Finite Element Method

Prof. C. S. Upadhyay

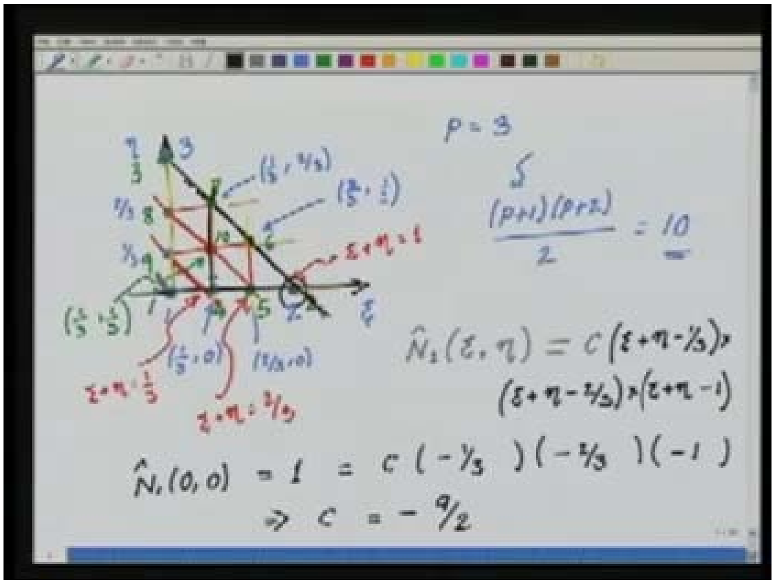
Department of Mechanical Engineering

Indian Institute of Technology, Kanpur

Module – 8 Lecture – 1

In this lecture we are going to go further with the definition of higher order shape functions for two-dimensional problems. We were working with triangular elements. How do I construct higher order approximation, that is, an order greater than two for a triangular element?

(Refer Slide Time: 00:45)



Here are the corners of the master element. Given the order, the shape functions should become the one dimensional shape function, when I project them on a given edge of the triangle.

If this is node 1, node 2, node 3 and if I am talking of the first edge connected to nodes 1 and 2; and if I am talking of P equal to 3, a cubic approximation, that approximation or those functions should become the one dimensional cubic shape functions on this edge. The one-dimensional cubic shape functions, in terms of definition - the Lagrangian functions, correspond to these four points on this edge, that is, these four equally spaced points. This point will correspond to the coordinate one-third, zero, this will correspond to two-third, zero and this is one, zero.

Similarly, on this edge, I should have the four equally spaced points given by these locations. Here the coordinate will be psi is equal to two-third, eta is equal to one-third, while this one will be psi equal to one-third, eta is equal to two-third because psi plus eta has to be one. On this edge I will have two more points, which are given by eta is equal to two-third and eta is equal to one-third. I have made 1, 2, 3, 4, 5, 6, 7, 8, 9 points. For P equal to 3 from the Pascal triangle we had drawn earlier, we needed (P+1) into (P+2) by 2 (number of monomials) for completeness. For P equal to 3, this is equal to 10. So nine points means nine definitions of Lagrangian shape functions. We need one more to give us ten independent basis functions.

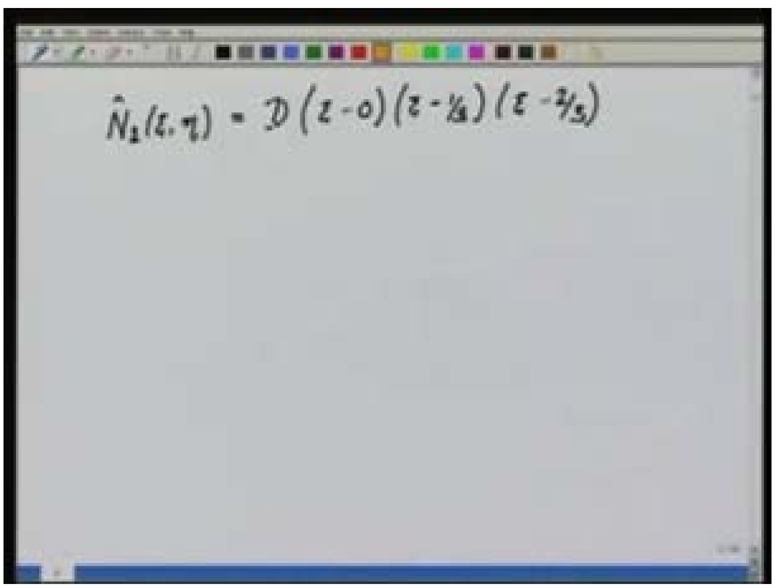
Let us say that I connect these points by straight lines. If I have drawn everything correctly, they should pass through like this. I have connected these points by a straight line, these two points by a straight line and then I draw straight lines parallel to the three edges. If I look at these straight lines, they intersect at an interior point. This interior point will have a coordinate. It lies on the line eta is equal to one-third and on the line psi is equal to one-third. So it will have a coordinate one-third, one-third. This is nothing but the centroid. This point is another node. This becomes the tenth node. Our shape functions are now defined. Now I am going to color the nodes with respect to these ten nodes and I will follow some numbering scheme. Let us say, as far as the definition of shape function is concerned, this is 1, this is 2, this is 3, this is 4, this is 5, this is 6, 7, 8, 9 and 10. I need ten cubic basis functions in order to have a complete set, which can completely represent any cubic polynomial.

How do I now go about defining the shape functions? Let us say I would like to define N_1 as a function of psi and eta. This function N_1 is with respect to this node such that N_1 is 1 at the nodes with coordinates zero, zero and zero at all other nodes.

How can N_1 be a cubic polynomial in terms of psi and eta and vanish at all other nodes barring node 1? I will put N_1 as psi. If I take the N_1 to vanish on this line, which is nothing but the line psi plus eta is equal to one-third, make it vanish on this line which is the line psi plus eta is equal to two-third and on this line which is psi plus eta is equal to one. So if I choose N_1 to vanish on these three lines, then automatically N_1 vanishes on all other nodes barring the first node. So I will say, it will have to vanish on the line psi plus eta equal to one-third. So psi plus eta is minus one-third. It has to vanish on the line psi plus eta equal to two-third, so psi plus eta minus two-third and it has to vanish on the line psi plus eta equal to one. So it is psi plus eta minus 1. This is going to give the N_1 such that N_1 hat at the point zero, zero, which is the first point is equal to 1. This is equal to c into minus one-third into minus two-third into minus 1. C implies c is equal to minus nine by two. So it is quite easy to find the definition of the first shape function.

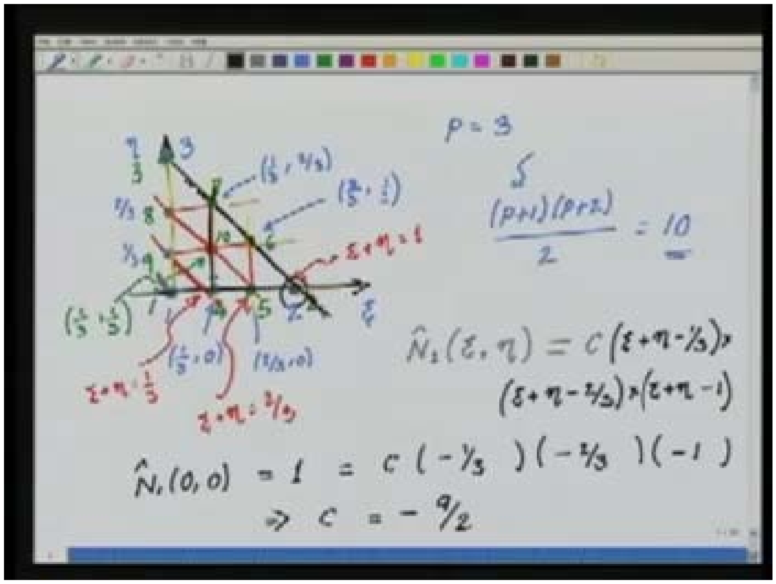
Similarly, if I want to find N_2, N_2 is a function, which is one at this point and vanishes at all other points. It simply means that it has to vanish along this line, vanish along this line, vanish along this line and vanish along this line. N_2 is now written in terms of equation of this line. Equation of this line is psi is equal to two by three; equation of this line is psi is equal to one by three; and equation of this line is psi is equal to zero.

