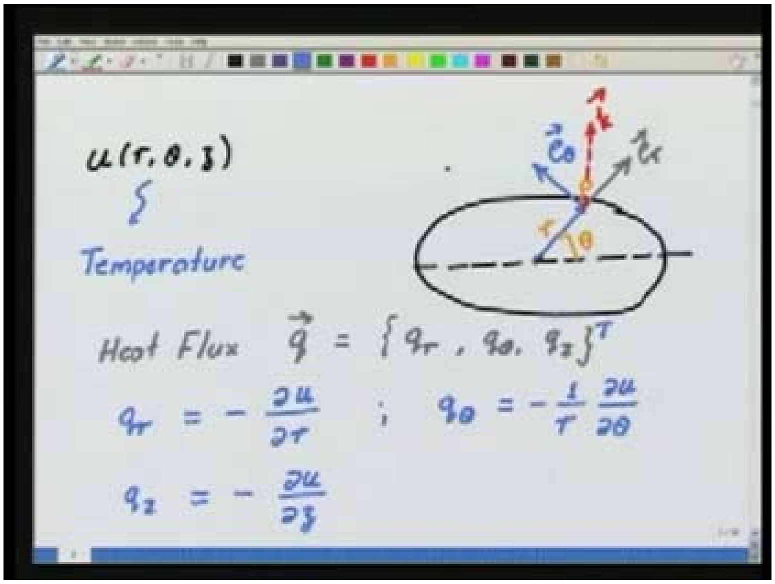

In this problem, the variable u is the temperature. So if I write the conservation law for the heat conduction problem in terms of the cylindrical coordinate system, then I will have to first define the heat flux. A heat flux is a vector q defined in this coordinate system which has components in the r direction, in the theta direction and in the z direction; where this (Refer Slide Time: 05:15 min) is my unit vector in the r direction, this is the unit vector in the theta direction and the one going upwards is a unit vector k in the z direction.

I have an orthogonal coordinate system here also with e_r, e_{theta} and k as the unit vectors and q_r, q_{theta}, q_z are the three components of the flux vector in these three orthogonal directions. Once I have this coordinate system, now we have the constitutive relationship for the flux quantities. So this transpose is will have q_r is equal to minus del u del r. Why this minus? Because, as we know, the heat will flow from a region of higher temperature to a region of low temperature. So it goes from the negative gradient direction. Similarly, q_{theta} is equal to minus 1 by r del u del theta and q of z is equal to minus del u del z. These are our components of the heat flux vector.

Given the components of the heat flux vector, I can define the conservation law.

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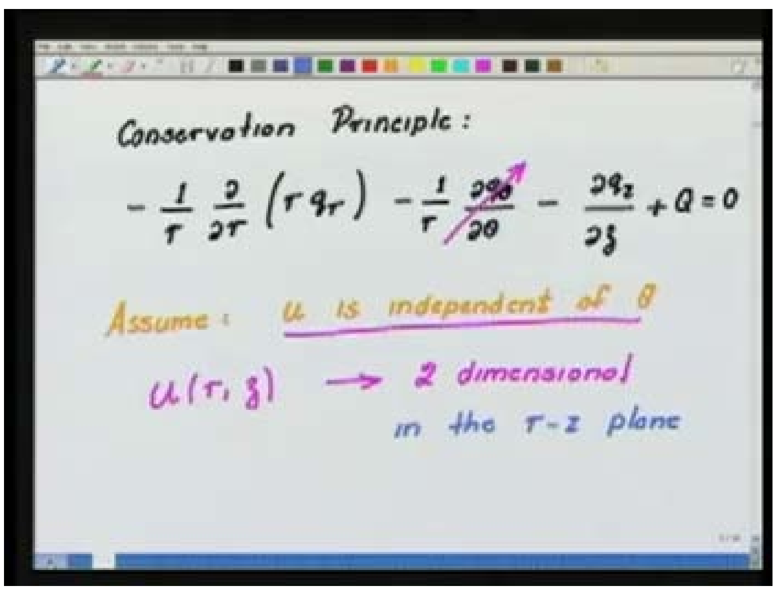

Essentially, it is a conservation of energy or the heat conduction problem. Conservation principle given as minus 1 by r del del r of r q_r minus 1 by r del q_{theta} del theta minus del q_z del z plus some source term is equal to 0. So, this is the differential equation governing the heat balance; the heat flux again plus the part which is developed by a source term is equal to 0; the total part.

Given this differential equation, now you see that this differential equation looks significantly different from the kind of differential equations that we had been dealing with; because, this 1 by r(s) are sitting there.

Let us now make a further simplification. We assume that u is independent of theta; that is, u does not depend on the angle of orientation theta. Similarly, the q also becomes independent of theta. All the derived quantities out of the temperature become independent of theta, which means, that in this expression I can knock of this quantity. (Refer Slide Time: 09:16 min)

So, I am left with the differential equation minus 1 by r del del r of r q_r minus del q_z del z plus q equal to 0. By making this assumption, which is generally, actually thought of as the axisymmetric problem, I have now reduced this problem in terms of the two variables r and z. This becomes a two-dimensional problem in the rz plane; that is, I come back to this plane that I have here. I am solving the problem over this plane, any arbitrary plane that I take.

Now the question is how do we go about solving this problem using the finite element method? In order to obtain the solution to this problem, the first step is to obtain the weak formulation. How do we obtain the weak formulation?

