**Finite Element Method**

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**Module – 8 Lecture – 2**

In the previous lecture we had stopped at the discussion of tensor product family of shape functions to be used with quadrilateral, square or rectangle type of elements. In defining the shape functions, we had looked at the Pascal triangle and we found that this tensor product family gives us representation of more monomials than the numbers, which are required to give a complete representation of a polynomial of a given order P.

Can we, without compromising on completeness, cut down some of these extra terms and what would be the advantage?

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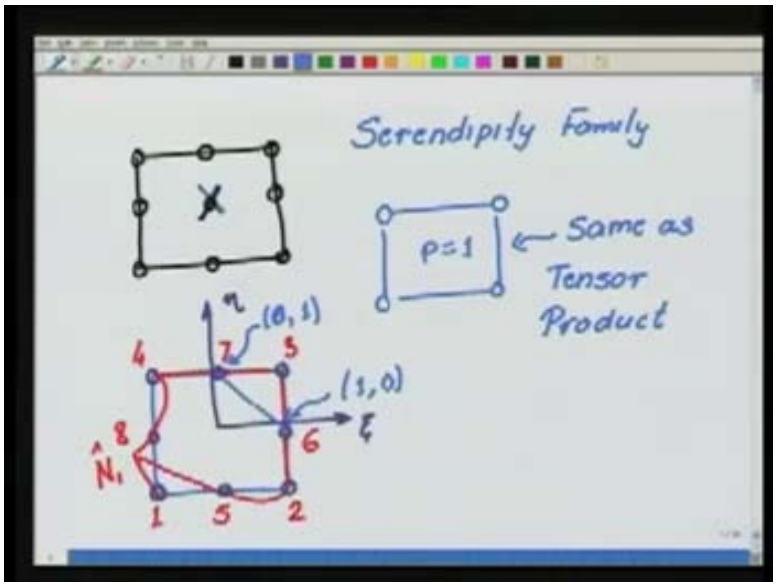


In the earlier case, in the tensor product family, there were nine nodes in the bi-quadratic. That is, I will have nine shape functions in an element for the tensor product family. Which means that, in order to represent the finite element solution in this element, I need

nine unknown coefficients to be determined. If I can in some way sacrifice one of the unknowns without compromising on completeness, then my computational task actually becomes smaller without compromising on accuracy. That is why we will talk about the serendipity family. What I am going to do today is talk of one type of element of the serendipity family, which is the quadratic serendipity element.

As far as the linear is concerned, the linear is the same as that we have for the tensor product. For P = 1, I have the same representation as that of a tensor product. Let us do the quadratic. In the quadratic the master element is always in terms of psi and eta. This is the psi line, this is the eta line. I have the four nodes at the corners and we add four more nodes. That is, from the previous one I remove the interior node for the tensor product. I want to define the shape functions using the Lagrangian definition, such that, they have a value one at one of the nodes and become zero at the other nodes. There is zero on this edge.

Let us number the nodes in this fashion: 1, 2, 3, 4, 5, 6, 7 and 8. I am numbering the corner or the edge vertices first and then number the internal one. The question is, how do I construct this shape function? Let us say this is $N_1$ in the master element, it is one here, zero here, zero here and zero in other places. Let us now do the construction.

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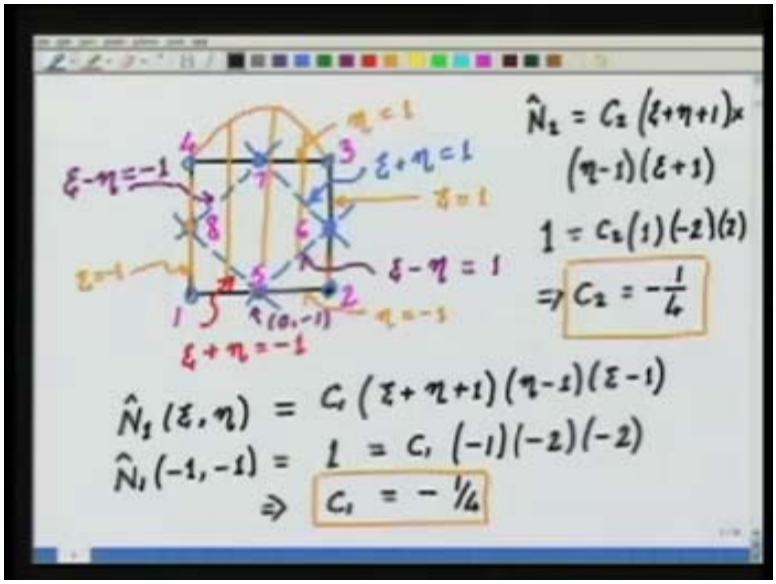

In order to do this construction, we have nodes. We now talk of these lines.

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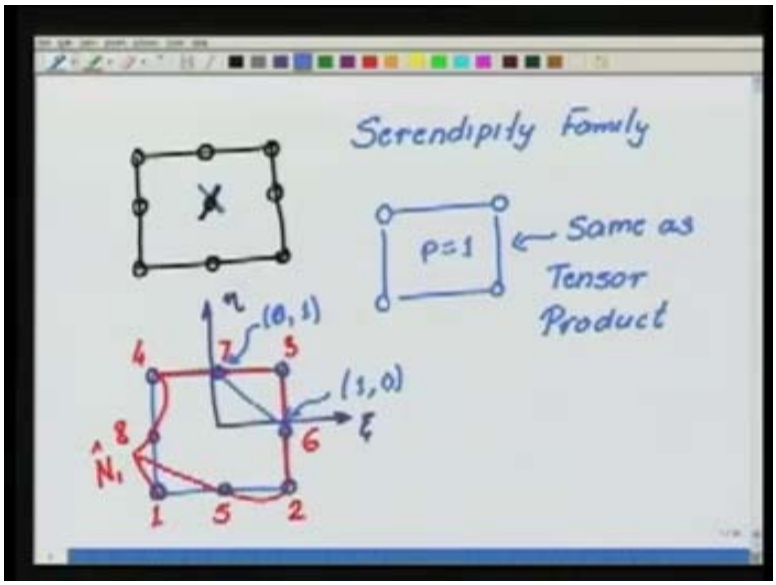


From the previous figure, in the psi eta coordinate system, this line is a line, which joins the points psi equal to plus one, eta equal to zero and the point psi equal to zero and eta equal to plus one. This is the line psi plus eta is equal to one.

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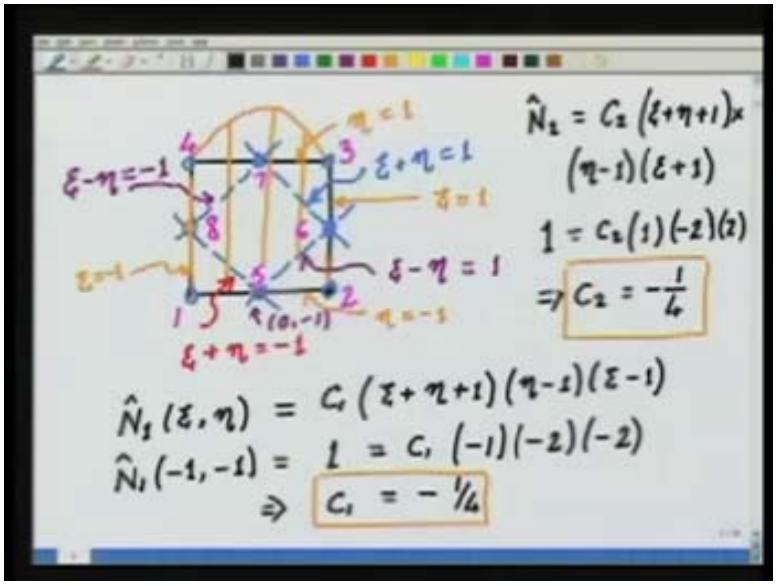

So this is the line psi plus eta is equal to one. Similarly, this line is psi plus eta is equal to minus one. Because this is the point which has coordinates psi equal to zero, eta equal to minus one and this is the point which has the coordinate psi equal to minus one eta equal to zero. Similarly, these two lines are perpendicular to these and will have the representation of these two lines as: psi minus eta is equal to one. (At eta equal to zero, psi is equal to one; when psi is equal to zero, eta equal to minus one). Similarly, the equation of this line is psi minus eta is equal to minus one. What are the equations of these four lines? Equation of this line is psi is equal to plus one, equation of this line is psi is equal to minus one, equation of this line is eta is equal to minus one, equation of this line is eta is equal to plus one. Given these definitions of these lines, can I now construct the shape functions?