(Refer Slide Time: 08:37)



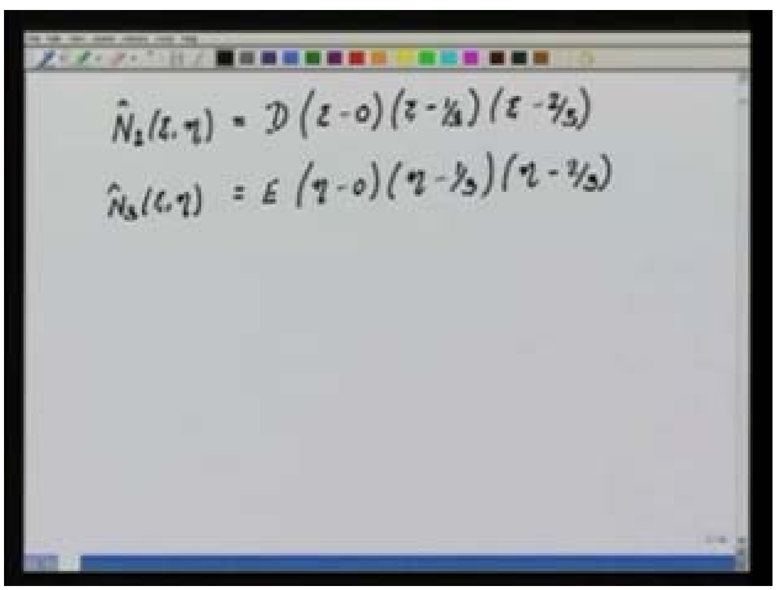
N_2 hat, which is a function of psi and eta is, let us say, some constant D. It has to vanish on the line psi is equal to zero, so psi minus 0, into (on the line psi equal to one-third) psi minus one-third, into (on the line psi is equal to two-third) psi minus two-third.

(Refer Slide Time: 09:00)



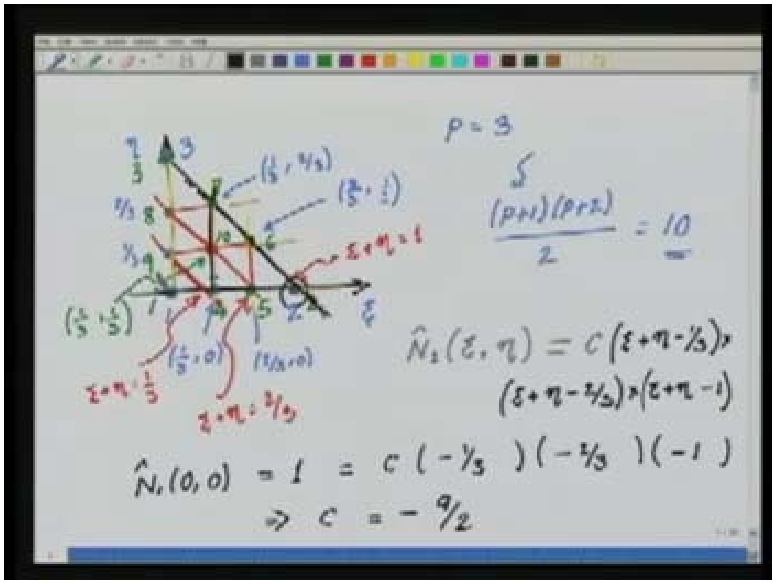
N_3 vanishes on this line, vanishes on this line and vanishes on this line. Equations of these lines are: eta is equal to zero, eta is equal to one-third, eta is equal to two-third. We should have a value one at the node three.

(Refer Slide Time: 09:21)



N_3 will be some constant E into eta minus 0 into eta minus one by three into eta minus two by three.

(Refer Slide Time: 09:43)

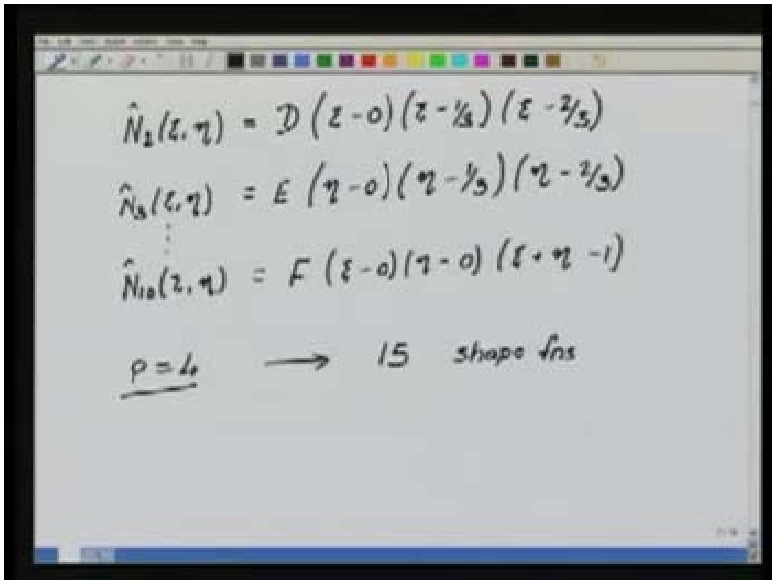


If I have to find N_4, N_4 has to be one at this point and zero at all other points. If N_4 has to be taken to be zero at all other points, N_4 has to be zero on this line. It has to be zero on this line and it has to be zero on this line. Equation of this line is psi is equal to zero. Equation of this line is psi plus eta is equal to minus one. Equation of this line is psi plus eta is equal to two-third.

By the same token, if I want to construct $N_{8,}$ N_8 will be 1 at the node 8 and it should be zero at all other nodes, which means that it has to be zero. If I take it to be zero on this line, zero on this line, which takes care of all these nodes and zero on this line, then N_4 is taken care of.

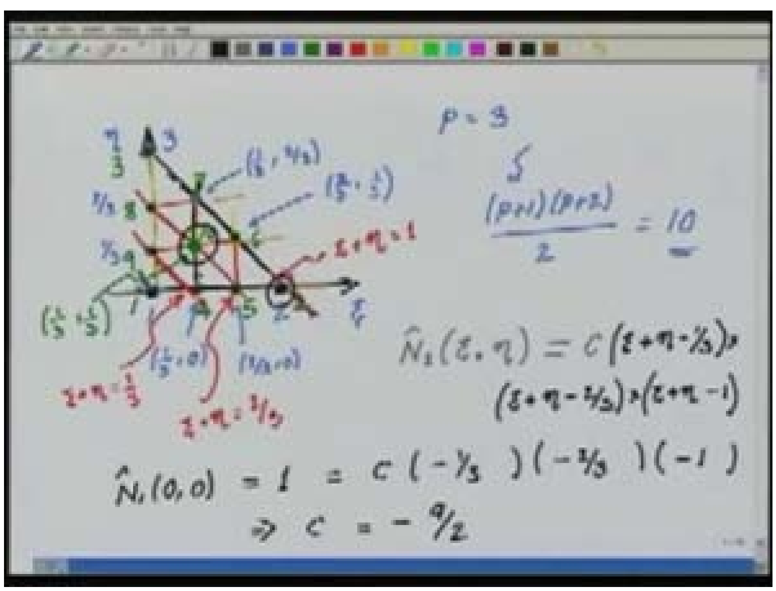
Similarly, I can define N_{10}. N_{10} has to vanish on the three edges of the triangle. If I have to talk of N_{10}, N_{10} will be zero along this line, which is eta is equal to 0, 0 along this line which is psi is equal to 0, 0 along this line which is psi plus eta is equal to 1.

(Refer Slide Time: 11:19)



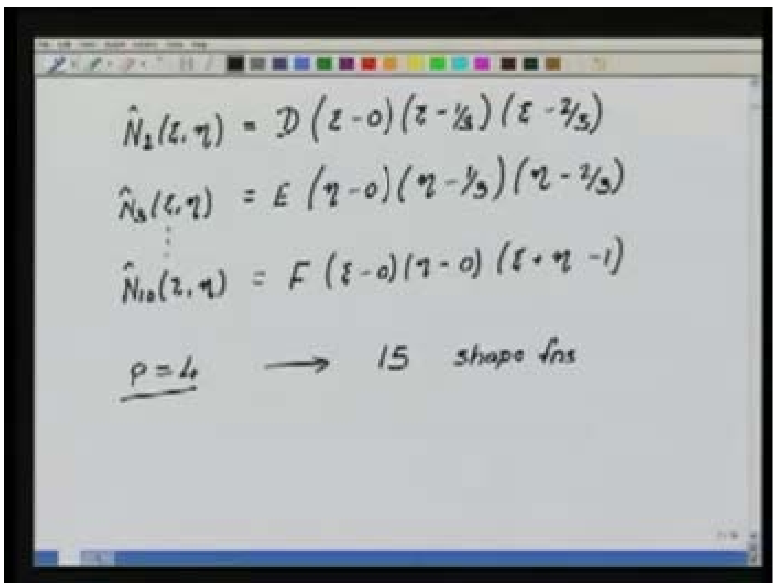
N_{10} is the function of psi and eta is equal to F into psi minus 0 into eta minus 0 into psi plus eta minus 1. I can construct this way, all the shape functions that we need in the master element. If I want to now go to P = 4, P = 4 will require, by our definition of (P+1) (P+2)/2 monomials, to define a fourth order polynomial. This will require 15 shape functions to be defined.

