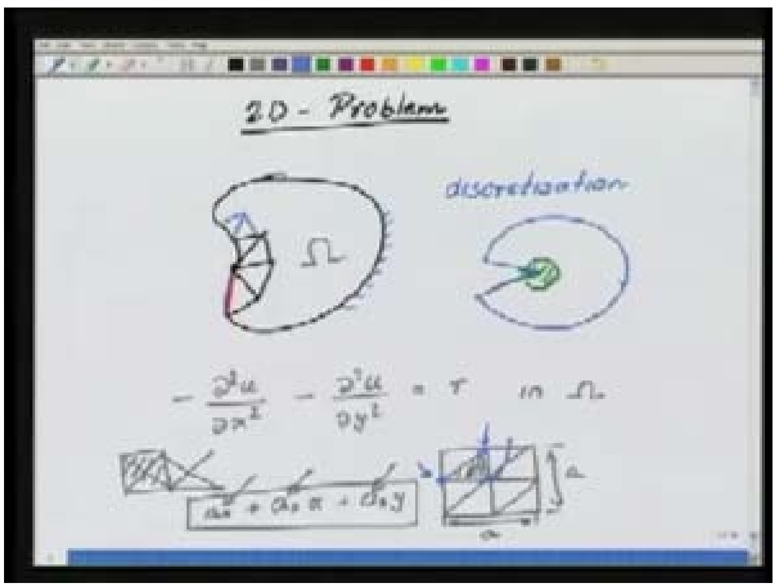
Finite Element Method

Prof C. S. Upadhyay
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Module 7 Lecture 2

In the previous lecture, we had started talking about the two dimensional problem, where, we had obtained the weak formulation for a given partial differential equation which arises in the two dimensional case.

(Refer Slide Time: 00:47)



We had considered the generalized problem of steady state heat conduction and from there we went ahead and used all the tools which we had developed for the one dimensional problem to obtain the variation formulation for the two-D problem. We saw what are the major differences between the one-dimensional problem and the two-dimensional problem.

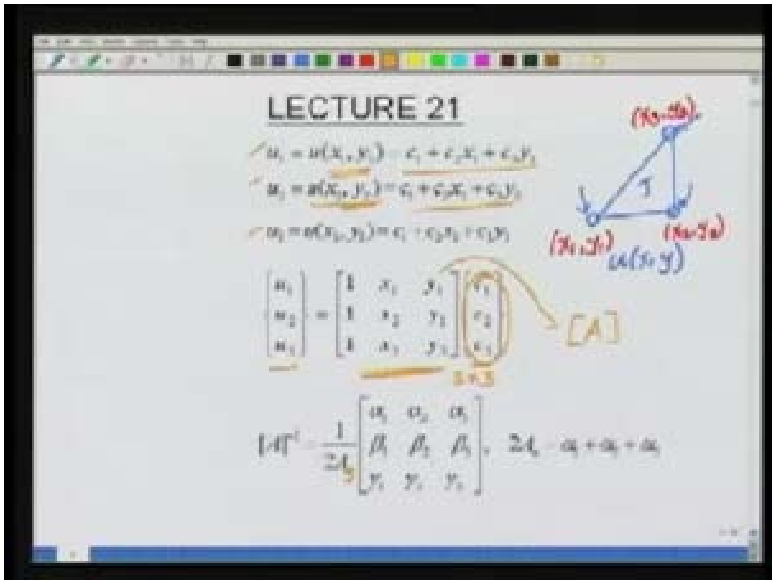
Let me recap. The major differences are that here we are talking of a two dimensional domain, so the geometry of the domain now plays a role and how well we resolve the geometry of the domain is another issue that has to be handled. For example, I mesh my domain with triangles. This is my domain. I know that these triangles with straight edges will only approximate this boundary curve. So I have to ensure that this approximation of the boundary curve is decent. That is, I have a sufficiently refined mesh, that is, there is sufficiently large number of small triangles at the boundary so that these curves are taken care of. We will also see other approaches of exactly representing these boundary curves wherever it is possible. The geometry was the problem and this leads to the discretisation error as far as the definition of geometry is concerned. Next thing was that in application of the boundary conditions in the one-D case, we had talked about the boundaries being two points, the end points of the domain. Here, the boundary could be a line, a curve, or a contour. So the boundary conditions have to be now imposed on a close contour. So this is another difference between the one dimensional and two dimensional cases.