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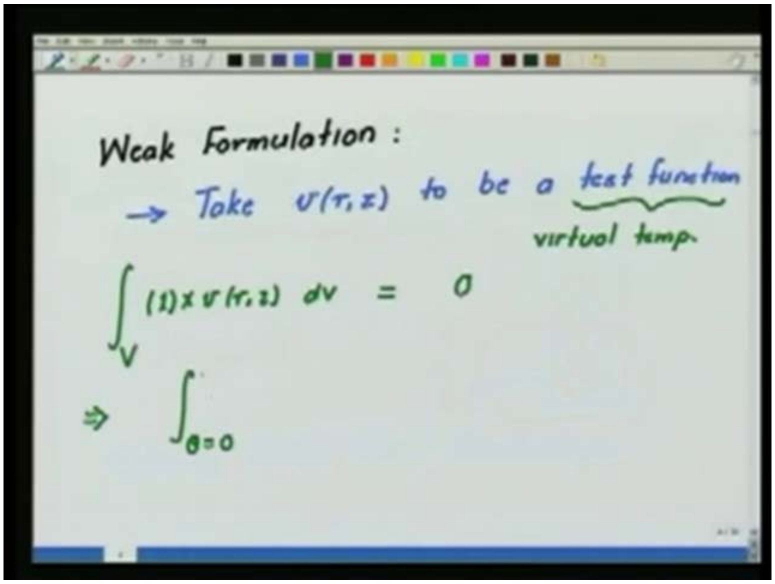

For the weak formulation, what do we do? We do exactly the same thing that we had been doing till now; that is, take v as a function of r and z to be a test function or you can think of it as a virtual temperature function; multiply the differential equation that we got in the previous page; multiply this differential equation (Refer Slide Time: 11:38 min); let me call it by the number 1. Multiply 1 by v and integrate over the area; over the whole volume actually, because, that is my body over which I have to solve the differential equation.

I take now 1 into v, which is a function of r and z equation 1, integrate over the volume of this solid of revolution, dv this has to be equal to 0. When I integrate over the volume, what do I get? Now how do I define dv? dv will be obtained as a variation of theta from 0 to 2pi; if I come back here, theta will go from 0 to 2pi.

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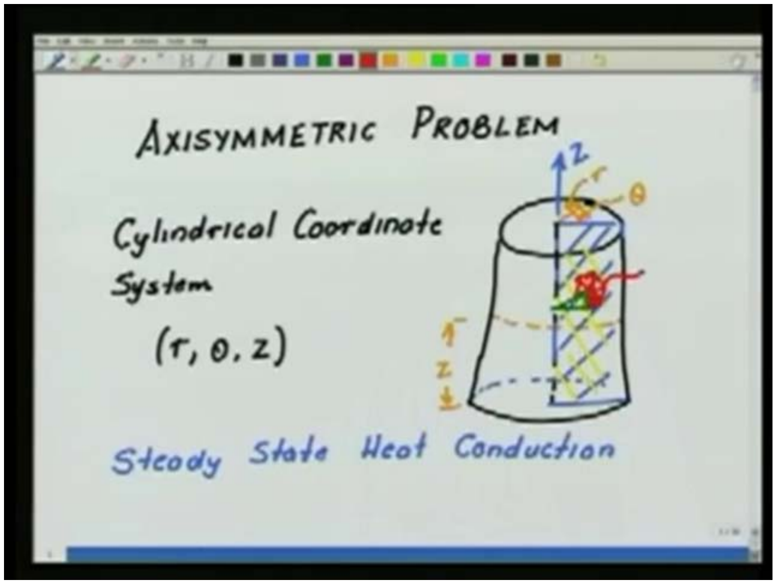

So dv will be in terms of, p is here multiplied by dz. This is going to be our dv. So, dv will be actually, if you see the size of this edge is rd theta, the size of this one is dr and size of this is dz. So r d theta dr dz, this is my dv.

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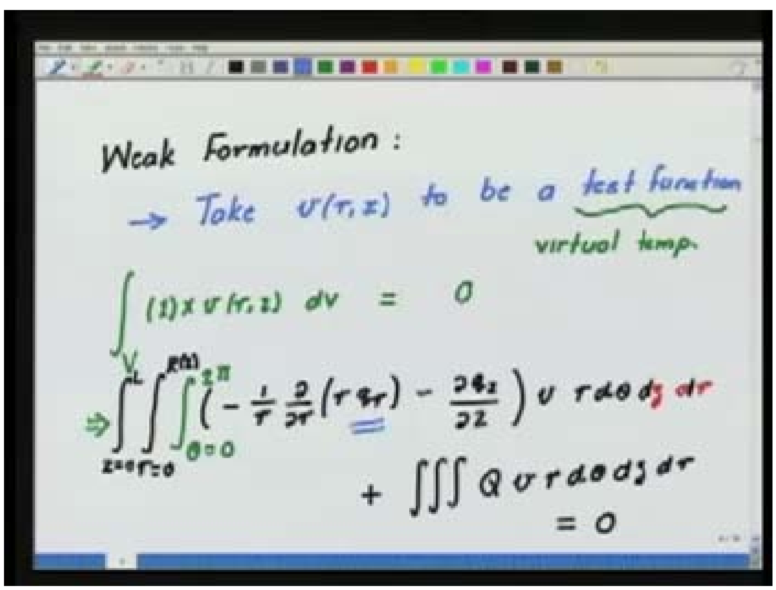

What I am going to do is I am going to take one which is minus of 1 by r del del r of r q_r minus, I have knocked off the theta term del q_z del z, whole thing into v. Now dv was rd theta dr dz. So then, I have the integral for theta going from 0 to 2pi.

Similarly, my r will go from 0 to big R of z and z will go from 0 to L. So I come back to my original figure (Refer Slide Time: 14:24) and specify this is z equal to 0 line and this is z equal to L line. For any location, this is my outer radius r which is a function of the z, depending on the profile of this solid of revolution.

This is quite easy; once I have this, now what do I do? Actually, in this integral I should switch that is ok; whatever I have done is ok. First, I should be integrating with respect to dr and then with respect to dz. (Refer Slide Time: 16:17 min) We have this integral, plus I have something more which I have to add, plus the same integral Q v r d theta dz dr this is equal to 0.

What was the next step that we did? The first step was to weight our differential equation by a test function and integrate it over the whole volume of the body of interest. The next step was to do an integration by parts. Why do we have to do that? Because, you see that here, q_r involves the first derivative of u with respect to r; similarly, q_z involves the first derivative of u with respect to z.

So del q_r, del r and del q_z, del z involves second derivative of u with respect to r and z, while v sitting as such; so, we would like to weaken the smoothness requirement on the u by doing an integration by parts, by which we shift one of the derivatives from u to v. So what do we do next?