(Refer Slide Time: 12:08)



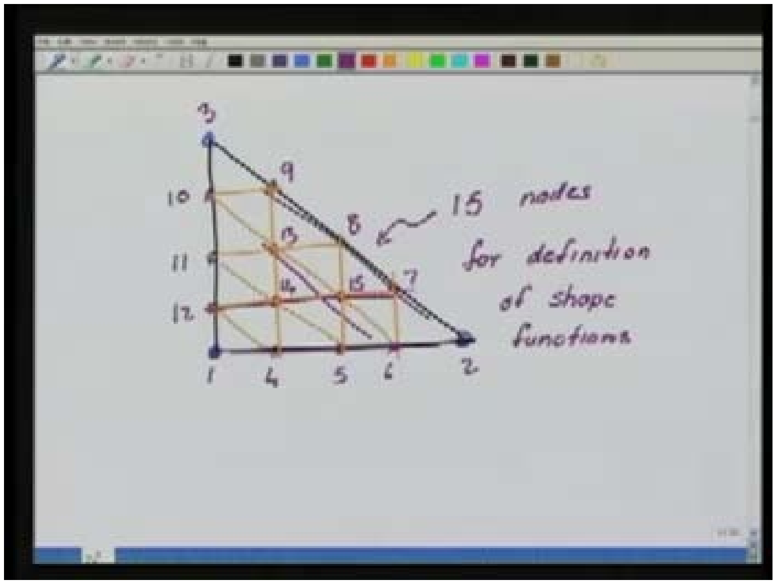
Let me give some notations. These corner degrees of freedom are called the vertex degrees of freedom. These degrees of freedom on the edge are called the side degrees of freedom and this interior degree of freedom, which is ten, is called the interior degree of freedom. Ten is non-zero in the given element. It is going to be zero outside this element. Ten is called an internal bubble function. These edge functions are going to be non-zero only in the two elements, which share this edge. So these are called side bubble functions. The vertex functions are non-zero in all the elements, which have this vertex as a common vertex.

(Refer Slide Time: 12:59)



Let us now see how to do the P=4 case.

(Refer Slide Time: 13:17)



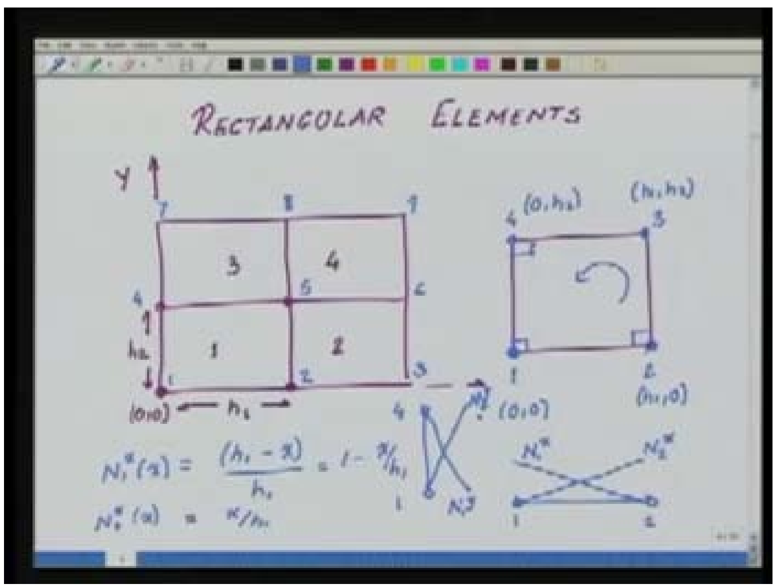
We will make these vertices. When I project the shape functions on the edges of the triangle, they should become equivalent to the one-dimensional fourth order shape functions (defined in the one-D case) on this edge. They are given by specifying five equally spaced points on the line P+1. So I have these three points here. Then I will have these three points here and similarly, three points here on these edges.

If we count all these points, I have 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. As far as the vertex and edge functions are concerned, there are only twelve side functions. So I need to have three more functions, which are now internal bubbles. We connect these points with straight lines on opposite edges. I have constructed these additional internal points in this grid. I define the shape functions for all these points. There are exactly three internal points and fifteen points with respect to which I define the shape functions. Here I have fifteen nodes for the definition of shape functions. I follow the same procedure as before to define the shape functions.

For example, if I have to find the shape function corresponding to this one, let us give the numbering 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 and 15. Let us say I am interested in the eleventh one. The eleventh one should be such that it is one at this point and zero at all other nodes. How can I choose four lines such that the eleventh shape functions vanishes on all other points? I can choose this line, I can choose this line, I can choose this line and I can choose this line. So the shape function should vanish on this line, in this line and this line and this line. Looking at the equations of these lines and multiplying them together, I will get the fourth order shape function corresponding to the eleventh point. I can keep on constructing the higher order approximations to any order that I wish by using this kind of a structure. It can be done in the master element and immediately used.

We are now, in principle, in a position to construct any triangular Lagrangian basis function of any order that we wish to obtain. And the beauty of these basis functions is that we have exactly the number of functions that we need to have a complete definition of the polynomial of the given order.

(Refer Slide Time: 17:31)



Now we are going to go to another family of shape functions or basis functions, which are very popular and are probably used more than triangles. Let us look at rectangular elements. Instead of meshing the simple domain that I had taken earlier with triangles, I have meshed it with rectangles such that the size here is h_1 and the size here is h_2.

Let us take a generic element here. This is element 1, 2, 3, and 4. So the rectangular element will have four corner nodes or four corner vertices. Everything has to be done or defined with respect to these corner vertices. The simplest thing we can do is to take a generic rectangular element.

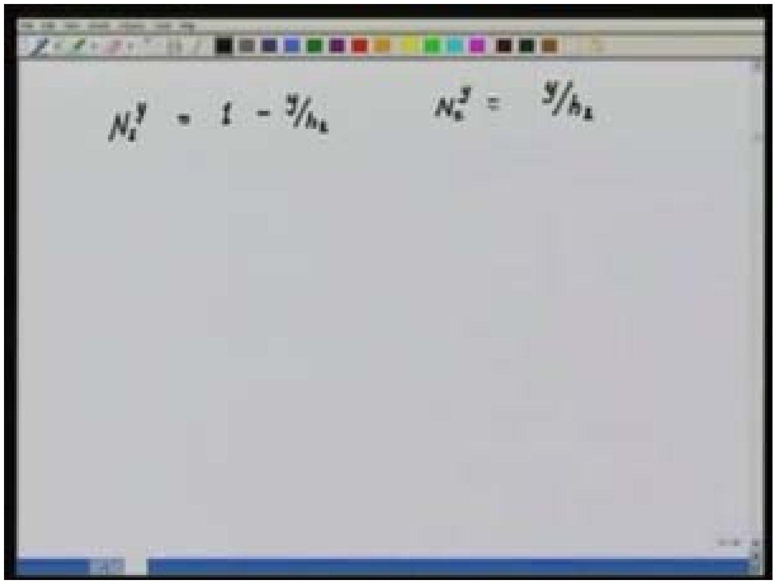
For simplicity, I am taking the first one. I will give a coordinate axis here, x and y, such that the first node has point locations (0, 0). This is the first node, second node, third node, fourth, fifth, sixth, seventh, eight, and ninth.

I take these four nodes and I will have 1 for the element, 2 for the element, 3 for the element, and 4 for the element. So I am taking the first element and by 1, 2, 3 and 4 here I mean the local numbering, just like we did for the triangles. The coordinates here are going to be (0, 0), (h_1, 0), (h_1, h_2) and (0, h_2). I am doing a counter clockwise numbering. What are the simplest basis functions or simplest element shape functions that we can define? They have to be defined with respect to these four vertices. Let us take this element apart because these angles are 90^0.

Let us imagine I have taken these two edges apart - this edge and this edge. Let us say this edge is with the nodes 1 and 2 and this edge is with the nodes 1 and 4. On each of these edges, as we had said for triangles also, the shape functions should be such for the two-D domain that the projection on the edge becomes a one-D shape function.

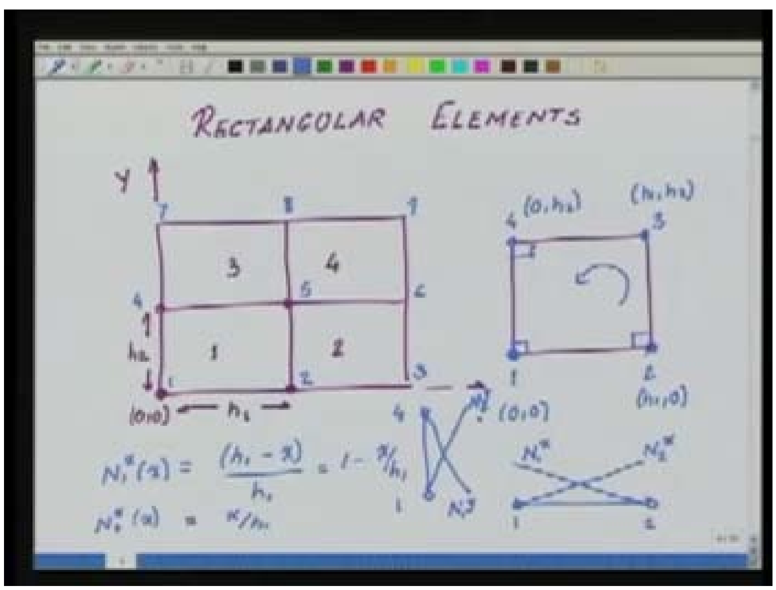
Let us take the same principle here. We have to define the shape functions in such a way that the projections on the edge become the one-D shape function. On this edge it will become this one-D shape function. Let us say these two linear are N_1 of x and N_2 of x. Similarly, here I will have this function. This will be N_1 of y and this will be N_2 of y. I have defined the linear on each of the edges. What is N_1 of x? As a function of x on this edge, it is quite easy to define the shape function. It should vanish at this point. It should be equal to 1 here. I will define it as (h_1-x) divided by h_1, which is 1 minus x by h_1. The second one, N_2 of x as a function of x (because this is a one-dimensional shape functions and along this edge there will only be functions of x), this will be equal to nothing but x divided by h_1.