Let us say I want to construct $N_1$ and I use some numbering 1, 2, 3, 4, 5, 6, 7 and 8. I want to find $N_1$ hat as a function of psi and eta. This I will say is equal to $C_{1;}$ on which lines should this function vanish to give me the value equal to one here and zero everywhere else? If it vanishes on this line and it vanishes on this line and this line, then it vanishes on all the points other than the first point. Therefore the equation of this line is psi plus eta plus one. What about the next one? It has to vanish at (08:32) these points, this line and this line. This equation is eta minus one and this line is psi minus one.

We see that by the choice of these products I will get this function vanishing at all other points in the element. How will I find $C_1$? It has to be one at the point one, I will have $N_1$ hat minus one minus one is equal to one, this is equal to $C_1$ (-1) (-2) (-2). This implies that $C_1$ is equal to minus one by four. Similarly, shape-function $N_2$ will have a value one at this point and it should vanish on all other points. By the same token, if it vanishes on this line running through the nodes 5 and 6 and it vanishes on the top line which is running through the nodes 3 and 4 and the vertical lines which is running from nodes 4 to 1, then it vanishes everywhere excepting the node 2.

If I have to write $N_2$ hat, I will write $N_2$ hat is equal to $C_2$ into (it vanishes on this line, so this line will have the equation) psi plus eta plus one into (it vanishes on this line, which is) eta minus one and (it vanishes on this line, which is) psi plus one. $N_2$ hat has to be one

at the point 2 which has coordinates psi equal to one eta equal to minus one. $N_2$ hat at psi equal to plus one, eta equal to minus one is one, which is equal to $C_2$ into (one plus one two minus one is) one, into (eta is minus one minus one) minus two into (and psi plus eta one is) two. This implies $C_2$ is equal to minus one by four.

In this way we construct first corner degrees of freedom or collar shape functions. $N_3$ has to vanish on the line joining the points 6 and 7 and the line joining points 4 and 1, and 1 and 2. Similarly, $N_4$ has to vanish on the line joining the points 7 and 8, that is, on the line psi minus eta is equal to minus one and on the lines eta equal to minus one and the line psi is equal to plus one. Once I have these definitions, it is very easy to go ahead and construct this corner shape functions. What about a mid-edge one or let us say function 5, 6, 7 or 8? I am going to form the generic function $N_7$. If I want $N_{7,}$ it should be such that on this edge these shape functions should become the standard one dimensional shape functions of order P. If we are talking of a quadratic on the edge, if I look at this top edge, the shape function four should look like the one-dimensional first shape function, shape function seven should like the second one and the shape function three should look like the third one-dimensional shape function. $N_7$ on this edge will look like this and then it comes down to zero at all other points.

This is how our $N_7$ is going to look. If I have to make $N_7$, it should be one at this point and vanish at all other points. How is it going to happen? We see that $N_7$ is sitting on this edge. If I take this edge, this edge and this edge, the equation of these three edges and multiply them together gives me this function, which, by definition, is going to vanish on this edge, this edge and this edge.

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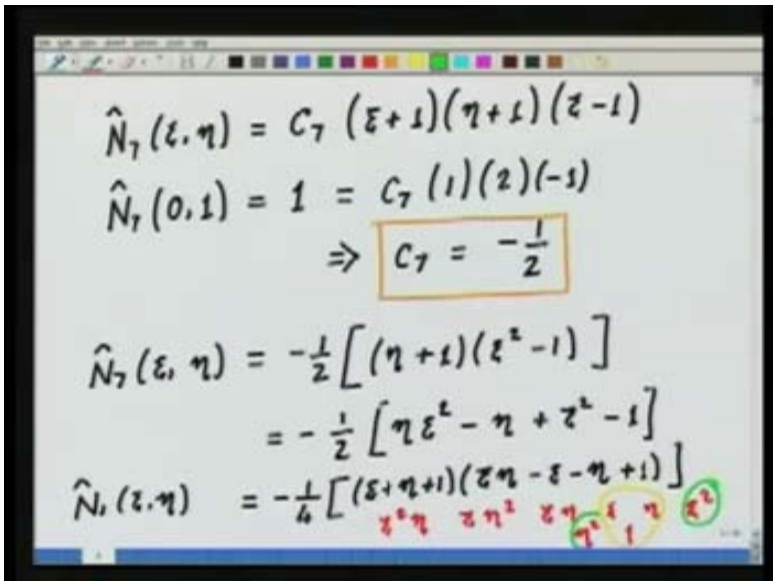


As far as the construction of $N_7$ is concerned, I am going to make $N_7$ vanish on the remaining three corner edges.

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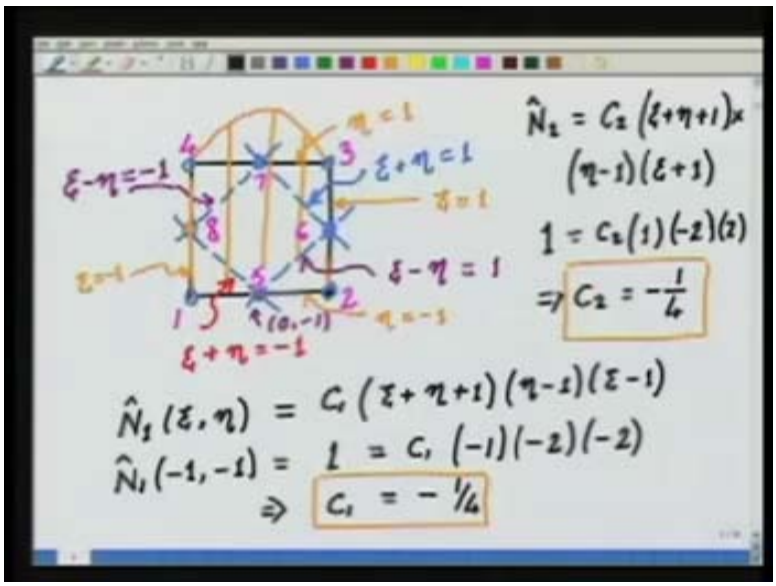


$N_7$ hat as a function of psi and eta is equal to $C_7$ into (psi plus 1) into (eta plus 1) into (eta minus 1).

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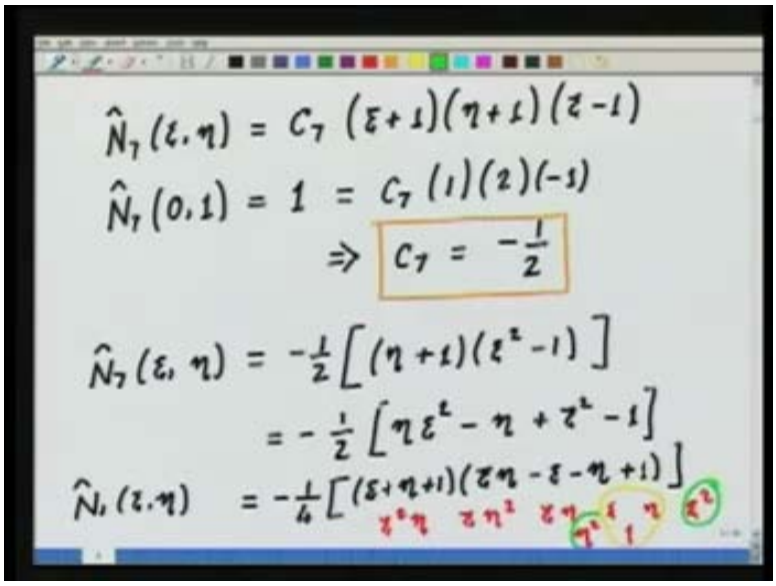


Also, I want $N_7$ hat at this point 7, which has coordinates (psi equal to zero, eta equal to one). $N_7$ hat at zero and one is equal to one, which is equal to $C_7$ into one into two and into minus one. This implies that $C_7$ is equal to minus one by two. Similarly, I can construct $N_8$. What will $N_8$ be? $N_8$ has to vanish on this edge, this edge and on this edge. Which means, $N_8$ will vanish on the edge eta equal to minus one, psi equal to plus one and eta equal to plus one. By the same token I can find the constant $C_8$.