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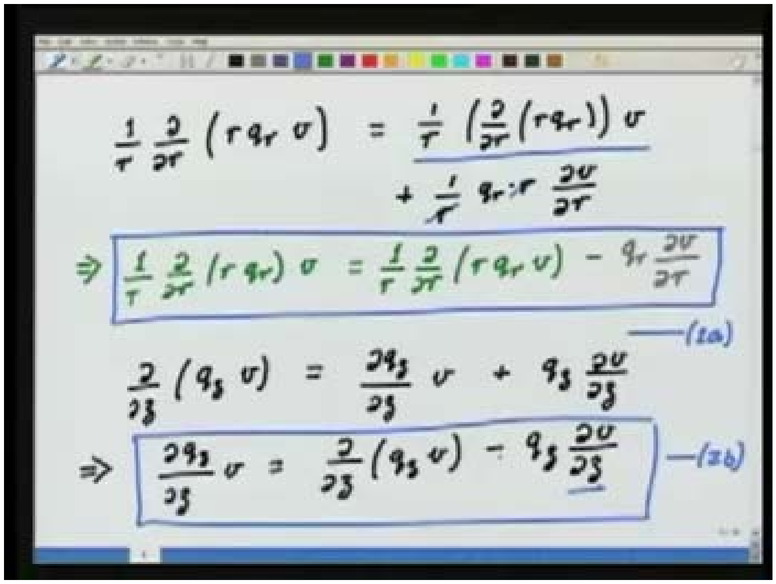

Next is we see that 1 by r del del r of r q_r v is equal to 1 by r del del r of r q_r into v plus 1 by r q_r into r del v by del r. This is what we had; if I go back, our expression was in terms of this quantity: 1 by r del del r of r q_r.

Now, we would like to replace this by this quantity and we will see why we want to do it. This implies 1 by r del del r of r q_r into v is equal to 1 by r del del r of r q_r v minus, here 1 by r will cancel so minus, I will have q_r del v del r. This is our first substitution that we are going to do; this I will call it as 2a.

Similarly, let us go back to our next expression this one: del q_z del z into v. This one we will say that del del z, this we have already done many times before q_z v is equal to del q_z del z into v plus q_z del v del z; which implies that del q_z del z into v is equal to del del z of q_z into v minus q_z del v del z; this is my second expression 2b.

You see what we have done. We have transferred through this the derivative from q_z to v the second expression. Now, what do we have when we go back and substitute this in our differential equation, in our weighted residual form that we have written, what we will get is minus integral, I will rearrange this properly; theta, this is our z, this is our r.

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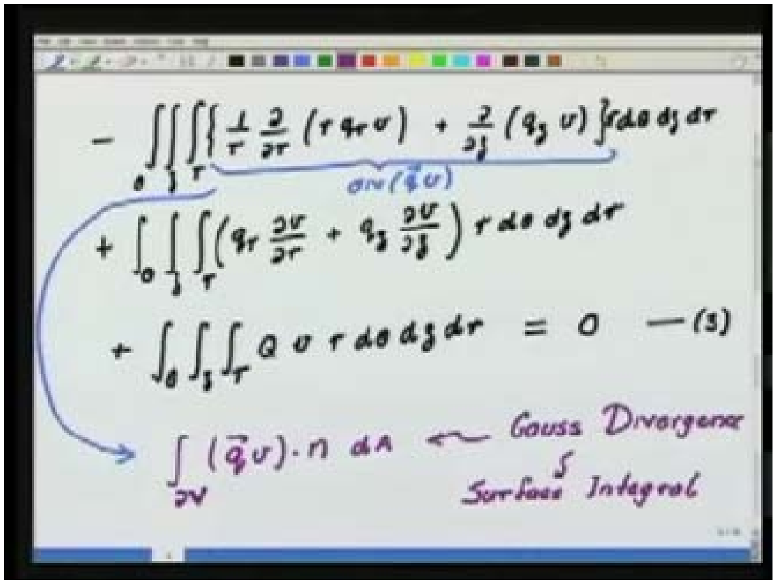

We will get 1 by r del del r of r q_r v plus del del z of q_z v; we will say this is d theta dz dr plus I will get integral over theta, integral over the z, integral over the r because minus, minus has become plus here q_r del v del r; it will be into r d theta dz q_r del v del r plus q_z del v del z whole thing into r d theta dz dr plus integral over theta, integral over z integral over r Q v r d theta dz dr is equal to 0. Who ever can, check if we have made any mistakes; this was a weak form; a weighted residual form; here we have substituted the next part 2a and 2b to get the form that you have written here; this was my expression; let us call it 3.

Now, this quantity here (Refer Slide Time: 22:56 min) is nothing but divergence of q into v; so divergence of vector q into v. Now the Gauss divergence theorem will give me this part, can be have written as integral over the surface delta v, outer surface of the body of qv dotted with n into dA. This becomes now by Gauss divergence gives us a surface integral.

So these are essentially the building blocks and see that barring this a little bit of complication handling the r and the z derivative, the whole approach is the same as what we have been doing till now.

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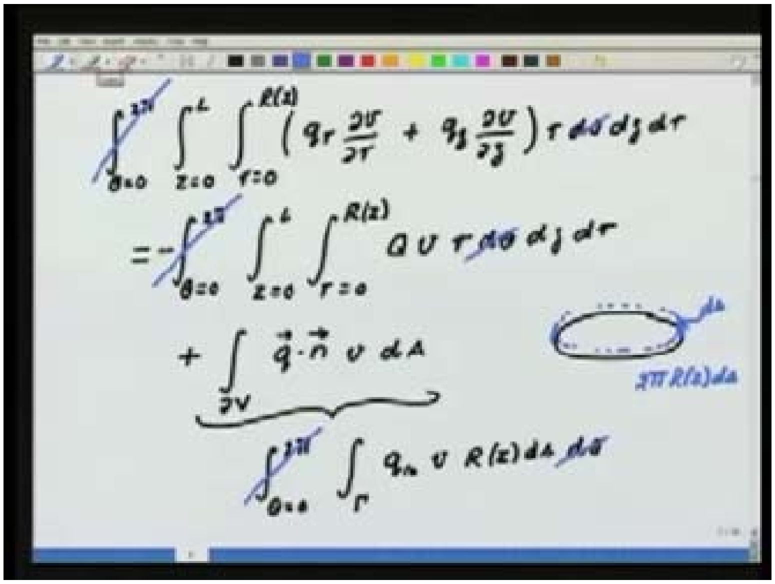

Then we will rewrite this thing as integral of theta going from 0 to 2pi, integral z going from 0 to L, integral r going from 0 to outer radius r of z of q_r del v del r plus q_z del v del z into r d theta dz dr is equal to integral theta going from 0 to 2pi, integral z going from 0 to L, integral r going from 0 to r of z of Q v r d theta dz dr. I will put minus sign here. From a previous expression, I will get a plus integral over delta v; I will get vector q dotted with outer normal vector n into v dA.