(Refer Slide Time: 22:11)



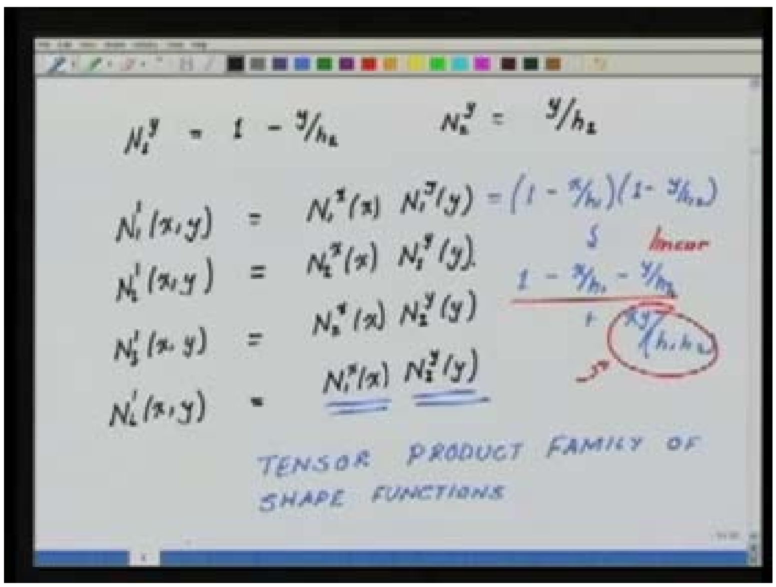
Similarly, N_1 of y should be 1 here, 0 here at the point y equal to h_2. By the same token, it becomes 1 minus y by h_2 and N_2 of y becomes y by h_2. This is the simple definition.

(Refer Slide Time: 22:43)



Using these one-D shape functions that we have defined along this line and along this line, we take the product of these functions to get all the functions. If I take the product of N_1 of x, that is, the function on this line corresponding to this node and function on this line corresponding to this node, which is N_1 of y, by the definition of N_1x and N_1y, this function is 1 at this node and vanishes at this node, which means it vanishes at this node also. It satisfies all our constraints that the function should have a value 1 at this node and 0 at all other nodes. Similarly, I go to this node. This node lies as a second node on this line, so I take N_2 of x. On this line, this is the first node and so I take $N_1$of y and by definition, $N_1$of y is going to vanish here and here; $N_2$of x is going to vanish here. So it vanishes at all other points and gives me a value 1 at the second. N_3 of x lies as a second node of this line. So $N_2$of x and it lies as a second node of this line and $N_2$of x into N_2 of y should do the job and the fourth one should be N_1 of x into N_2 of y.

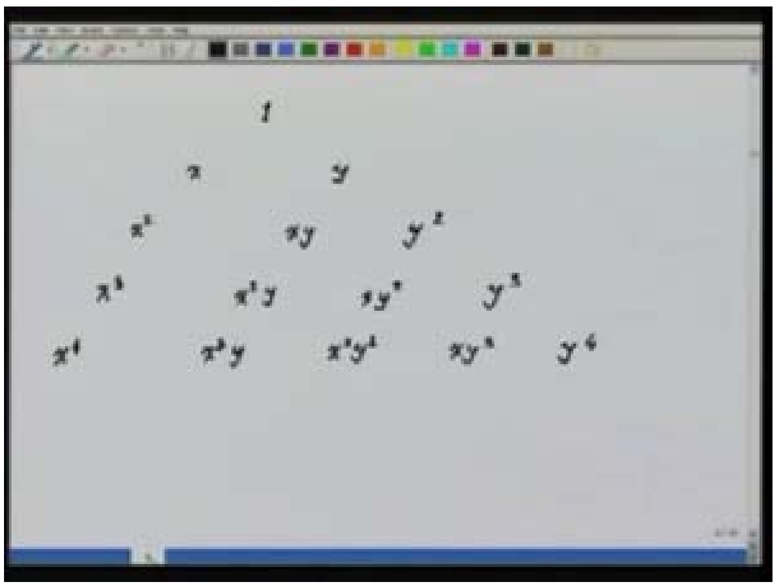
(Refer Slide Time: 22:43)



We are going to define our shape functions in terms of this: N_1, which is a function in the element x and y in the element 1, equal to N_1 of x into N_1 of y which is the function y. N_2 in the element 1 has a function of x and y is equal to N_2 of x into N_1 of y. N_3 in an element 1 is equal to N_2 of x which is a function of x into N_2 of y which is the function of y. N_4 in element 1 which is the function of x and y is equal to N_1 of x and $N_2$of y. I have defined these four shape functions with respect to the four vertices of the rectangle and this is the minimum we can do, because these functions have to be defined by the logic that we have been following with respect to these four vertices. By definition, these are products of functions in the x direction and functions in the y direction. Since, they are defined as products of one-D functions defined in the x and y directions, they are said to be Tensor Product Family of shape functions.

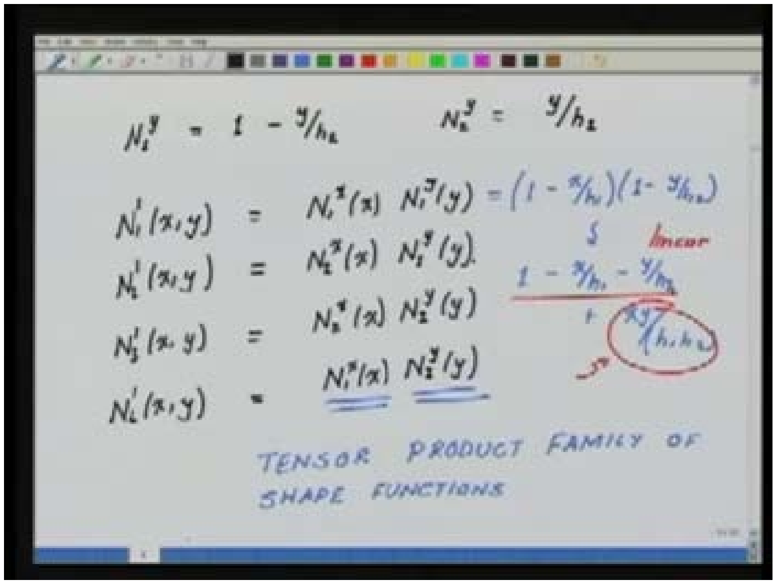
Let us look at some features of it. N_1 of x is nothing but 1 minus x by h_1 and N_1 of y is 1 minus y over h_2. So if I take this product, this is going to be 1minus x by h_1 minus y by h_2 plus xy by h_1h_2. This part is linear in x and y. This part is bilinear, that is, it is not linear, it is more than linear in x and y. The basis functions or shape functions seem to represent more than the set of functions we need to represent completely or to define a linear. That is, they contain more terms than just the linear.

(Refer Slide Time: 27:33)



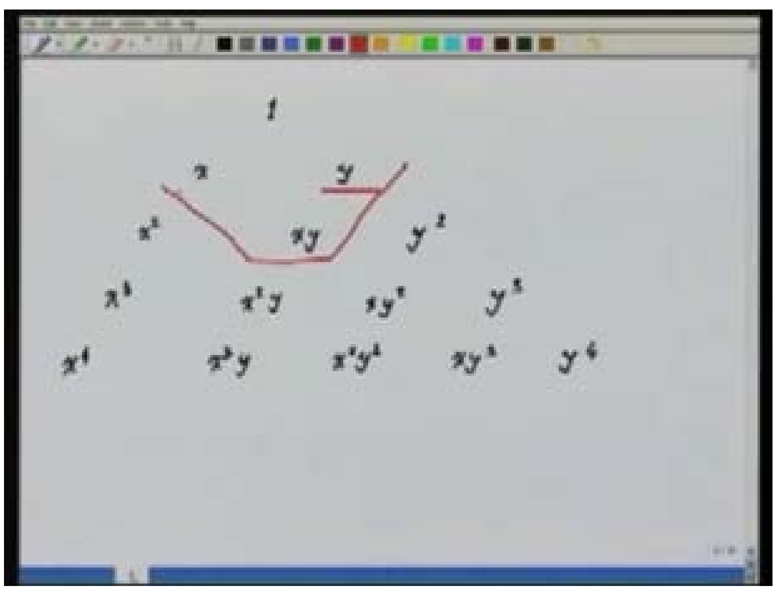
We have the Pascal triangle here: 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, (I will write up to the 4th order) x^4, x^3y, x^2y^2, xy^3 and y^4.