Another thing we had discussed is that if I have a domain like this, you see that because of this corner here in the domain, there is going to be a singularity in the solution that you obtain. Irrespective of what the loading is, I am going to get, depending on the boundary condition on these two edges, I am going to get a singularity in the solution that you obtain in the vicinity of the corner. So these are certain things we have to keep in mind. Now let us go ahead and build the finite element approximation for the model problem we have taken. For the sake of simplicity, here I am going to consider the simplified version of that model problem, that is, del to the power of 2 u by del x to the power of 2 minus del to the power of 2 u by del y to the power of 2 is equal to r, which is what we know as the parson equation. I have deliberately chosen, with respect to our earlier model problem, a_{11}=1, a_{22}=1, and a_{12}=0. Two is just to fix ideas. Everything else we will follow. We will take a rectangular domain. In the rectangular domain, we had said, we are going to make a mesh of triangles. This is the simplest possible domain that I can have. Let us say it is a square domain not even a rectangular domain. It is a square domain with sides ‘a’ and ‘a’ and I have made these triangles.

What we had said is, the first thing that we are going to do is construct the crudest possible approximation; the crudest possible approximation was an extension of the linear approximation that we had in the one-dimensional case. So we would now like to create a linear approximation in the two dimensional case. The linear approximation in the two dimensional case would be of this form now. So, the linear in the two-D case will be $a_0+a_1x+a_2y$ and in this linear approximation, what we are going to do is, we are going to construct the basis functions over these elements that we have, over this domain, such that, I have a piece-wise linear approximation in each element, like we had in the one-D case. Over each element here, I would like to construct a piece-wise linear approximation. Piece-wise linear approximation means that over the element, my basis functions, the truncation of the basis functions are linear polynomials. Now how many independent functions do we need in an element in order to completely define a piece wise linear? The answer is that, since this piece wise linear incorporates three unknowns, we need three independent functions. In the two-D case, we required two independent functions in an element. Now we need three independent functions. How do we define these three independent functions?

We had said that we are going to define these three independent functions with respect to the three corner vertices of the element or the three corner nodes of the element. These are the nodes of the element of interest. In the two-D case, I have the triangles as the elements and the corners of the triangles are the nodes. So we are going to define these piece-wise linears with respect to these elements. Let us now go and see how we do this job. What we say is that, let the polynomial, let any function ‘u’ be represented by a linear in the element of interest.

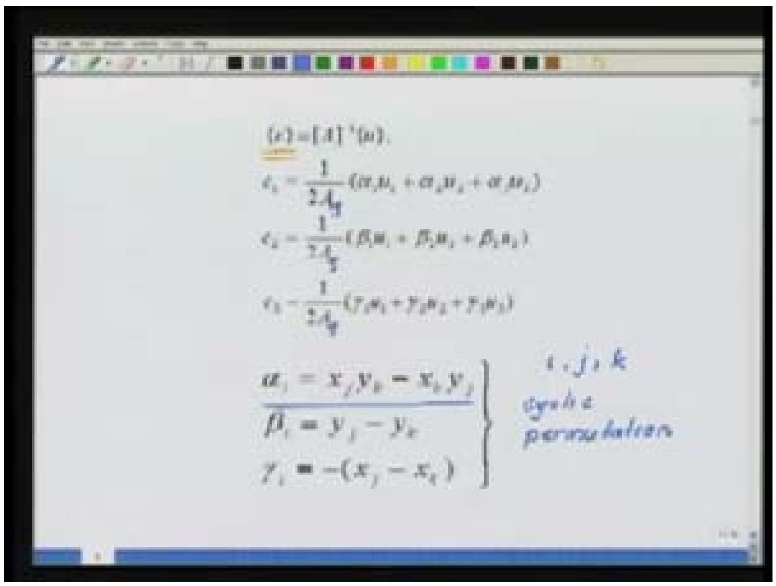
(Refer Slide Time: 07:26)