Now what do we mean by the outer normal n? If you see the surface, outer normal will be the unit normal on the outward face. It will be like unit normal on this outward face of the given surface. Here it will be something along this face; here it will be actually in the z direction if it is a flat cut and so on.

Now what? Now you see that here, if I look at this expression, this is an integral over the surface. How can I write it? I can write it as this is this integral is equivalent to theta going from 0 to 2pi integral over the outer surface s; I will say outer contour gamma; we will define what this gamma is: q_n v into R at that particular z ds d theta.

What we mean is that if I take this outer surface (Refer Slide Time: 27:49 min), I take a thin strip around this outer surface, this strip this is dimension ds. So, the total area is ds into the length of the perimeter of this circle. Perimeter of the circle will be nothing but integral of theta from 0 to 2pi or we can say 2pi or at that particular z into ds; this is the area. So that is what we are writing. You see from here, this part can be easily knocked off, the integral over theta could be eliminated out of this. So what we are left with finally is, if I am allow to erase here I will be left with this expression (Refer Slide Time: 29:11).

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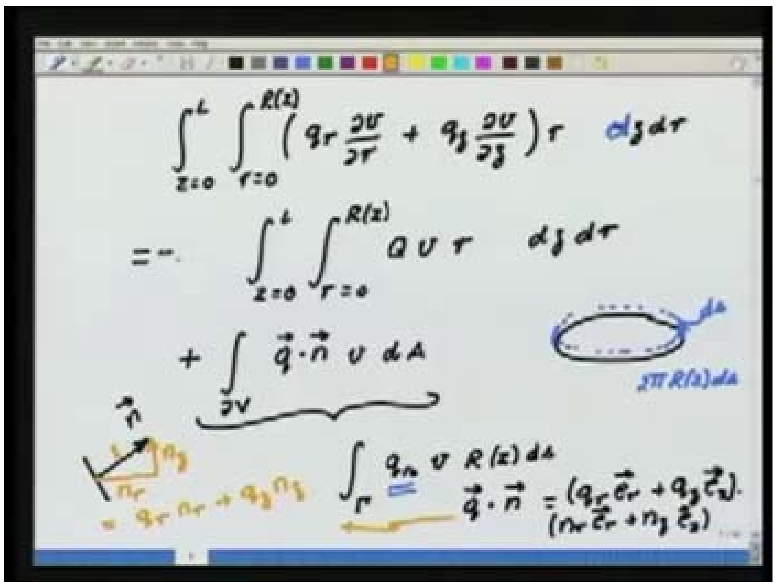

This is the expression that we have as far as this particular problem is concerned and you see now that we are only playing with z and r being our variables of interest. So use a function of z and r; how did I get this q_n? q_n is actually let me do it from first principles q dotted with n this is equal to q_r e_r plus q_z e_z dotted with n_r e_r plus n_z e_z; that is, the components, if I take this face, the components of the normal, this is my normal n, its component in the r direction and its component in the z direction; this is n_z; this is n_r and this normal has size 1.

This dot will give me, this is equal to q_r n_r plus q_z n_z; e_z is nothing but k; so this is how we define q.

Now the question is what is the given data and what is not the given data? First of all, what do we mean by this gamma?

If I go back to this figure here that we have (Refer Slide Time: 31:03) simply [poach] this figure.

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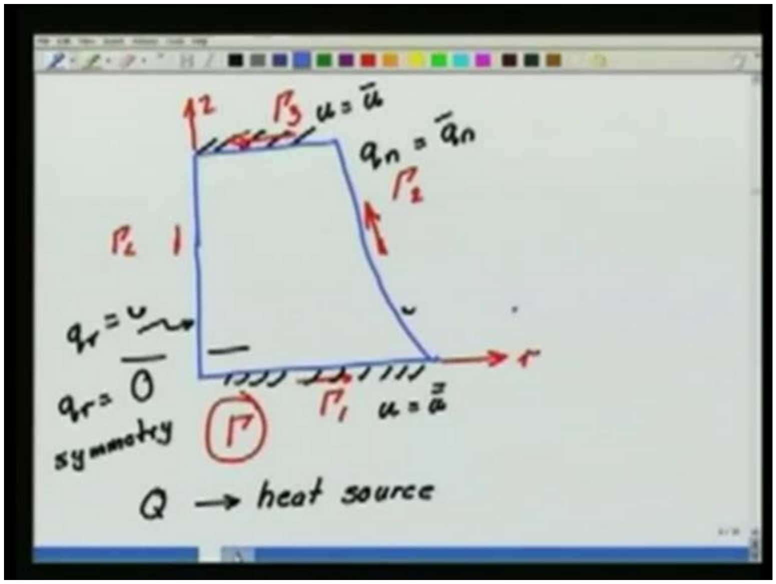

Now say, that since this problem is independent of theta, I take that plane, that particular plane like this, I may have the surface like this; this is the line, I will say that this is my z-axis and I can say this is my r-axis.

Here, if I take this plane, this surface (Refer Slide Time: 32:01 min) is the outer surface gamma; this I can call as gamma$_1$; this I can call as gamma$_2$; this I can call as gamma$_3$ and this one I call as gamma$_4$.