(Refer Slide Time: 28:12)



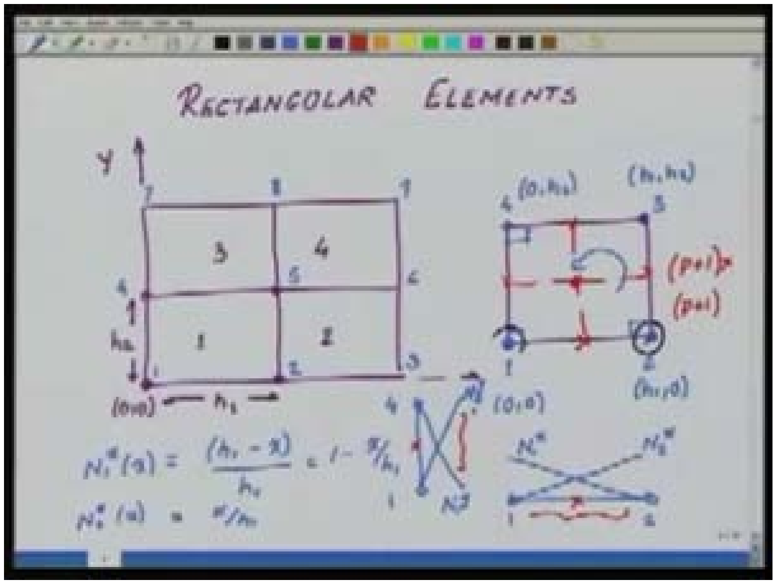
If I expand all of these functions N_1 N_2 N_3 $N_{4,}$ I will find the same feature that they will contain the linear part in terms of 1, x and y plus they will also contain the product of x and y.

(Refer Slide Time: 28:26)



If I come to the Pascal triangle, what happens is, if I look at the representation of these functions, they will not only represent the linear, they will also contain the xy part. That is more than what is really required to define the linear polynomials.

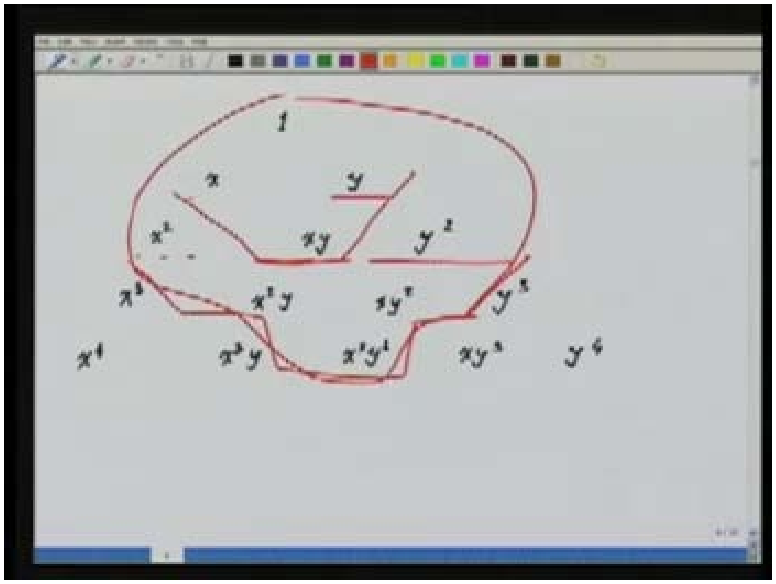
(Refer Slide Time: 28:52)



Similarly, if I define the quadratic shape functions, I will simply add intermediate points on each of these edges. We connect these points just like we had done for the triangles. I will get an additional node in the middle. There are nine such functions that are products: three functions on this edge, three functions on this edge; three into three and I have nine and these nine functions are nicely given here in terms of these nodal values. If these nine functions are to be given in terms of the quadratic, I will define the quadratic shape functions on the edge and multiply for the two edges to get the quadratic shape functions for the element.

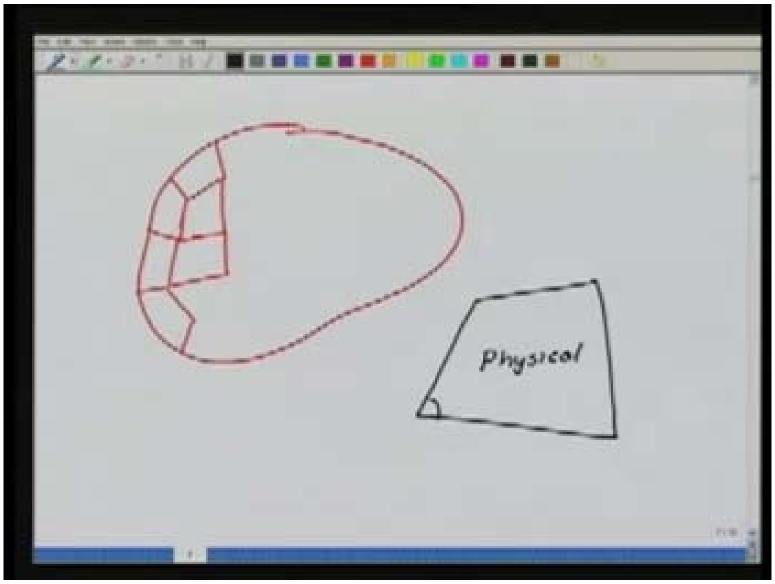
Here the feature is that I get P+1 functions in one direction into P+1 functions in the other direction. That is, I will get P+1 squared functions. For the linear I had 2 into 2 which is 4, for the quadratic I will have 3 into 3 which is 9, for the cubic I will have 4 into 4 which is 16. I am increasing the number of functions as I am increasing the approximation.

(Refer Slide Time: 30:30)



If we look at the quadratics by the same formula that we have followed, that is, the same line of approach, the quadratics should represent this much to be complete, but it turns out that they not only represent this much, they go and represent this whole set. That is, the quadratic definition will include all these functions. It will go beyond the quadratic and have these extra functions. These additional functions are also in the representation above the requirement of completeness. This is a feature of the Tensor Product Family.

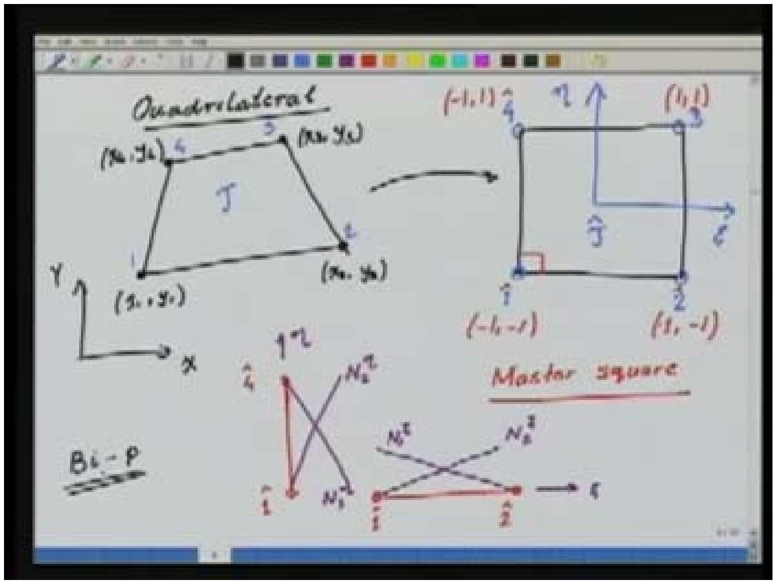
(Refer Slide Time: 31:25)



It is not always that we will have such a nice rectangular domain. Let us take this kind of a domain. If I try to mesh this domain with four nodded entities, this mesh will have quadrilaterals. It will be a mesh of quadrilaterals, not of rectangles. When I have a mesh of quadrilaterals, then we do not have the luxury of having these two perpendicular edges for which we define these individual functions and take the product. I cannot do anything at the physical level, that is, at the level of the physical element.

For that we map this physical element to a master element. So we will have to define the master element now for these quadrilaterals. We define the master element in such a way that I can use this tensor product representation to get the shape functions in the master element.

(Refer Slide Time: 32:41)



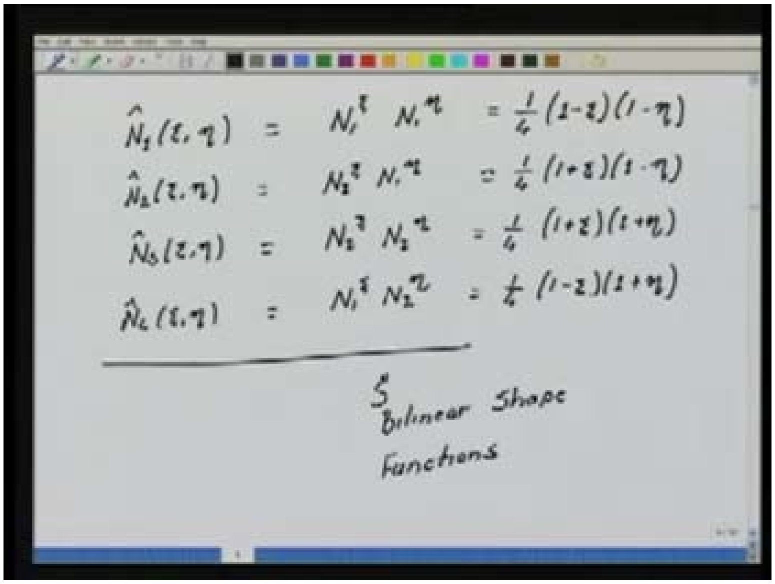
Let us say this is the physical element, which is a generic quadrilateral, which has n nodes (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4). Let us say, I have the xy coordinate system somewhere. We are going to take it to a master element, which is defined as this. This is psi. This is eta. This is node 1 in the master element, node 2 in the master element, node 3 and node 4. This is the generic element tau. This is the master element tau hat and this is the node 1 in the physical element, node 2, node 3 and node 4.