So my element would be this and as we had said, these are the corner vertices of the element. So this is my element top and I give it some name. So what we want, as we had done in the one-D case is, to interpolate the given function u with a linear in this element. In order to interpolate this function u with a linear in this element, what we do is: u is given by some linear which is $c_1+c_2+c_3y$ in this element. Then what is a value of this linear at these nodes of the element? The nodes of the element have coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3). So, we want a representation of the function that we have taken at these nodes. So the value of the function at these nodes is u at (x_1, y_1) u at (x_2,y_2) and u at (x_3,y_3). What we want now, going by what we had done in the one-D case is, the linear interpolation of this function to match the value of this function at the three nodes. When we have this linear interpolation, we want c_1 plus c_2 x_1 plus c_3 y_1 to be equal to the value of the function at the point x_1 y_1. Similarly, I want c_1+c_2 $x_2+c_3y_3$ to be the value of the function at the point (x_2, y_2) and at the third point. These values of the function at these three points I am going to call as mu_1, mu_2, and mu_3. I assume that I know my function, I know the values mu_1, mu_2 and mu_3, which can be written in terms of the coordinates of the three points and these unknown constants. We want to find these unknown constants c_1 c_2 and c_3. So the three values of u are given as a combination of these unknown constants c_1 c_2 c_3. Now the job is simple: it is a three by three matrix, so we have to invert this matrix to get c_1 c_2 c_3 in terms of mu_1, mu_2 and mu_3. That is about all we have to do.

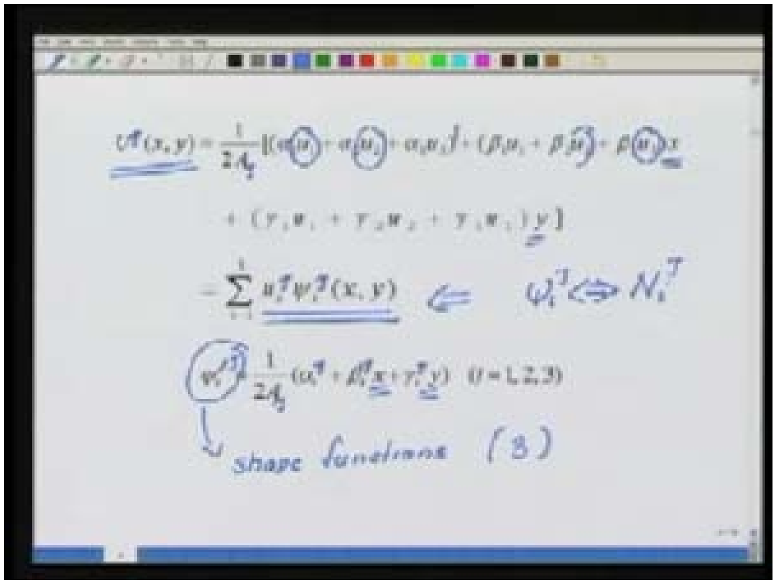
What we end up getting is: I call this matrix as matrix A, so A inverse will be one by twice the area of the triangle, we have called it tau, into some coefficients $alpha_1$, $alpha_2$, $alpha_3$, $beta_1$, $beta_2$, $beta_3$, $gamma_1$, $gamma_2$, $gamma_3$. We can write the area of the triangle A tau in terms of these $alpha_1$, $alpha_2$, $alpha_3$.

(Refer Slide Time: 10:54)



What do we get then? The unknown constants c become as A inverse into this vector mu and after expanding this out, the constant c_1 is 1 by 2 area of the element tau into $alpha_1$, mu_1 (mu_1, is a value of the function mu at the first node which has coordinates (x_1,y_1)) plus $alpha_2$, mu_2, plus $alpha_3$ mu_3. Similarly, I can find c_2 and c_3, where these alpha 'i's are given like this in terms of the three nodal coordinates, that is, $alpha_1$ will be x_2 y_3 minus x_3 y_2, where, i j k are written in a cyclic permutation. Similarly, $alpha_2$ will be equal to x_3 y_1-x_1 y_3 and so on. Similarly, I can find what is $beta_1$ and what is $gamma_I$ in terms of the x_1 x_2 x_3, y_1 y_2 y_3. Once I have these quantities, I can rewrite: the linear interpolation of u over the element tau in terms of the values of the function at the three points u_1 u_2 u_3 as one by twice area of the element tau into $alpha_1$ u_1 plus $alpha_2$ u_2 plus $alpha_3$ u_3 plus the part multiplying x plus the part multiplying y.