I have these now in the rz plane, the outer surfaces of this rz plane, not the outer surfaces now this is contour; the outer contour of this surface is given by gamma$_1$, gamma$_2$, gamma$_3$, gamma$_4$ or the total is 1 [on this] gamma.

Now the question is what are the knowns? The heat source term is the known and on part of the boundary, I may be given what is the heat fluxing in to the body or out, from that particular boundary or on part of the boundary I may be given the temperature. (Refer Slide Time: 34:48)

For example, let us say here on this boundary, I want to say that my u is a known, u bar; on this boundary I want to say that q_n is equal to q_n bar which is a known; this could be coming out of conduction, convection or radiation boundary conditions. On this boundary, again, I can say that u is equal to u double bar. On this boundary, for this particular problem, you see that because the solution is independent of theta, so from the solution on this side of the r equal to 0 line and on this side are the same.

So as far as taking the full plane, if I took the full cutting plane here, this whole plane (Refer Slide Time: 34:05 min), then the solution would be symmetry with respect to r. Since the solution will be symmetry with respect to r, the derivative with respect to r at this point will be 0. So along this surface I will have the q_r is equal to 0. So on this surface q_r is equal to 0. Now, once I have q_r is equal to 0 on this surface and this is from symmetry I have the boundary conditions given on each of the four edges of this particular two-d domain in the rz plane.

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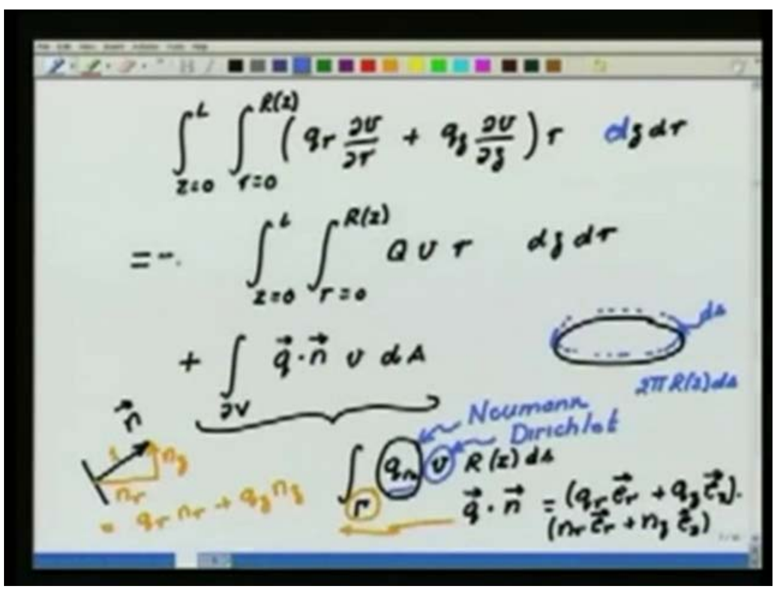

So if I have this boundary conditions given; so let's go back to our weak form and you see here that which is our force type of boundary condition or the natural boundary condition or the Neumann boundary condition, it is q_n.

If the normal flux is given on the edge, we call it a Neumann condition. Similarly, if the temperature is given on the edge, then that edge has so-called essential boundary condition or the Dirichlet boundary condition.

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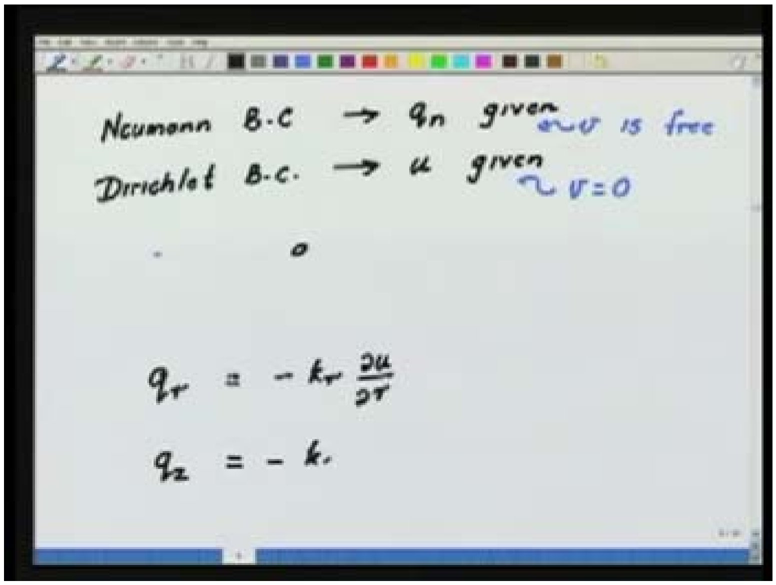

So, on the edge we will have here that I can have Neumann boundary condition q_n given. So, on an edge where the normal heat flux is given, I cannot specify the temperature; its either or. A Dirichlet edge is one on which u is given; on the Neumann edge, v is free; v is allowed to be anything; on the Dirichlet edge v is equal to 0.

You see that everything is exactly the same as what we have been developing till now; only thing is now we are talking of a problem in a cylindrical coordinate system. Once we have this, then essentially, we have formed the full weak form of this particular problem. Now we go and substitute for q, our q_r is equal to minus k_r del u del r; qz is equal to minus k_z del u del z. I go and substitute it back in my expression here and I go ahead and solve the problem. This is what we have to do as far as solving this problem is concerned.

Now as far as the mesh generation, what is the mesh that I am going to form? I am going to now, make a mesh over this rz plane.