Node 1 will have coordinates (-1, -1). Node 2 will have coordinates (1, -1). Node 3 will have coordinates (1, 1) and node 4 will have (-1, 1). The four nodes in the master element – 1, 2, 3, 4, map to, let us say, 1 hat, 2 hat, 3 hat and 4 hat nodes with these coordinates. The master element is now a square and is called a master square.

How do I define the shape functions with the master element? The master element has edges that are perpendicular to each other. So I will take these edges out with the nodes 1 hat, 2 hat, 1 hat and 4 hat and then on these edges, I can redo the whole job as I had done earlier. I will define these linear functions on this edge, which is in the direction psi, on this edge, which is in the direction eta. I will define this linear again.

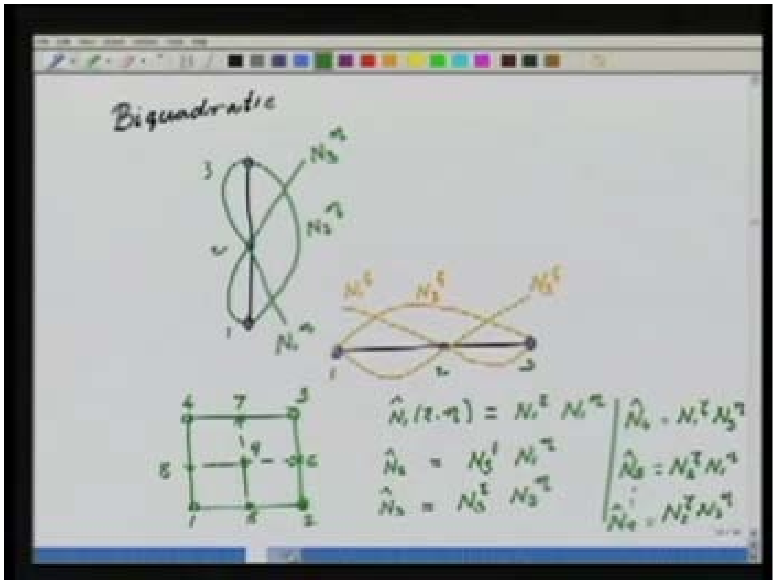
This is N_1 psi. This is N_2 psi. This is N_1 eta. This is N_2 eta. This way I can define these one-D shape functions on the psi edge and the eta edge. In terms of the products of these one-D shape functions, the shape function for the master element can be defined. If I have to define the bilinear or the linear approximation, it is a tensor product and it is called Bi-P approximation. Which means, it is P in one direction and P in the other direction. It is a product of these two Ps.

(Refer Slide Time: 36:38)



When P is equal to 1, it is called a bi-linear approximation. N_1 hat, which is a function of psi and eta, will be nothing but N_1 psi into N_1 eta. N_2 hat as a function of psi and eta is equal to N_2 psi, N_1 eta. N_3 hat, which is a function of psi and eta, is N_2 psi, N_2 eta. N_4 hat psi and eta is equal to N_1 psi into eta. We can define the bi-linear shape functions here. It will turn out to be 1/4th of (1 minus psi) into (1 minus eta). This one will be 1/4th of (1 plus psi) into (1 minus eta). This one will be 1/4th of (1 plus psi) into (1 plus eta) and this will be 1/4th of (1 minus psi) into (1 plus eta). So this way, I can define the bi-linear shape functions for the master element. Similarly, we can define the quadratic shape functions by quadratic shape functions.

(Refer Slide Time: 38:27)

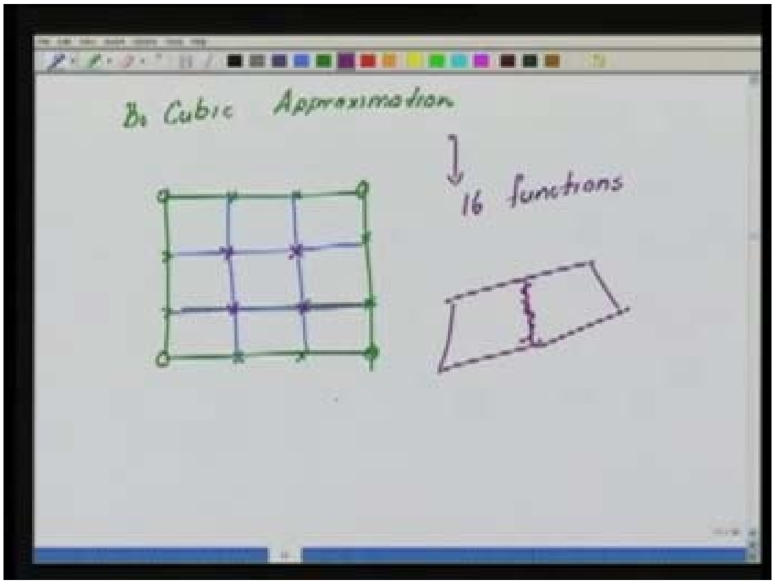


Let us look at biquadratics. As we had done earlier, this is the psi edge. This is the eta edge. I will add these additional nodes on the edge at the mid side and I will define these one-D quadratic functions on the edge. This becomes N_1 psi. This becomes N_2 psi. This becomes N_3 psi. On the eta edge, I will similarly define N_1 eta, N_2 eta and N_3 eta. I have three functions on each edge. So I have these nine functions defined on the master element with respect to these nine points. I will call this point 1, this is 2, this is 3, this is 4, and this is 5, 6, 7, 8 and 9. Depending on which line the point lies on, I will find the corresponding shape function.

For example, N_1 hat as a function of psi and eta should be equal to (it has the first node of the psi side, first node of the eta side) N_1 psi into N_1 eta. Similarly, N_2 hat, lies on the third node of the psi side and the first node of the eta side. So it will become N_3 psi into N_1 eta. To define N_3 hat: (N_3 hat lies on the third node of the psi side and the third node of the eta side) N_3 psi into N_3 eta. N_4 hat is equal to N_1 psi into N_3 eta. Similarly N_5 will be equal to (see it is a second node for the psi side and first node for the eta side) N_2 psi into N_1 eta.

To get N_9 hat: N_9 hat is equal to (it lies on the second node of the psi side and second node of the eta side) N_2 psi into N_2 eta. I am able to define all the nine basis functions with respect to these definitions of N_1 N_2 N_3 and so on, depending on P on the edge.

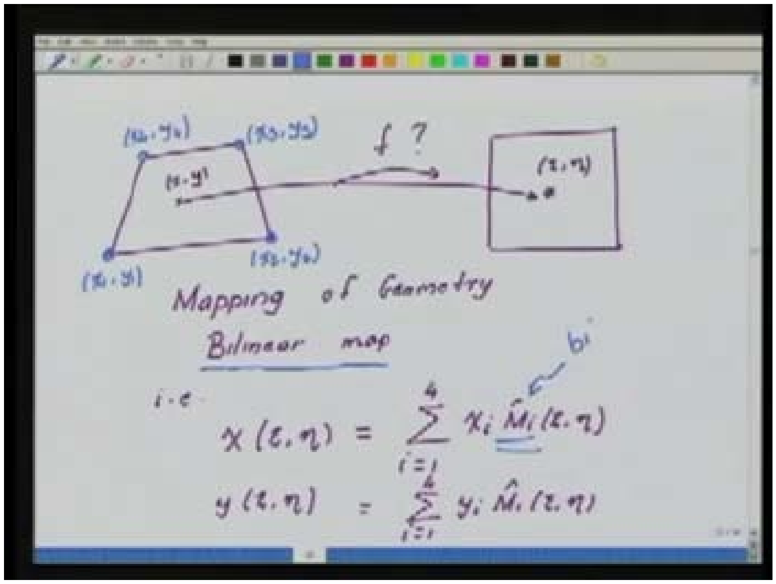
(Refer Slide Time: 42:41)



Similarly, I can construct bi cubic approximation. If I look at the master element, I will have these corner vertices. On this edge I will have two more equally spaced points, just like we had done in the triangles. Connect these by lines. We will get four more points lying in the interior and we will get a grid of sixteen points. Bi Cubic will have sixteen functions. We can continue this to whatever order we want.