If I can do these things I have constructed the eight shape functions corresponding to the serendipity family. Let us now look at what are the shape functions and look at the expansion. My $N_7$ hat as a function of psi and eta is equal to minus half into (eta plus one into psi squared minus one). This is minus half into (eta psi squared minus eta plus psi squared minus one). Let us look at $N_1$. $N_1$ hat as a function of psi and eta will be equal to minus one fourth into, (if I do this one (16:51), it will be) (psi plus eta plus one) into (psi eta minus psi minus eta plus one). The bottom line: I will get psi squared eta, psi eta squared, psi eta, psi, eta, psi squared, eta squared and some constant, some coefficient into these terms. If I expand all these terms we see that it will contain the linears, the quadratics and these two extra terms, psi squared eta and psi eta squared.

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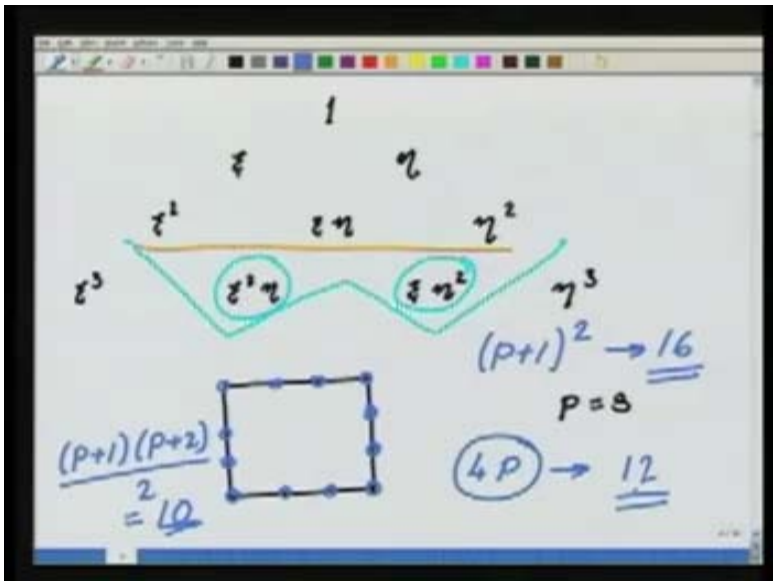


The Pascal triangle in terms of psi and eta was one, psi, eta, psi squared, psi eta, eta squared, psi cubed, psi squared eta, psi eta squared and eta cubed. If I look at this, the Pascal triangle for the quadratic will have only this part. This part is completely represented by our serendipity family, plus, it has some more terms, which are these terms (18:54). The serendipity family has two extra terms as compared to three extra terms that we had for the tensor product family.

Similarly, when we go and define, we can easily continue this job and talk of the serendipity family of order three. In that case the shape function should be such that on the edge they should represent the given order one-D shape function. I will construct them using the four points on the edge. In this case I will have four P terms. The number of shape functions corresponding to this is 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11and 12.

I have four P shape functions, which, for P = 3, is 12. If I had taken the tensor product I would have (P + 1) squared number of terms, which, for P = 3 would be 16. If I had taken triangular elements then I would have (P + 1) (P + 2) divided by 2, which would be 10 terms. If I had used triangular elements of order 3, I would get ten unknown coefficients to solve for in each element. If I use tensor product of order 3, that is, bi cubic, I would need 16. If I use the serendipity of order 3, I need 12. Depending on what cost we want to

pay, one can use one type of function or the other. We have now defined the major families of shape functions that will be needed to start off any finite element computation in two dimensions.

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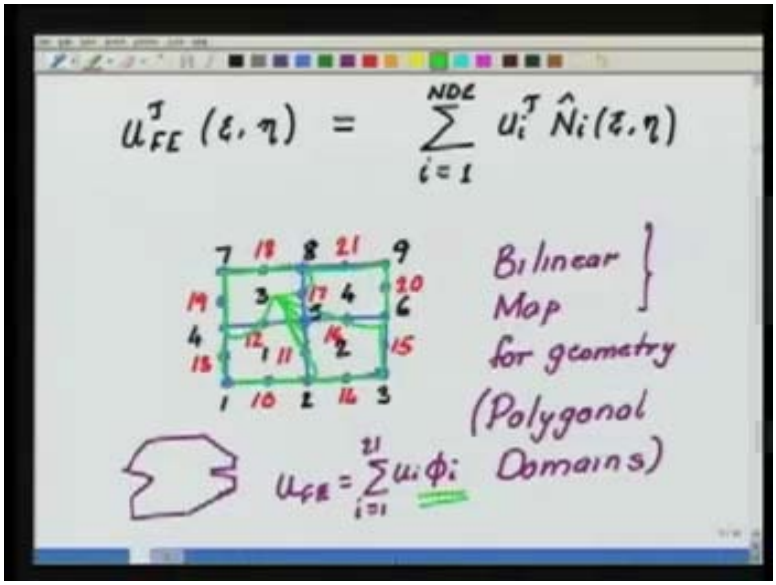


The next job is that, given the shape functions, I can have the u finite element in the element tau written in terms of psi and eta in a generic element given in terms of i = 1 to number of degrees of freedom in the element, $u_i$ tau for the element $N_i$ hat as a function of psi and eta. This is the representation in the element. What is a global representation? Let us say I have a domain with four elements. I have a serendipity that is quadratic. I formed this domain or mesh of four elements of quadratic serendipity. I number the degrees of freedom as 1, 2, 3, 4, 5, 6, 7, 8, 9 and I call this 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, and 21.

In the four-element mesh that I have, I will have twenty-one unknowns when I use the serendipity quadratic shape functions. I have deliberately done this kind of numbering, so that I first number the corner degrees of freedom. As far as mapping of the geometry is concerned, the geometry is always mapped using the bi linear map. It does not matter whether I use the tensor product family or the serendipity family. I always use the bi linear map for the geometry. I start it off with a polygonal domain or even non-polygonal

domain. In a polygonal domain, all edges of all the elements are straight edges. The elements are initially quadrilaterals, not with curved edges, but with straight edges and these are mapped to the master squares. For that the bi linear map is good enough. Let us say I am talking of polygonal domains. What do we mean by polygonal domains? When one domain is right in the picture, another domain I can have is this kind of a domain, where the bounding edges are all straight lines.

We see that I have the total finite element solution given as $u_{FE}$ is equal to sigma i = 1 to 21 $u_i$ $phi_i$. What is this $phi_i$? If I look at the fifth phi for example, the phi corresponding to the fifth global node (by our definition the phi's are defined with respect to the global nodes), it will be one here, it will die of like this in this element and then it becomes zero on these edges. In this element it will have this kind of a behavior, in this element it will have this kind of a behavior and then it will vanish here, vanish here and here. This phi is like a kind of tent.

Similarly, by piecing together these element shape functions, we can have the global phi definition, which we have discussed many times over and there is no ambiguity. Given this, we also have to construct the local to global enumeration. Let us say I am talking of element 2.