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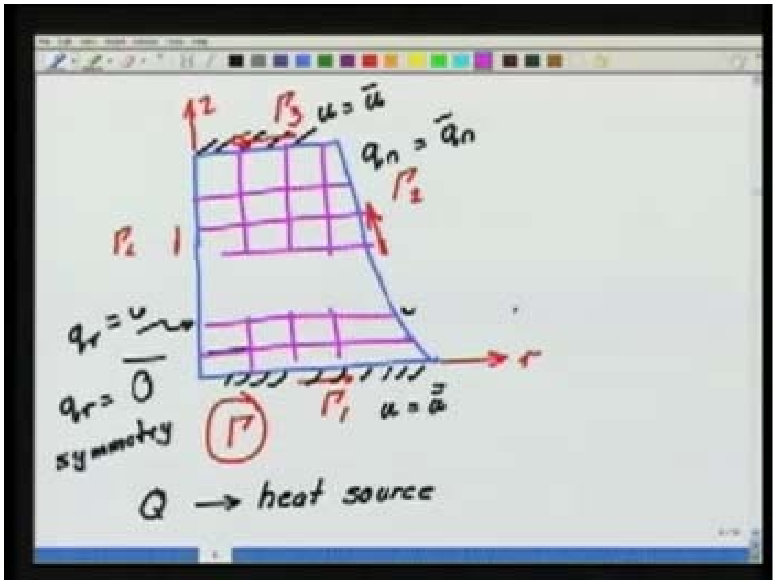

So I could make a mesh of any type: it could be a mesh of squares or triangles or quadrilaterals, whatever, I wish; that is my choice. I will make my mesh, solve the problem exactly the way that we have being doing till now and I have my solution to the axisymmetric problem. So we substitute this and we solve this particular problem and that is it.

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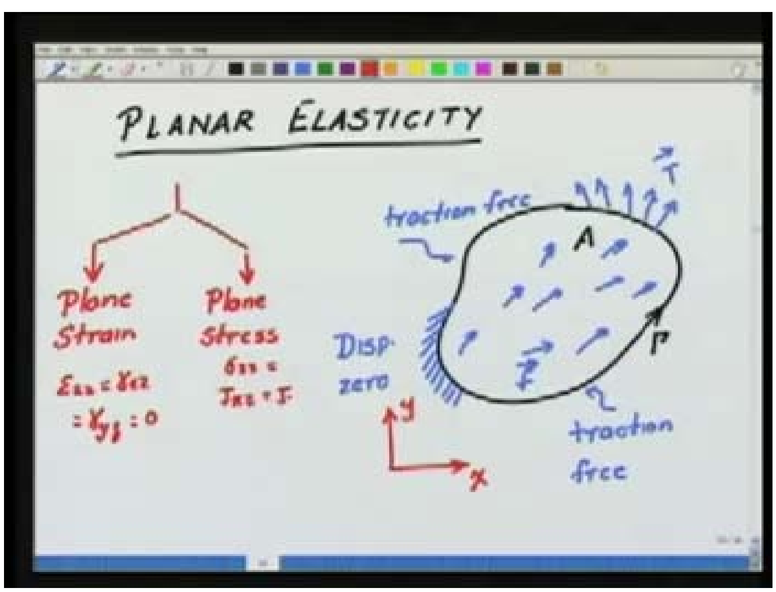

Next what we are going to look at is the problem of planar elasticity. This is another problem of great interest in mechanics.

Here we have some domain which has an area A and it has a boundary. I will call it gamma as a boundary. In this case, what is given to us is that there is a distributed body force, vector F. So the body force is given by the vector F. On part of the domain, I could be given the tractions; traction is given by the vector T. On some part of the domain I may fix the displacement; on the rest of the domain I could have a traction free condition. So, this could be the situation for a planar elasticity problem. In the planar elasticity category, we have two types: plane strain where the components of the strain in the transverse direction become 0 and from there we have a reduced constitutive relationship or I have a plane stress.

Let us say this domain lies in the xy plane; that is, the components in the case of plane strain my E_{zz} is equal to $gamma_{xz}$ is equal to $gamma_{yz}$; $gamma_{yz}$ is equal to 0. Here I would have $sigma_{zz}$ is equal to tau_{xz} this is equal to tau_{yz} is equal to 0. So these are two different situations that we are going to consider here.

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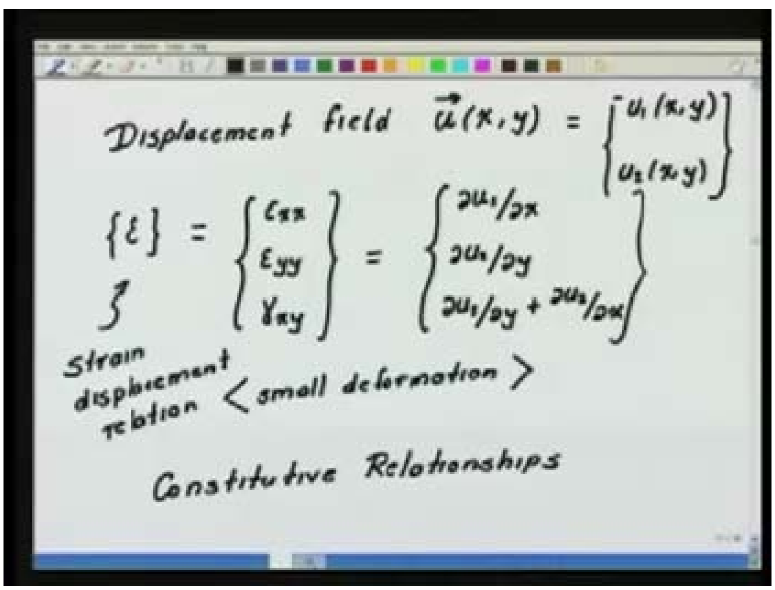

For the planar elasticity problem, the displacement field is now a vector field u; it is a function of x and y, because it is a planar problem we do not take it to be a function of z. This is a vector given by u_1 which is a function of x and y and u_2 which is a function of x and y.

For this displacement field, now we have a corresponding strain. The definition of strain, I will write the strain in the engineering notation; strain is written as a vector this is equal to E_{xx} E_{yy} gamma_{xy}. This is del u_1 del x del u_2 del y del u_1 del y plus del u_2 del x, this is the definition of the strain. This is called the strain displacement relationship. This is for small deformation theory; remember that, we are not talking of large deformations; here we are talking of small deformation theory. Further, we will have what we know as the constitutive relationships, which tell us what are the stresses in terms of strains.