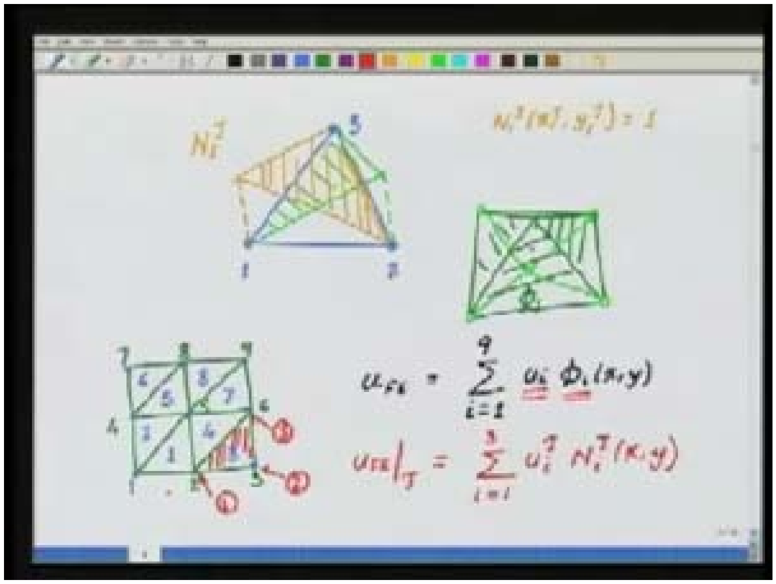
(Refer Slide Time: 12:21)



So this you can say is the part multiplying one, the part multiplying x and the part multiplying y. So what is the next thing? We would like to group all the coefficients or all the monomials multiplying u_1 together, all the monomials multiplying u_2 together, all the monomials multiplying u_3 together. Rewrite u in the element tau in terms of the values of the function at the three nodes into some polynomial Psi$_i$. Psi$_i$ tau is equivalent to what we know as element shape function N_i tau. It turns out that Psi$_i$ tau is given as one by area of the triangle into alpha$_i$ tau plus beta$_i$ tau x plus gamma$_i$ tau y. These Psi$_i$ taus are the shape functions in the element.

How many do we have? We have three of them. So we have the three linear shape functions in the element obtained by doing this procedure. You see that each of the shape functions is a linear showing that this set is a complete set, that is, it can represent any linear polynomial exactly by taking a linear combination of these and showing that these are linearly independent is now relatively an easy job. One can do exactly the same thing that we had done in the two dimensional case. Let us go further; so what will these functions look like?

(Refer Slide Time: 14:55)



Let me just take an example. Let us say this is node 1, this is node 2 of the element and this is node 3 of the element. Here I would like to plot the function psi or N_1. If I take N_1, as you will see, the N_1 has a value one at the node one. So, we will see N_1 tau at x_1 tau y_1 tau is equal to one. It will have a value one at the first node and it will have a value zero at the second node and the third node. So this function is going to be like this. It is like a slanting roof. It has a value one at the first node, zero at the other two nodes. That is, it is zero along this whole line. Similarly, this is function N_1 tau or as I have written in the previous slide as psi_1 tau. Similarly, if I want to draw N_2 tau, N_2 tau will be a function, which has value one at the second node and value zero at the first node and the third node. So this is my function N_2 tau. So it is a roof slanting down from the second and similarly I can draw the third function. These are our element shape functions and you can see that these shape functions for the ith node will correspond to a tent like structure. That is, by piecing together all the shape functions, I get the global basis function phi_i for the ith node of the mesh that we have and this function vanishes on these edges.

It vanishes on this black edges and it is one at the node with respect to which it is defined and it is piece-wise linear and each of the elements shares that node. This, you will remember again, that in the two-dimensional case, a node may be shared by many more than two elements. In this figure that I have drawn there are four elements sharing a node.