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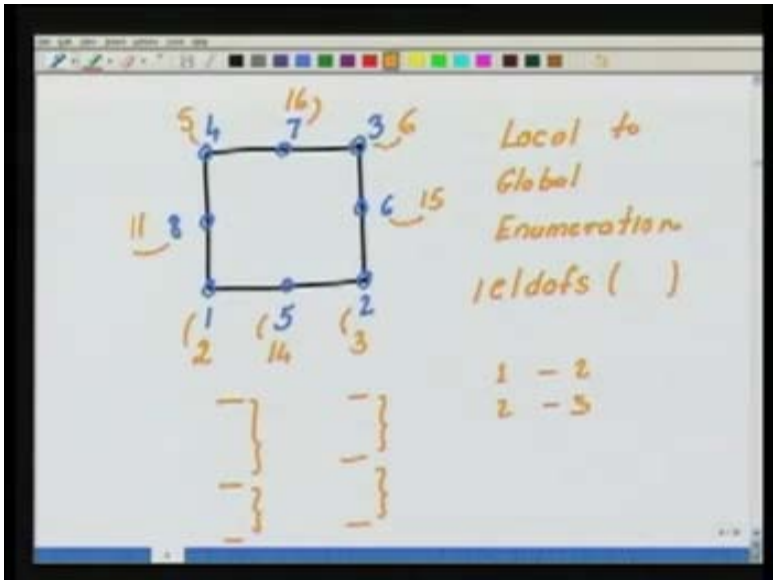

Let us do the local enumeration for element 2. The local is 1, 2, 3, 4, 5, 6, 7, and 8. Which global ones does it correspond to?

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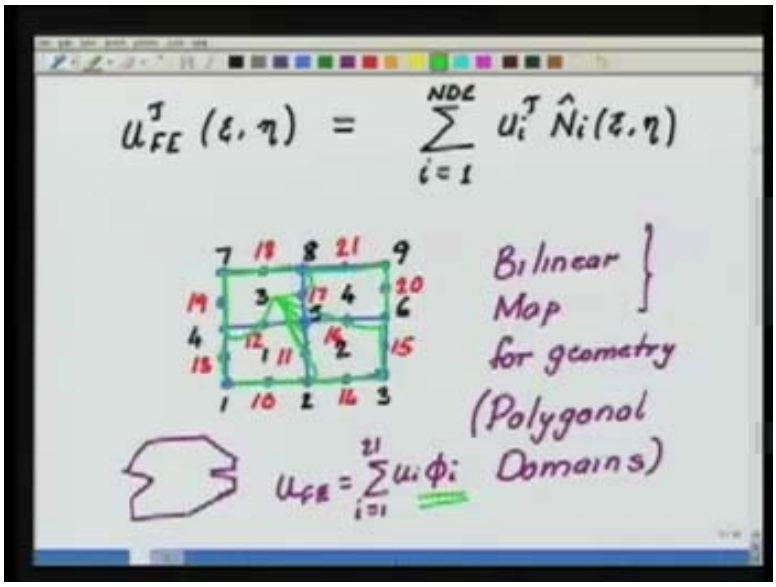


The one of element two corresponds to global two, the two corresponds to global three, three corresponds to global six and four corresponds to global five.

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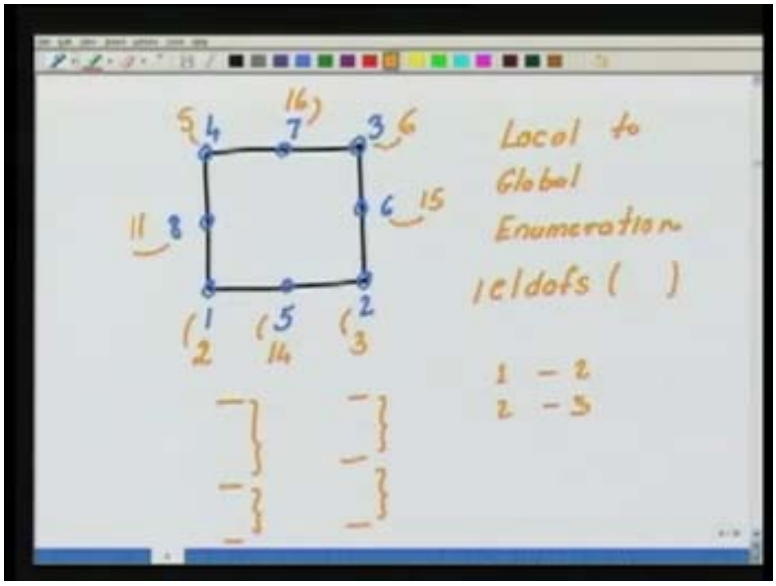

One corresponds to global two, two corresponds to global three, three corresponds to global six and four corresponds to global five.

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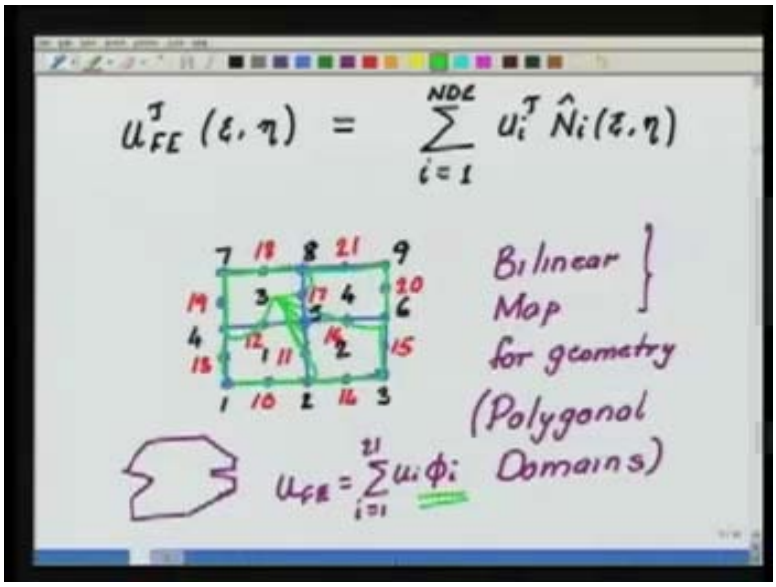


Five corresponds to global fourteen, six corresponds to global fifteen, seven corresponds to global sixteen and eight corresponds to global eleven.

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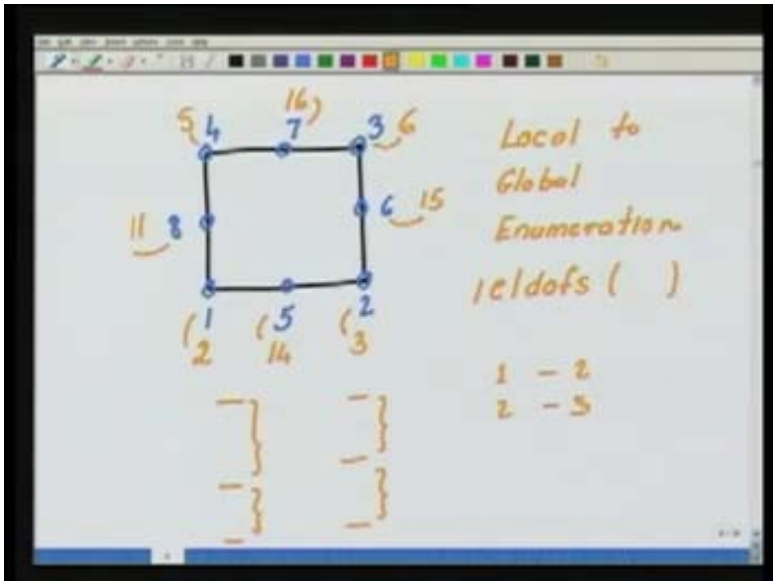

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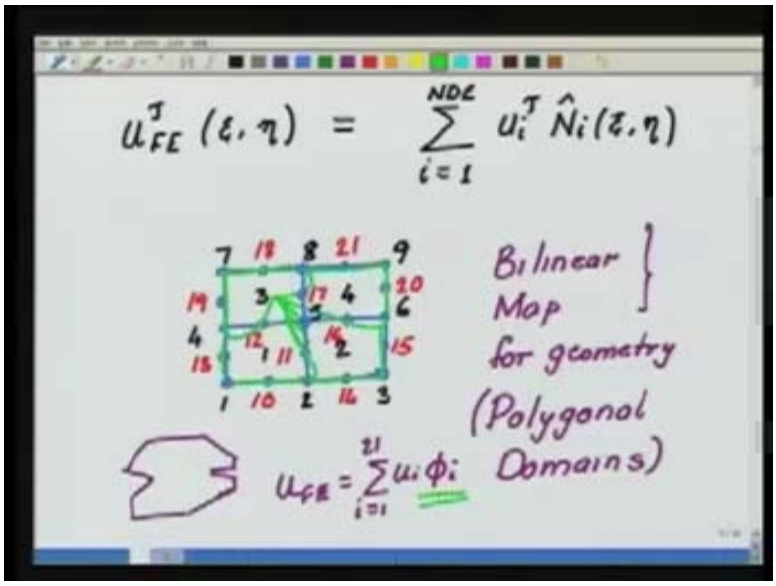


I first went ahead and numbered these corner degrees of freedom, then I started looping over each element and I started numbering all the side degrees of freedom of the first element. I am giving them global numbers. First I had the 9, then, I had a 10, 11, 12 and 13. Then I come to the second element, look at the mid side degree of freedom and if it has not been numbered, then give it a number, here it's 14. If the next side is not numbered, give it a number 15, next side 16, come to the fourth side, it already has a number, so retain it. This way we go ahead and construct the numbering for the degrees of freedom. One can choose many other ways but I have chosen this.