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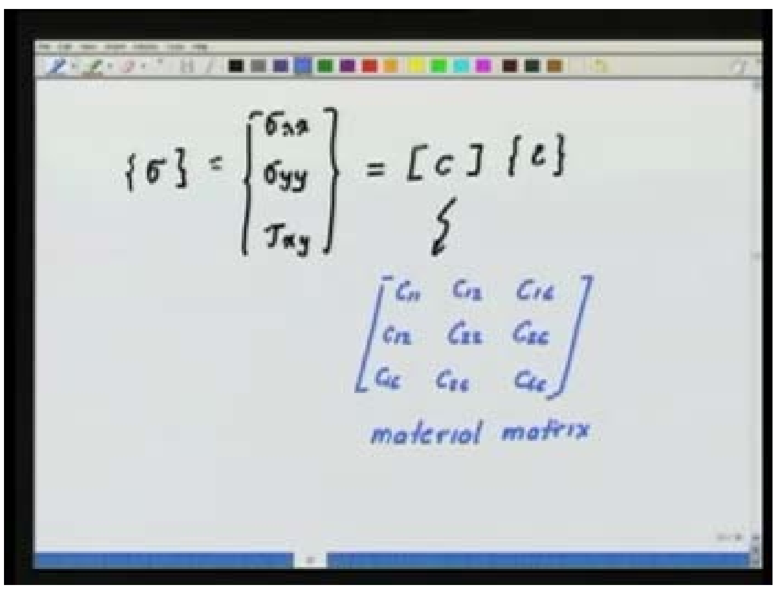

Then I will write here - stress again I am writing as an engineering vector notation - it is sigma_{xx} sigma_{yy} and tau_{xy}. Stress is assumed to be symmetric; not true all the time, but we are assuming that there are no distributed body movements. So the stress tensor is a symmetric tensor. This will be equal to some material matrix C into the strain vector psi; where C has components c_{11}, c_{12}, c_{16}; c_{12}, c_{22}, c_{26}; c_{16}, c_{26}, c_{66}; this is the elastic or the stiffness matrix, material matrix. We are talking of the problem of linearized elasticity. So stress is a linear function of strain given by this material matrix C.

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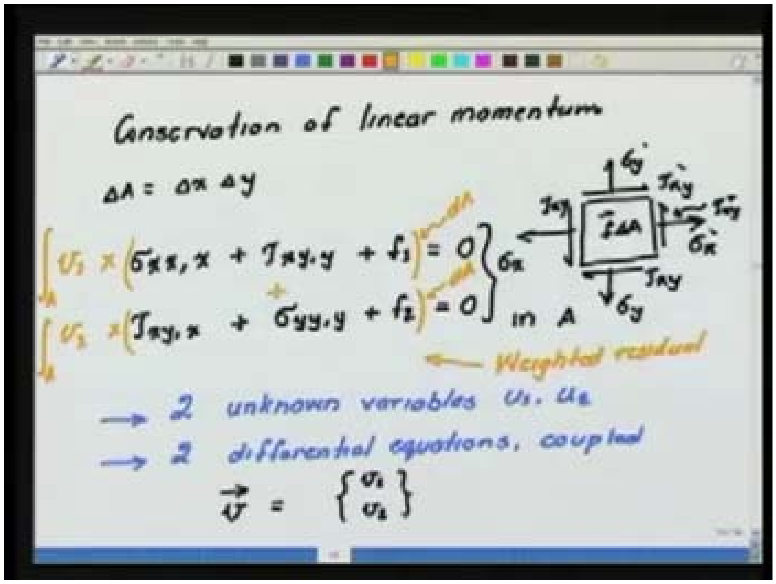

Now for this problem, the equilibrium is written in terms of angular momentum, conservation of linear momentum, conservation of angular momentum. Conservation of angular momentum has for these problems given as stress symmetry; conservation of linear momentum is what is left to us.

If you remember, it is given in terms of sigma_x star, sigma_x, sigma_y star, sigma_y, tau_{xy} star, tau_{xy} star and tau_{xy} and here is a contribution due to f; f delta A; delta A is delta x delta y.

So if I take this piece, this infinitesimal piece out of my domain, look at the equilibrium of this piece under the action of these stresses, of the forces on these faces, essentially for a static problem I will get (sigma_{xx}, x) plus (sigma_{xy}, y) were I am using tau. So let me use tau here; (tau_{xy}, y) plus f_1 is equal to 0. Similarly, you will have (tau_{xy}, x) plus (sigma_{yy}, y) plus f_2 is equal to 0; all this in the area A.

Now, you see certain features of this problem: it is two unknown variables; variables u_1 and u_2. There are corresponding to these, now there are two equations, governing differential equations. You see that these differential equations are coupled; that is, you cannot separate the u_1 and u_2 out; both the equations are in terms of both u_1 and u_2.

So, these are the two features of this particular problem and this is a deviation from what we have been doing till now; we have been looking at a single variable problem almost throughout the whole course till this time; now, we are going to a multi variable problem. Now, how do you handle this particular problem? So today, we are going to only look at development of the weak formulation. What is the basic idea? You have all done principle of virtual work in your mechanics courses, we will borrow from that.

So we will say that let this system be in an equilibrium. Now, I apply a virtual displacement v; a virtual displacement v, which is a vector which has components v_1 and v_1. This is virtual displacement or this is the test function, but this is the vector node.

So what will do now, as we had done earlier, we simply take both the equations and multiply by the corresponding component of this vector. So this I will multiply with v_1; this I will multiply with v_2. Then what we had done? We had said earlier that we are going to integrate it over the area. So we are going to integrate both of these over the area. You see that each one of them, if I integrate will be equal to 0. So, if each one is 0, I am simply going to add the two. I will get now one expression in terms of an integral, that is integral over the area of v_1 into ($sigma_{xx}$, x) plus (tau_{xy}, y) plus f_1 plus integral over the area of v_2 into (tau_{xy}, x) plus ($sigma_{yy}$, y) plus f_2 this is equal to 0. So this is our weighted residual form, when I sum this is the weighted residual. (Refer Slide Time: 52:02 min).