Constructing this basis functions in the generic quadrilateral element has to be done at the master element level and we follow the same principle that we had followed for the rectangular element and we get the shape functions. Question is will it give me a continuous approximation? Answer is yes, because, if I have the next element setting here and if I have two elements like this, the shape functions of both these elements should become equivalent to the one-D shape function on the edge. Both these functions on this edge will match from either side because they have to become equivalent to the same one-D shape functions. The approximation will match because the functions match. Continuity is not a problem here in this definition.

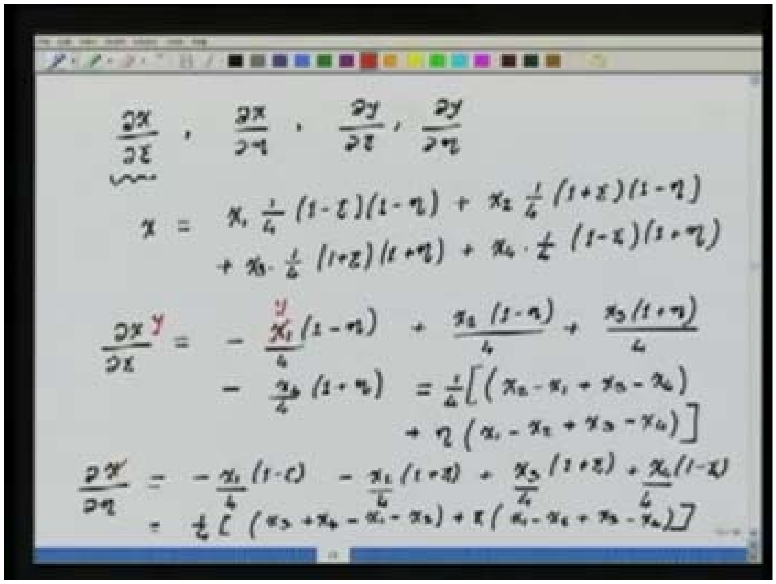
(Refer Slide Time: 44:46)



We have not discussed how we map this quadrilateral to the master element. What is the kind of mapping that gives me this? This is a domain with straight edges. For the mapping of the geometry, in this case, as long as I do not have curved elements, we are going to use bilinear map. That is, x at any point psi and eta in the master element will be sum of i is 1 to 4, x_i M_i (I am deliberately writing it as M_i) as a function of a psi and eta, y as a function of psi and eta is sum i is 1 to 4, y_i M_i hat as a function of psi and eta. Where are these x_is, y_is? This is the (x_i, y_i), this is (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4).

Why have I written M_i? Because I want to bring out clearly that mapping need not always be of the same order as the approximation. Here, Irrespective of whether I am using a cubic approximation or a fourth order, that is, bi cubic or bi quadratic, the mapping is always going to be bilinear for this kind of a domain, where the quadrilateral is a straight edged quadrilateral. So the mapping should always be done with respect to the bilinear shape functions, while the approximation can be done with respect to any shape function that I wish. These M_is are nothing but the bilinear shape functions.

(Refer Slide Time: 47:45)



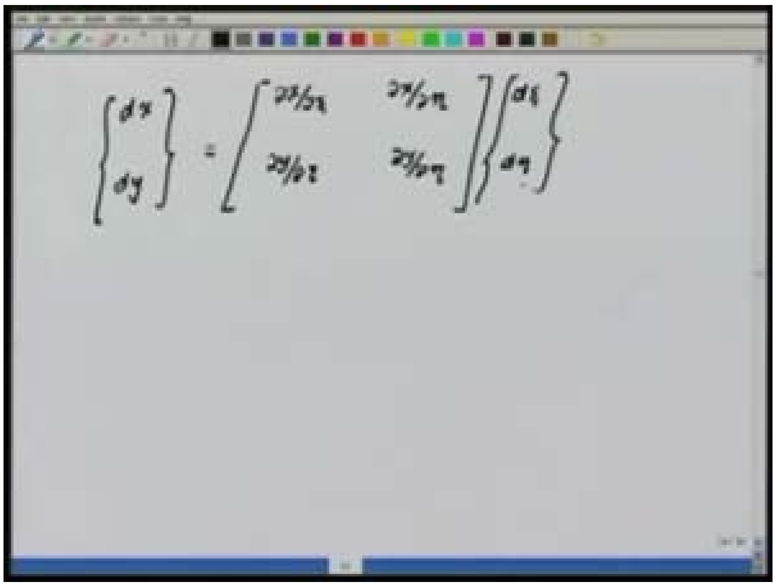
We see that as far as our computations are concerned, we finally want to put everything in a computer program. As far as the computations are concerned, we need quantities like del x divided by del psi, del x divided by del eta, del y divided by del psi and del y divided by del eta because, we have converted, we have mapped all our functions from the physical domain to the master domain. So why not do all the integration and other procedures which are required to get the stiffness matrices and the load vectors in the master element? These quantities are required. How do I find these quantities?

X will be equal to x_1 into ¼ of (1 minus psi) into (1 minus eta) plus x_2 into ¼ of (1 plus psi) into (1 minus eta) plus x_3 into ¼ of (1 plus psi) into (1 plus eta) plus x_4 into ¼ of (1 minus psi) into (1 plus eta). y is also written similarly, in terms of the y. So these are our Ms - M_1, M_2, M_3 and M_4.

Del x divided by del psi becomes equal to minus x_1 into (1 minus eta) by 4 plus x_2 into (1 minus eta) by 4 plus x_3 into (1 plus eta) by 4 minus x_4 into (1 plus eta) by 4. So this one will be equal to, if I write it in terms of psi and eta, $1/4^{th}$ of (collect all the constant parts) (x_2 minus x_1 plus x_3 minus x_4) plus eta into (x_1 minus x_2 plus x_3 minus x_4). Similarly, del x divided by del eta is equal to minus x_1 into (1 minus psi) by 4, minus x_2 into (1 plus psi) by 4, plus x_3 into (1 plus psi) by 4, plus x_4 into (1 minus psi) by 4. If I collect, it will be 1 by 4 into ((x_3 plus x_4 minus x_1 minus x_2) plus psi into (x_1 minus x_2 plus x_3 minus x_4)).

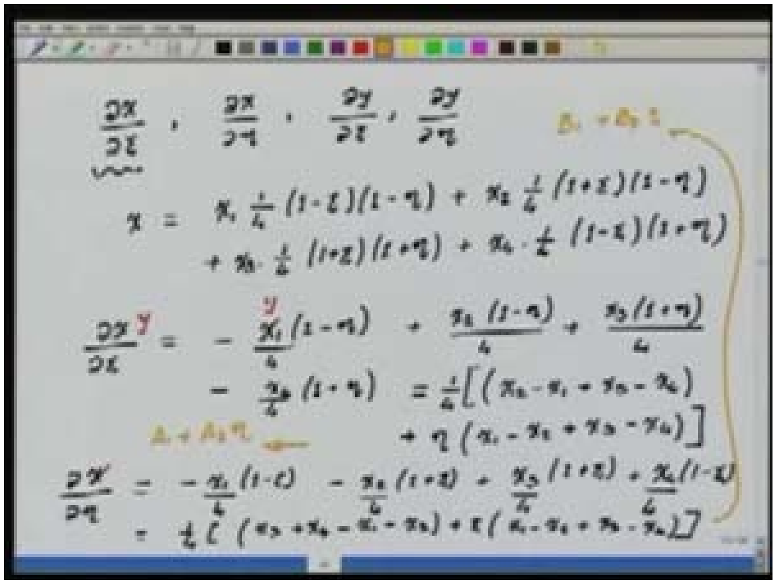
I can get the quantities corresponding to y by replacing this by y and replacing this by y_1, y_2 and so on. Similarly, I replace this one by y and these by y_1, y_2 and so on. So del x divided by del eta, del x divided by del psi, del y divided by del eta, del y divided by del eta can be now obtained. Once I have obtained these quantities, I use them to get the matrix of the transformation.

(Refer Slide Time: 52:12)



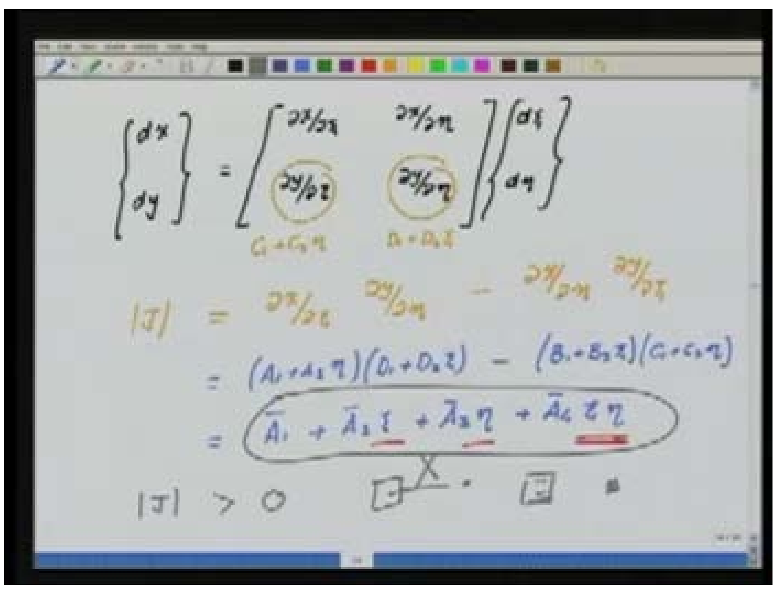
I would like to write dx, dy is equal to (del x divided by del psi, del x divided by del eta, del y divided by del psi, del y divided by del eta) into (d psi, d eta).