This basis function has to be defined with respect to all the elements which share that particular node. With this, now, we have obtained our representation of the linear shape functions. Let us now construct the finite element solution, if I have to do that in an element. I define the finite element solution globally. So, given this phi_i, let me again make a mesh. For our own convenience, let us take an example of a mesh of eight elements. You can take the simplest possible method. So let us say, this is my node 1, this is my node 2, I am doing a numbering of the nodes in the mesh, node 3, node 4, node 5, node 6, node 7, node 8, and node 9.

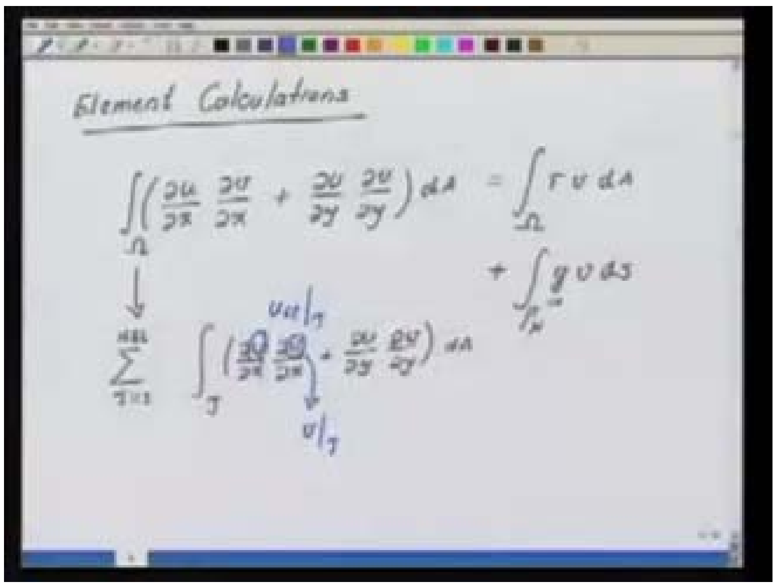
Similarly, I will number the elements, this is element 1, this is element 2, element 3, element 4 and elements 5, 6, 7, and 8. So, I have numbered the nodes and the elements. Now how many basis functions will I have formed by piecing together these piece-wise linears? I will have nine basis functions corresponding to the 9 nodes of the mesh. If I want to make a finite element solution, which is a linear approximation over this domain, then, u_{FE} would be equal to sum of i is equal to 1 to 9 u_i phi_i, which is the function of (x, y). Now, I want to look at the restriction or the part of this finite element solution over some generic element. Let us say I am looking at it over this third element. So, u_{FE} over a generic element tau will be given in terms of the value of this coefficients u_i at the three nodes, which are the N vertices of this particular element, multiplied by the restriction of these basis functions over this element.

So, it can be written as, sigma i is equal to 1 to 3 u_i in the element tau N_i tau (x, y). By this we understand that if I am in the third element, I will have a local numbering for the nodes. If you recollect, in the one-D case too we had defined the global numbers and the local numbers. So this will be my local node 1, local node 2 and local node 3 for the element. Then I see that the u_1 of the element tau is equivalent to the global u_2, u_2 of the element tau is equivalent to the global u_3, u_3 of the element tau, which is the third element, is equivalent to the global u_6.

For every element, I have to obtain the local to global numbering. So remember that same data structures have to be continued here. Local to global enumeration has to be done, which tells me which global degree of freedom does my local degree of freedom correspond to. Similarly, the N_1 tau for this element is equivalent to the phi_2 in this element N_2 tau is equivalent to the phi_3 and in this element N_3 tau is equivalent to the phi_6. We now know the one to one correspondence between what is the global solution and what is that we are doing locally.

Once I have the representation of the finite element solution in the element then I can go and do the next step. That is, I can do my element calculations.