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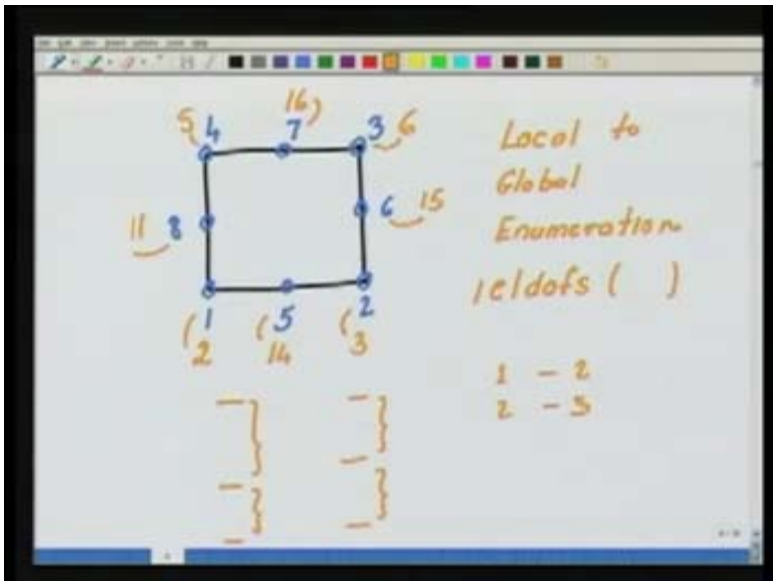


This correlation is our global to local enumeration or local to global enumeration and this will go in the ieldofs array, where we will stack one below the other, the correspondence of the degrees of freedom from the local level to the global level for each element. For this element the local one corresponds to global two, local two corresponds to global three and so on. I will do this in such a way that the stack of the eight degrees of freedom for the first element is put first, then the next eight corresponds to the second element, then the next eight corresponds to the third element and then the fourth element.

In this we have stack of eights because every element has eight degrees of freedom defined in the local (29:41). Through suitable pointers we can address these locations and find out the local to global enumeration. This information has to be constructed in the two-D case, as we had done in case of one-D.

With this information, for any generic element, which is either a serendipity type or the tensor product type, we can go ahead and compute the entries of the stiffness matrix for the element, which is integral over the area of the element tau.

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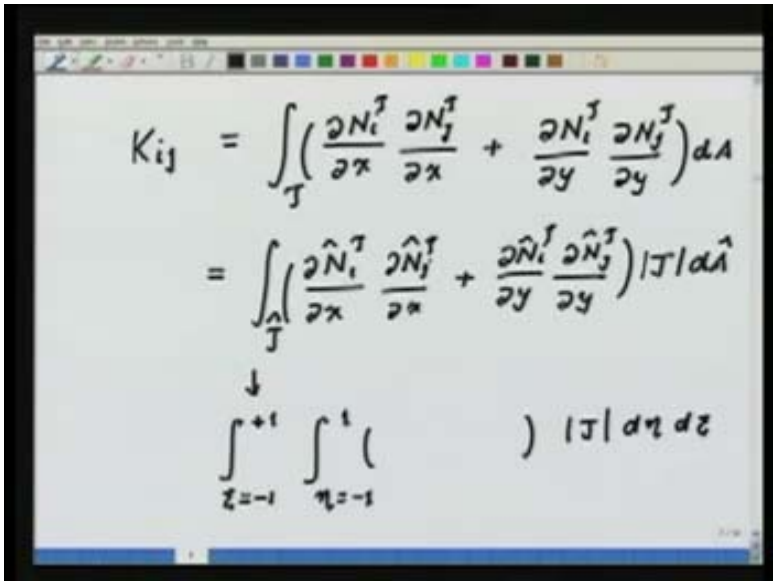


We have taken the simple definition of the model problem: integral over the area of the element tau into (del $N_i$ tau divided by del x into del $N_j$ tau divided by del x plus del $N_i$ tau divided by del y into del $N_j$ tau divided by del y) into dA. This is equivalent to doing the integration of the master element tau hat of del $N_i$ tau divided by del x into del $N_j$ tau divided by del x plus del $N_i$ tau divided by del y into del $N_j$ tau divided by del y, represented as functions of psi and eta into Jacobian into dA hat. What will happen to dA hat? It will become an integral of psi going from minus one to plus one, an eta going from minus one to plus one, this integrant into the Jacobian into d eta d psi.

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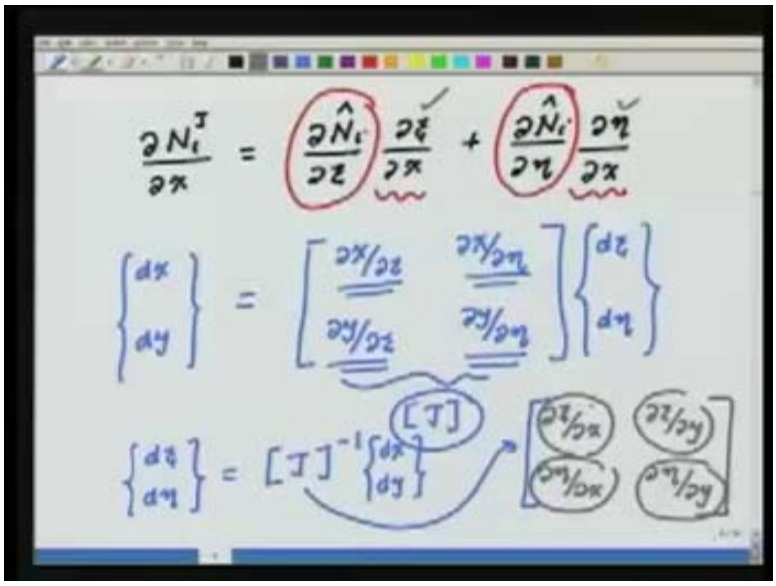

It is the standard way of getting these integrals now in terms of the master element and as we had done before, I need to find del $N_i$ tau divided by del x, which is equal to del $N_i$ hat divided by del psi into del psi divided by del x plus del $N_i$ hat divided by del eta into del eta divided by del x. These quantities can come out easily from the definition of the shape functions. When we are define the shape functions, at the same time we will define the derivatives of the shape functions with respect to psi and eta. Our shape function routine should return the value of the shape functions and the value of psi and eta for a given point psi and eta. We have to get these quantities from mapping. Remember that whether it is serendipity or tensor product or P order, we are talking of only linear maps of geometry. The mapping is independent of the order approximation for the problem cases that we are talking about.

As we had done in the previous class, dx dy is equal to (del x divided by del psi, del x divided by del eta, del y divided by del psi, del y divided by del eta) into (d psi d eta). This is the Jacobian matrix and we can compute these quantities by using our bi linear mapping: x in terms of $x_1$, $x_2$, $x_3$, $x_4$ and the psi in functions of psi and eta; and similarly, y in terms $y_1$, $y_2$, $y_3$, $y_4$ and functions of psi and eta. We can simply write d psi d eta. Note that Jacobian is now going to be a function of psi and eta.

This will be equal to Jacobian inverse into dx divided by dy. This Jacobian inverse is actually equal to del psi divided by del x, del psi divided by del y, del eta divided by del x, del eta divided by del y. With these components, it is very easy to find Jacobian inverse. That is, one by Jacobian (change the sign of these terms) I get these quantities.