(Refer Slide Time: 52:42)



If I go back and look at this one, I can write this expression as A_1 plus A_2 eta and this expression becomes B_1 plus B_2 psi.

(Refer Slide Time: 53:08)

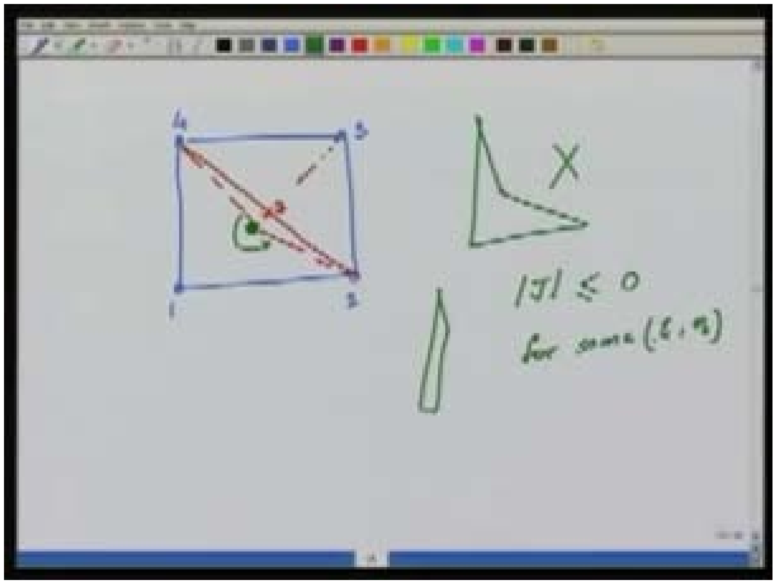


This one is going to be C_1 plus C_2 eta and this one is going to be D_1 plus D_2 psi. Now I have to compute the Jacobian, because Jacobian is needed for the transformation of the integrals from the physical domain to the master domain. This will be equal to (del x divided by del psi into del y divided by del eta) minus (del x divided by del eta into del y divided by del psi). Del x divided by del psi is A_1 plus A_2 eta. This one is D_1 plus D_2 psi, where the $D_{1,}$ D_2 are again obtained from the previous expression. Del x divided by del eta is B_1 plus B_2 psi and the second one is going to be del y divided by del psi, that is, C_1 plus C_2 eta.

In the product, it will be A_1 bar plus A_2 bar psi plus A_3 bar eta plus A_4 bar psi eta, where the bars are obtained like this for us: A_1 bar is A_1D_1 minus B_1C_1 and so on. The bottom line is that the Jacobian is no longer a constant. Even with this bilinear map, this is no longer a constant because it is now a function of psi eta and the product psi eta. While in the triangular element, when we did the linear map, the Jacobian turned out to be a constant.

When we have to do the numerical integration, we have to account for this part of the Jacobian. Jacobian is also a bilinear in terms of the psi and eta and because it is a bilinear in terms of psi and eta, we would like as such, that the Jacobian should always be greater than zero, because that is what is physical and that positive area maps to a positive area. It is not that the area can become negative or a given area cannot map to a point. An area will map to another area, but this will be a positive number, the ratio may be less than one or greater than one but nevertheless, it is a positive number and the ratio of the two areas is nothing but the Jacobian. When we say Jacobian is zero, it means the area maps to a point, to which we not aligned, we cannot have. It is unphysical. We would like to avoid elements, geometries of quadrilaterals for which this Jacobian can become a negative or zero at a point. Who stops this from becoming negative or zero? I can have some combinations of psi and eta given this $A_{1,}$ $A_{2,}$ $A_{3,}$ and A_4 that are the four nodal coordinates, and will define A_1 bar, A_2 bar, A_3 bar A_4 bar in such a way that the Jacobian could be negative at a point.

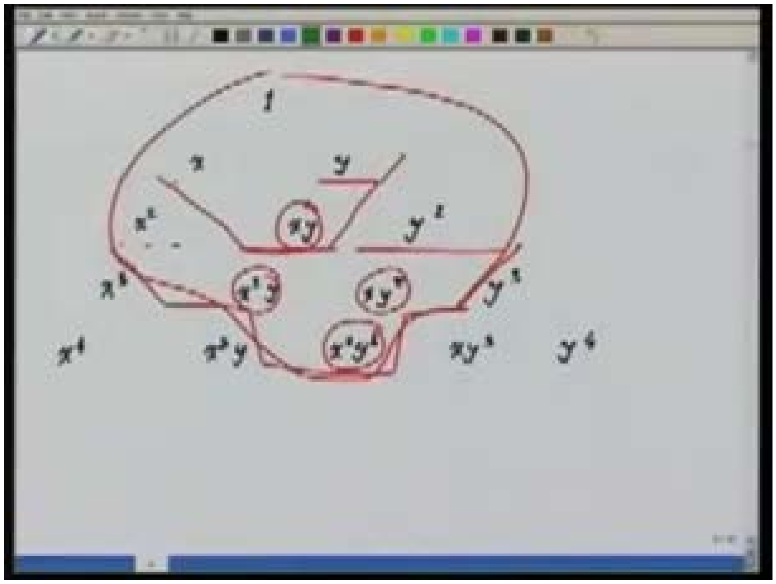
(Refer Slide Time: 56:44)



It is quite easy to show that this can only happen, when (let us take this as the initial physical element) I have 1, 2, 3, 4, these four points and I decide to start moving point 3 inwards. It will turn out that point 3 could lie on a triangle, degenerate triangle. If I go beyond this, that is, if point 3 moves to this point here, then the Jacobian would be negative at some points. Geometries of quadrilaterals for which one of the angles becomes an obtuse angle are not allowed.

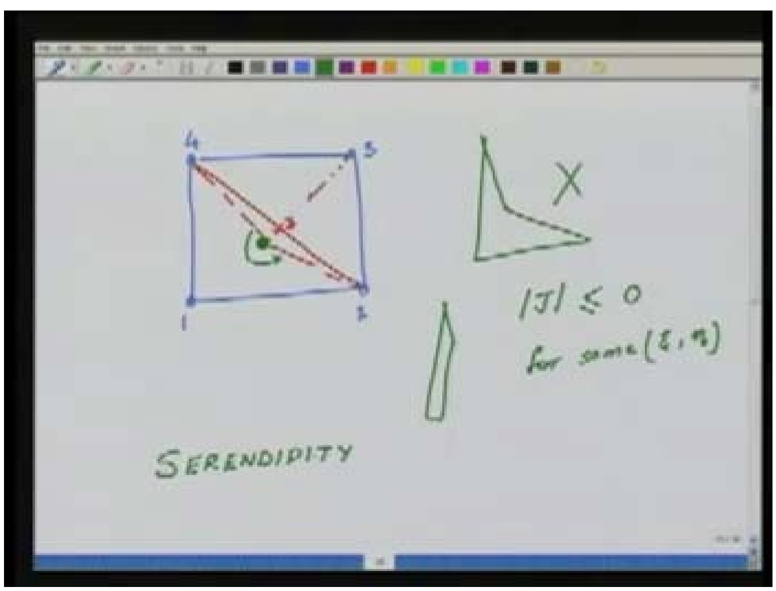
One has to be very careful when making the mesh and ensure one does not get this kind of a quadrilateral domain. We can always convert it into two quadrilaterals for which the quadrilateral is convex, that is, the angle is certainly lesser than or equal to 180 degrees. We do not want quadrilaterals for which angles are large. We should not have these kinds of quadrilaterals – very large or very small. In general their performance will be pretty bad. So while making the mesh we have to ensure that these quadrilaterals are as good as possible. The angles are not too large and the angles are not too small. In this case, we will have a point where Jacobian is less than or equal to 0 for some psi and eta. This is a very important point that has to be kept in mind, when we are using quadrilateral elements.

(Refer Slide Time: 59:00)



As far as the definition of the quadrilateral is concerned, these rectangular elements, we have over-done the job. That is, we have in the Pascal triangle, accounted for more terms. The question is can we redefine our basis functions or the shape functions over the quadrilateral in such way that we do not over-do the job. That is, can we cut out some of these extra terms?

(Refer Slide Time: 59:25)



The answer is yes. That comes by using what we call the ‘serendipity family of functions’. Completeness is guaranteed and these extra terms are cut down.

So in the next class, as far as the approximation is concerned, we are going to discuss the serendipity family of shape functions. Then we are going to now consolidate everything in terms of the finite element approximation in an element. We will then move on to look at how we are going to use this in a computational regime. That is, how numerical integrations are done for the quadrilateral or for the triangular element.