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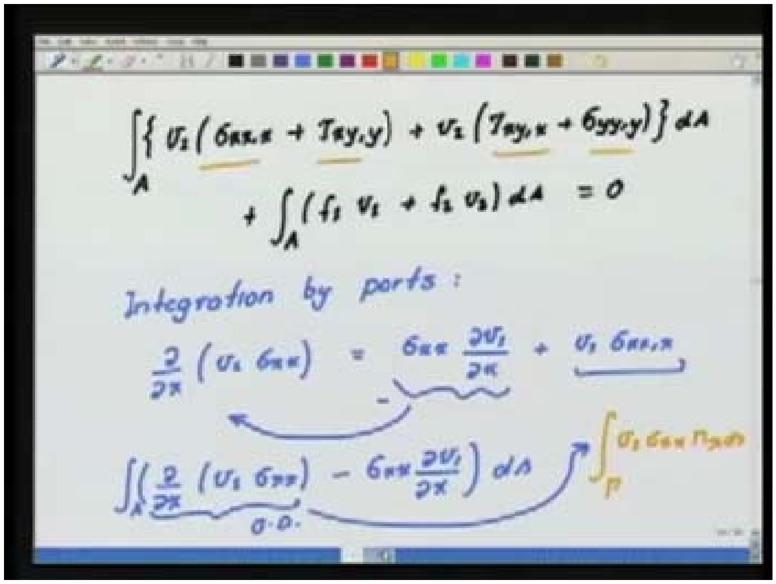

Once I have this, then I will go and again now look at this expression. I have this expression v_1 into (sigma$_{xx}$, x) plus (tau$_{xy}$, y) plus v_2 into (tau$_{xy}$, x) plus s (sigma$_{yy}$, y) dA plus integral over A f_1 v_1 plus f_2 v_2 dA is equal to 0; where f_1 f_2 are the components of the body force. Now again, you see, that if I go back and look at my expressions for the stress in terms of the strain, stress is a linear in terms of the strain, strain is in terms of the derivatives of u, as we have here. So in our expression here actually second derivative of u is sitting and similarly, v is sitting without any derivatives. So we want to now weaken the requirement of smoothness on u_1 u_2 by transferring a derivative to v. So now we do integration by parts.

In the integration by parts again the same thing we can do that del del x of v_1 sigma$_{xx}$ is equal to sigma$_{xx}$ del v_1 del x plus v_1 (sigma$_{xx}$, x). I transfer this thing from here to here with a negative sign, that will be this one in terms of this minus this.

Similarly, I can do for the others and then what do I see that integral over area of this quantity would have been let me do it, del del x of v_1 sigma$_{xx}$ minus sigma$_{xx}$ del v_1 del x dA.

Now look at this expression (Refer Slide Time: 54:52 min) this is the partial of a given quantity, of a given expression. That again, I can use the Gauss divergence theorem to give me this. This expression is actually equal to integral over the boundary of v_1 $sigma_{xx}$ into the normal component in the x direction; the x component of the normal on the particular edge into ds.

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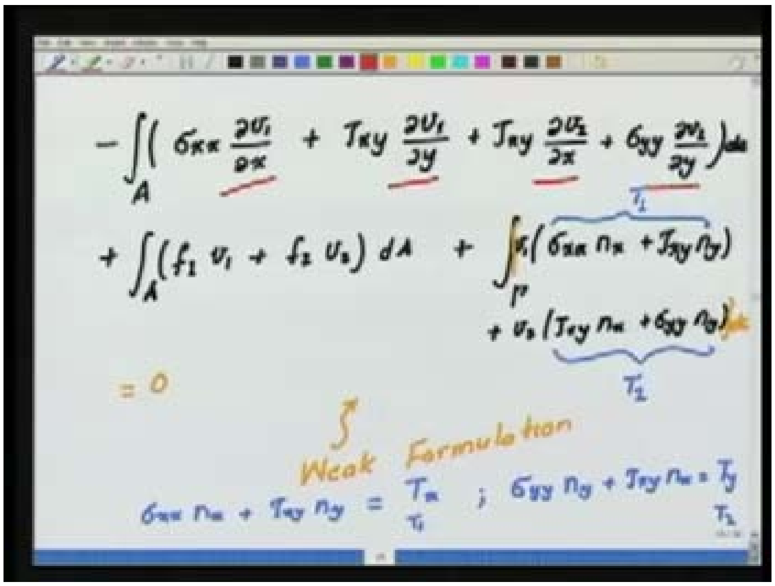

If I do this process for each of these components here – this, this, this, this, I will end up getting integral over the area $sigma_{xx}$ del v_1 del x plus tau_{xy} del v_1 del y plus tau_{xy} del v_2 del x plus $sigma_{yy}$ del v_2 del y whole thing dA plus integral over the area, this will be minus integral over the area f_1 v_1 plus f_2 v_2 dA plus integral over gamma. You will see you will have $sigma_{xx}$ n_x plus tau_{xy} n_y into v_1 plus v_2 into tau_{xy} n_x plus $sigma_{yy}$ n_y whole thing ds; ds integral over the edge. This is equal to 0; this is the weak formulation that I was talking about (Refer Slide Time: 58:48)

Now from basic mechanics we know that on the boundary $sigma_{xx}$ n_x plus tau_{xy} n_y is equal to the x component of the traction vector on the boundary; $sigma_{yy}$ n_y plus tau_{xy} n_x is equal to the y component of the traction vector on the boundary.

So these things can be given in terms of the x component or one component of the traction vector, y component or two component of a traction vector. I will call this also T_1, this is also T_2

Now on the boundary either the traction is given or the displacement is given so we will have to talk about what are the possible boundary conditions that can be applied for this particular problem, in terms of the displacements or in terms of the given traction forces on the boundary, but what we have obtained here in this expression is the complete weak form; I go and substitute the expressions for $sigma_{xx}$ tau_{xy} and $sigma_{yy}$ in terms of the strains, strains in terms of the displacements, I will get the weak form in terms of the displacements.

Notice one more thing that here, these are all in terms of del v_1 del x del v_1 del y del v_2 del x del v_2 del y and not in terms of the strain. This we will talk about in the next lecture, that how it gets converted to the strain for this particular type of problem and then will work from there.