These are going to be functions of the given psi and eta. I evaluate these quantities at every point, from this inverse mapping and that is why we see that the inverse mapping should have the Jacobian greater than zero. Otherwise, this can become singular. Determinant of the Jacobian matrix can become zero and the inversion may not be possible. That is, many points are going to one point or vice versa. That is, the mapping is not one to one.

That is why we need to have the shapes of the elements or the quadrilaterals in such a way that the angles are always less than 180 degrees. We can determine these quantities at every point and the only question is how do we go ahead and put it in a program? We will have to do the integration using integration points in a program. That is, replace the integral with a summation.

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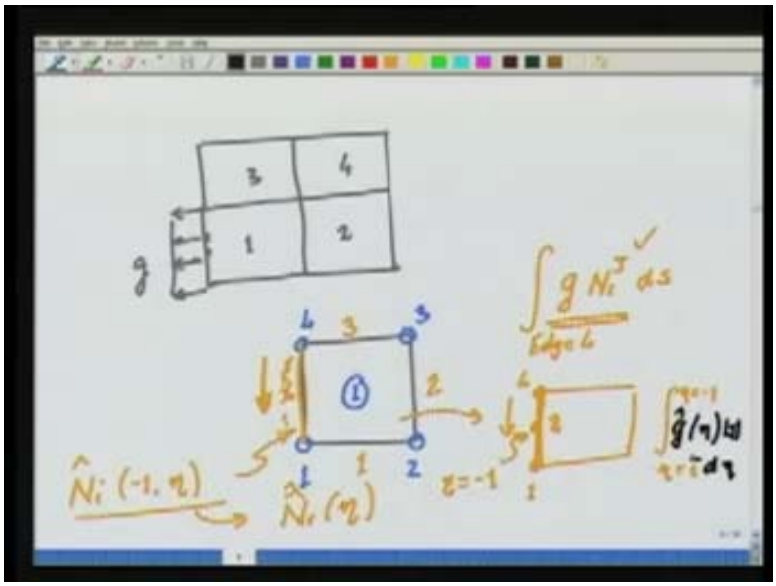

Let us talk about how to apply boundary conditions. Let us say, on this edge I have given some kind of a flux boundary condition. On this edge I have specified g. This edge lies on the element one. If I look at element one we are going to number the edges for the generic element in terms of the local nodes for element one. This will be the edge 1, this will be edge 2, this will be edge 3 and this will be edge 4. Edge 4 of element one has g specified. As far as the integral is concerned, I have to integrate g against $N_i$ tau ds on the edge 4. This is what I have to do as far as the integral is concerned. In the master element, this edge corresponds to psi is equal to minus one and again on this edge, using the quadratic serendipity, the first one for the element is non zero, the fourth one for the element is non zero and the eighth one is non zero.

Only these shape functions are going to participate as far as this integral is concerned and on this edge these shape functions are the one-D shape functions. On this edge the shape functions are $N_i$ hat as a function of minus one and eta. They become a function of psi and eta but psi is equal to minus one. They are left as functions of eta only.

Here I am left with these points being functions of $N_i$ double hat of eta. I map it to this master edge. The edge of the master element is a function of eta. The problem is that we have to do the integration with respect to eta, but since this direction is downwards we will have to integrate for eta is equal to plus one to eta equal to minus one. If I integrate from eta equal to minus one to plus one, ds will be minus d eta.

The integral of this quantity g has a function of eta on this edge into Jacobian for this edge d eta and I start from the pointers ds is minus d eta. With this I go ahead and do my integral on this edge. Similarly, if the boundary conditions are specified, I do the projection for the other edges and the integral, which is the one-D integral. I can thus construct the integrals on the edges of the element. After I compute integral of r $N_i$ hat d psi d eta into the Jacobian over tau hat, I compute the boundary integrals if the edge lies on a Norman boundary and simply add the integral to these corresponding entries.

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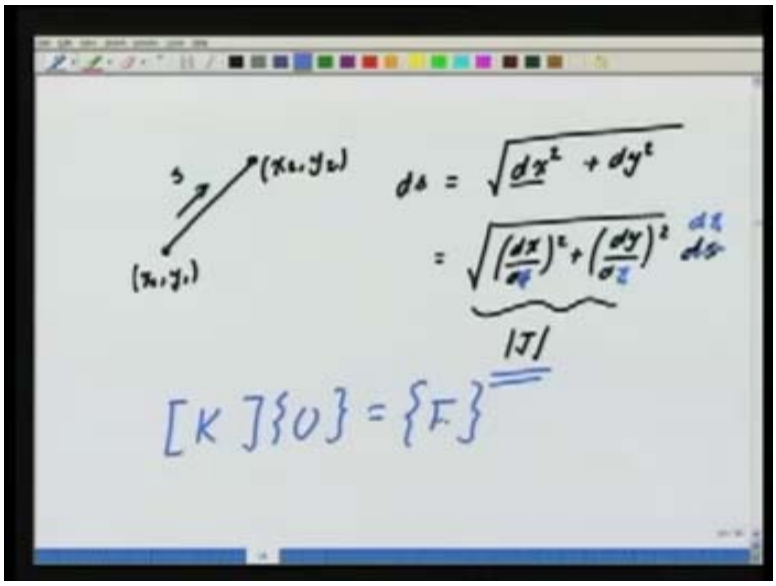


If there a generic edge and s goes like this, then we see that this has point ($x_1$, $y_1$) and this has point ($x_2$, $y_2$). In terms of length ds is equal to square root of dx squared plus dy squared for this edge. This is equal to square root of dx divided by d psi whole squared plus dy divided by d eta whole squared d psi. Jacobian is equal to this (42:11).

ds is given in terms of d psi and this is nothing but the scaling of the lengths, which is the Jacobian for the edge. This is not the same Jacobian as that for the area. This way I can complete all my integrals that we have to do for the element and given the local to global enumeration, we can assemble the stiffness and the load vector components in the respective global stiffness and the global load vector entries [K] {U} is equal to {F}; apply the essential boundary conditions by fixing those known degrees of freedom to the known values and solve the problem.

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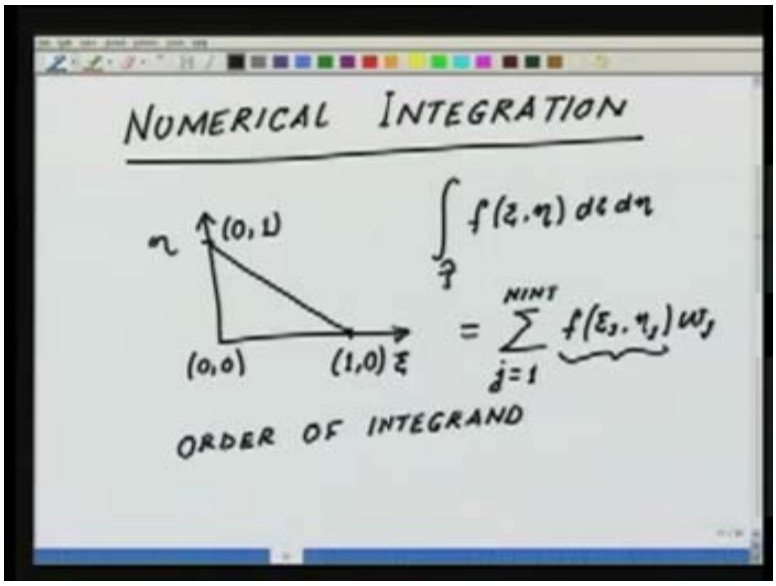


How do we do the integration? Integration in a computational domain involves numerical integration. We will discuss the integration rules for triangles and quadrilateral elements. In the case of the triangle, we are doing the integral over the master triangle. This is zero zero, one zero and zero one. I want to replace integral of some psi and eta d psi d eta over tau hat is equal to summation over from j is equal to one to number of integration points, F evaluated at these points psi j, eta j, into a weight function corresponding to these points. Thus, we replace it by a summation.

Depending on the order of the integrand, that is, the integrand polynomial order (here we are dealing with only polynomials defined over the master element) will define what order integration rule I should use and how many points would give me an exact integral for that given order. The number of points is fixed by the order of the integrand. We are now going to give a sample Fortran routine, which gives these integration points for triangles. The construction of these points for triangles is not so easy. This is something that is obtained from a research paper written long ago in the 1970's on how to obtain these points for triangles.

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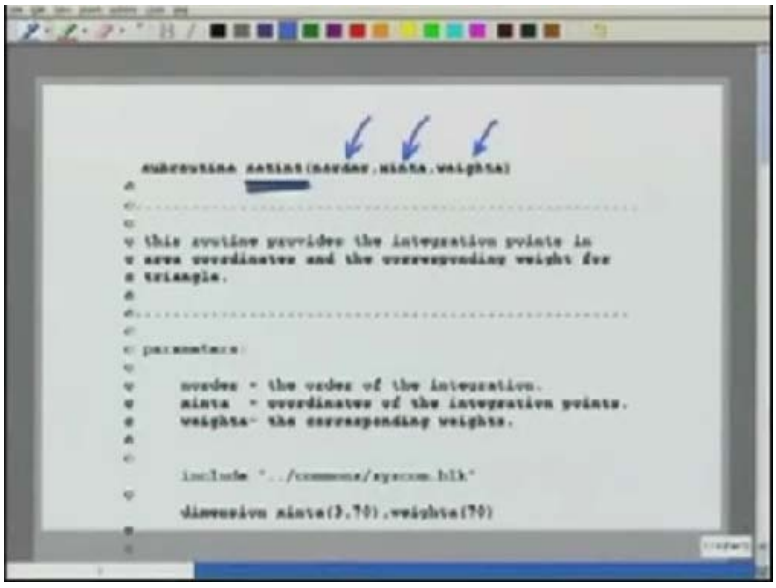


This is a routine called second. We input to it the order of the integrand, which we have to compute from the terms involved in the integral, which is exactly the way we had done in the one-D case. Given the order of the integrand, it returns the values of the coordinates of the integration points given for this particular order and the corresponding weight function through vita.

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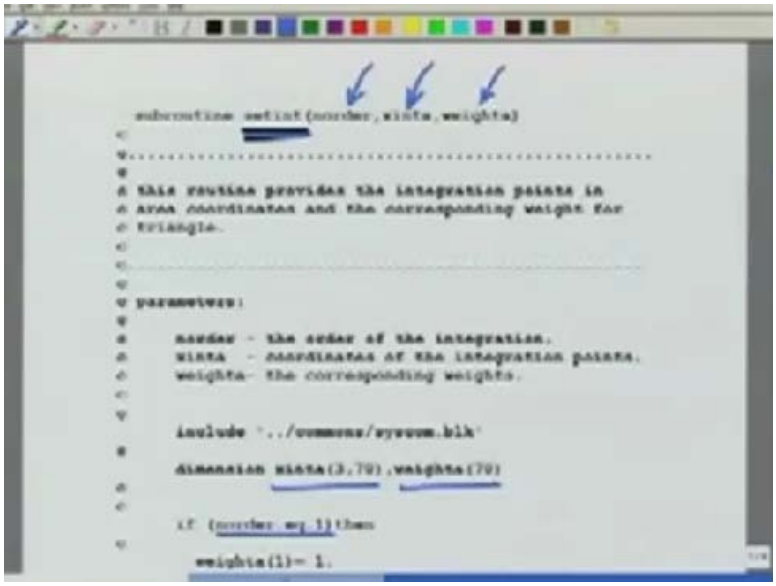

Things arranged in this page are: I have an array giving the coordinates of the shape functions of the integration points and a vector weight, which gives me the values of the weight functions.

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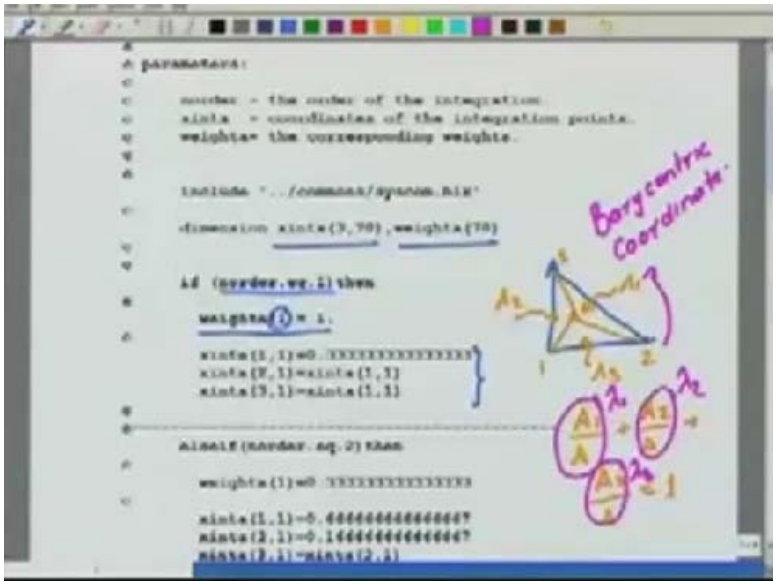


If my order is equal to one, my weight function has a value one. For this order equal to one, I have one point integration root. That is, there is only one point for which the weight is equal to one and the x int is now given as three coordinates, though we had only psi and eta. These three are actually given in terms of what is called the area coordinates or the barycentric coordinates. Let us take any point. This is node one, node two and node three. Connect these points. This becomes area $A_1$. Given any point in the middle or anywhere inside the master triangle, I connect that point to the three vertices. The area opposite to a given vertex is given the name $A_1$. This area becomes $A_2$. This area becomes $A_3$. $A_1/A + A_2/A + A_3/A = 1$. These quantities are given names as lambda one, lambda two and lambda three. For any point, you can define unique values of lambda one, lambda two and lambda three. This is sometimes referred to as the barycentric coordinate system. This was classically used when triangular elements were used. Now we write everything in terms of psi and eta.

If we look at lambda two, lambda two is zero when I am on this edge (49:00) and when I have a point on this edge. It is one when I have the point here. When the point is here then lambda two is one. Which of our psi's and eta's are one here and zero here? Lambda two is actually equal to our global psi. Similarly, lambda three is actually equal to a global eta and lambda one is nothing but one minus psi minus eta. It is quite easy to relate this barycentric coordinates to psi and eta; that lambda two is the psi lambda three is the eta. As far as we are concerned, this is our psi and this is our eta. Whenever we see these three points given in terms of lambda one, lambda two, lambda three, corresponding to this integration point, we are only interested in the psi and eta values that will do the job. Similarly, if an integrand is of order two, then we see that we are going to have actually a three-point rule.

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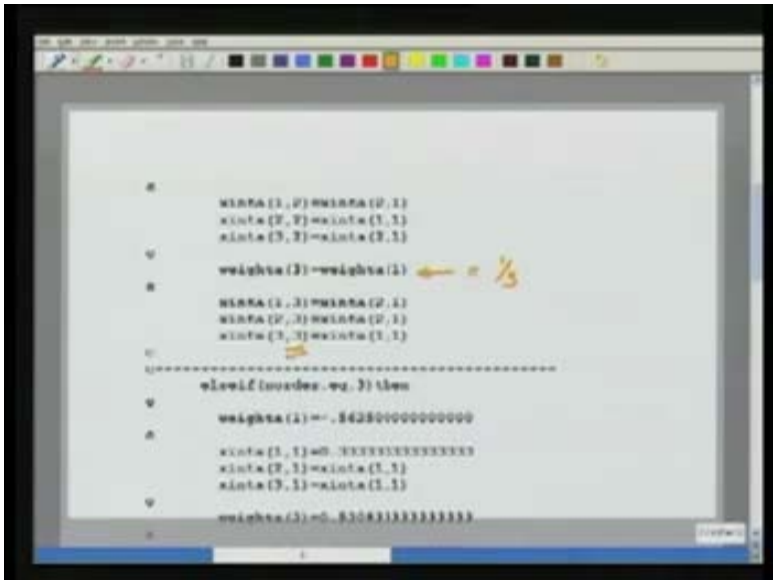

When the integrand is of order two then I have three points involved and the weight of the first point is one third, weight of the second and third point is two third and one sixth. Similarly, the coordinates are given like this. The coordinates of the other two points of the first point is: psi is equal to one sixth, eta is equal to one sixth, the weight of the second point is also one third, weight of the third point is also one third; so weight of the second and third point is one third and the coordinates are inherited in this way.

This way I can continue and create the integration rules of the requisite order. For example, when the order of the integral is three, then I think I have a four-point rule.

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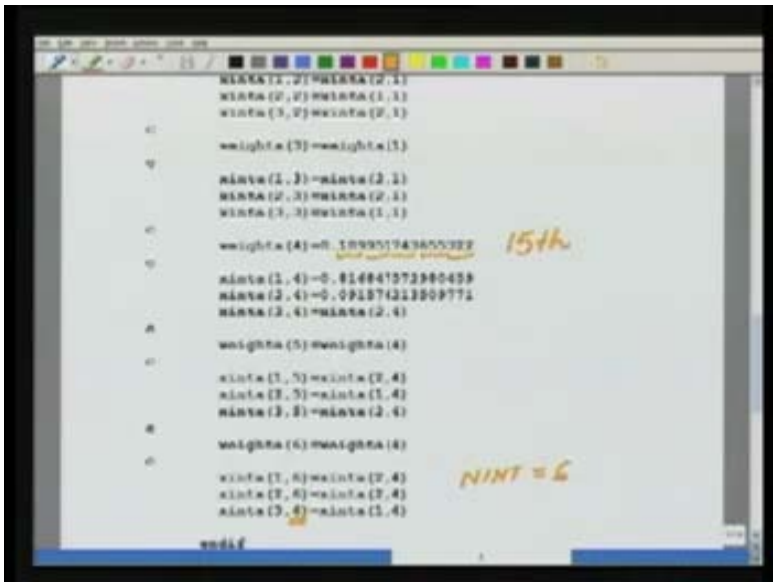


We see that there are four points involved and the weights and the coordinates are given like this, the weight of the first point is this, weight of the second point is this, weight of the third point is the same as that of the second point and of the fourth point is the same as that of the second point when the coordinates are given in terms of the barycentric, which we can easily convert to our psi eta form.

Similarly, when the order is four, we will have six points. There are six points, that is, NINT for order four is equal to six. We see that the number of points cannot be obtained directly as a formula in terms of P. It is a little complicated, nevertheless, we have these points available to us and we see that the coordinates are given up to the fifteenth decimal digit. They are given in double precision. So one can use these points and define the integration rule for a triangular element.

We have now obtained all modules that are required to do a finite element computation in two-D for the single variable problem

We have defined the integration rule for the triangles. How do we define it for the quadrilateral?

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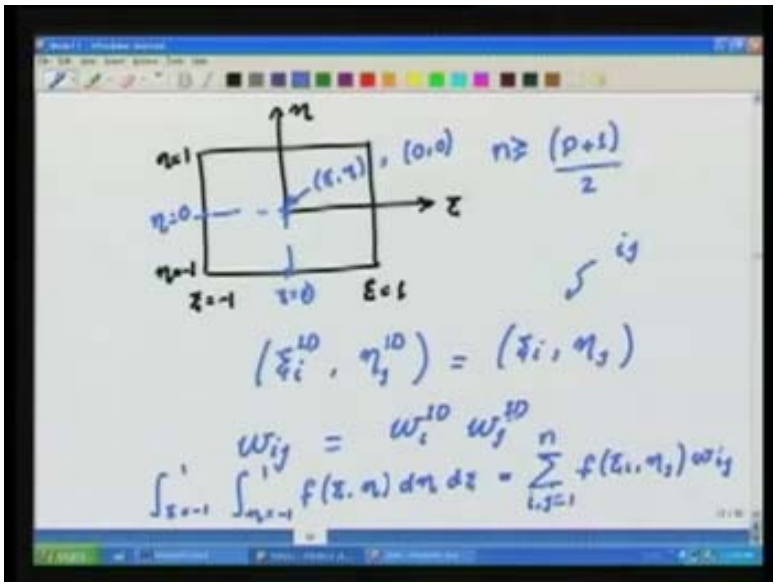


For the quadrilaterals, I have the psi and eta directions like this. Here psi is equal to minus one and here psi is equal to plus one. Here, on this line, eta is equal to minus one and here eta is equal to plus one. I take the tensor product of the one-D integration rules for the psi direction and eta direction and that is going to give me the integration rule for the two-D. One-D rule is already known. If I have linear in psi and a linear in eta as my integrand, that is, it is a bi linear integrand that is linear in psi linear in eta, then the rule is governed: n is greater than equal to P plus one by 2. Where P is in each direction, what is the order of P? If P is equal to one in the psi direction and one in the eta direction then essentially P + 1 is two. Two by two is one, so I can have a one-point rule.

For the psi in this direction, the one-point rule corresponded to the point psi equal to zero. In the eta direction, this one point rule corresponds to the point eta equal to zero. For our element we take the corresponding coordinate as the one which has psi eta equal to zero. What is the weight? I take (psi i 1D, eta j 1D) as (psi i eta j), corresponding to the ijth (we do a double index integration point). The weight for this point is equal to weight i 1D into weight j D1. That is, the integral over psi is equal to minus one to one integral of eta

going from minus one to one of f which is a function of psi eta, d eta d psi is equal to summation of i comma j going from one to n f at the point psi i, eta j into $w_{ij}$. This is the integral for the quadrilateral elements. I replace the integral by the summation, while the summation is understood as summation in the psi direction, that is, a tensor product of the psi and eta directions. I have the one-D functions, which we have already created in the one-D problem. We can easily construct the two-D functions.

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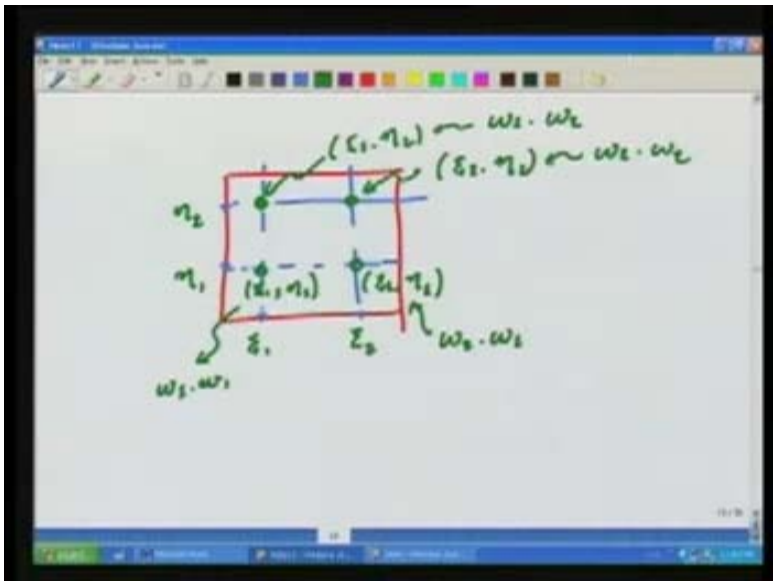

Let us take a case for the two-point rule. The two-point rule corresponds to psi eta is equal to plus or minus one by root three. eta is also plus or minus root three. So I will take our integration points in the two-D domain as this. Our integration points are going to be this, this, this, this (57:34), where this is psi one, this is psi two, this is eta one, this is eta two. This is psi one eta one point, this is psi two eta one point, this is psi two eta two point and this is psi one eta two. The weight here is $w_1$ into $w_1$, weight here is $w_2$ into $w_1$, weight here is $w_2$ into $w_2$ and weight here is $w_1$ into $w_2$. This is how I can do the integral. We have essentially defined how to do the numerical integration inside the area.

In the next lecture, we are going to look at some of the properties of the finite element solution that we have obtained as far as convergence is concerned, how to improve the convergence, we will talk about those aspects of the finite element solution because that

is very important in the two-D case. The shape of the domain itself, that is, the geometry is going to decide, to a large extent, the smoothness of the solution.