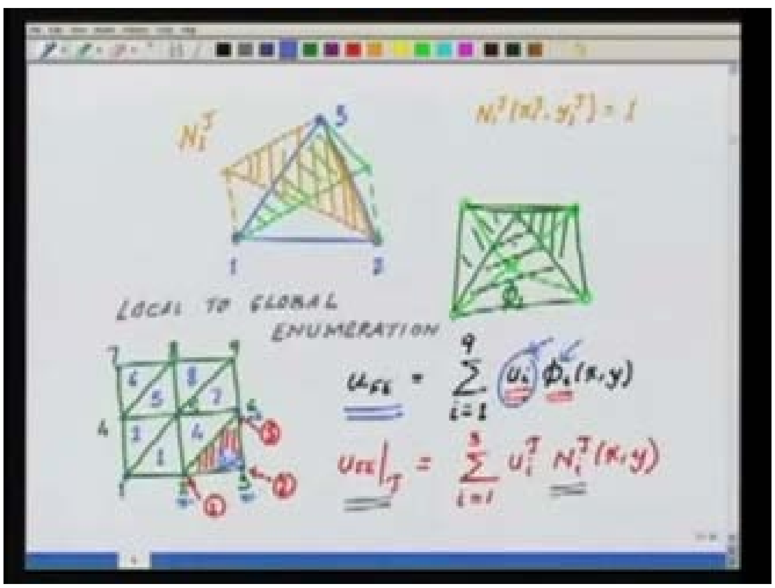
(Refer Slide Time: 23:07)



What do we do for the element calculations? We should remember, if we take the weak form that we obtained and put it for this particular model problem that we have taken, the weak form would be del u del x del v del x plus del u del y del v del y dA would be equal to integral over the domain r v dA plus if you remember we had talked about the Norman part of the boundary, the part were the flux conditions are specified, you will get g v d s. g is the specified boundary flux and v is our test function or the weight function that we have used.

This is going to be our weak form, which is obtained as a specialization of the general weak form we had obtained earlier. Now, what do we know from this? We can now partition as sum over all the elements: sum 1 to number of elements integral over the elements tau of del u del x del v del x plus del u del y del v del y dA. Similarly, we can do the same for the right hand side. Now from the finite element point of view, we replace u with u_{FE} . u_{FE} on the element is given by u_{FE} tau. Similarly, what we do as far as v is concerned is, we take v to be the global basis function itself. So v would be again given as v over tau. Here we are talking of the global basis functions that are going to be non-zero in this element or the global basis functions that correspond to the N_i tau, that is, the shape functions of the element. Essentially, what happens is, as far as the contribution of the element is concerned, the element is going to only contribute to the rows in the global stiffness matrix or in the global equations, which correspond to the phi_is, which are non-zero in the element. Let us go back to our previous figure.

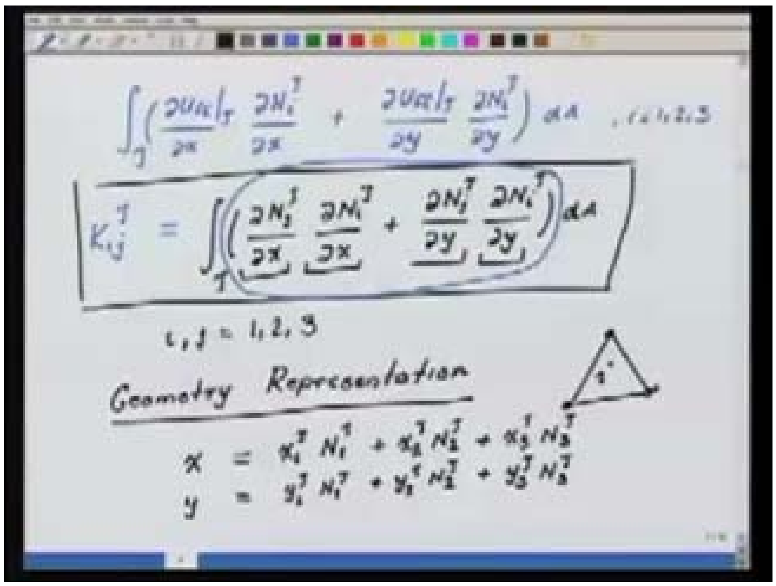
(Refer Slide Time: 26:10)



If I am in element three, then, corresponding to the u_{FE} that we have constructed, I will have a 9 by 9 global system, in terms of the nine unknown u_is. So, as far as element 3 is concerned, phi_2, phi_3, and phi_6 are non-zero in element three. All other phi_is are zero. So my element 3 is only going to contribute to the global equations corresponding to phi_2, phi_3, and phi_6, that is, it is going to contribute to the second, third and the sixth global equation and similarly for the other equations. It is also going to contribute to the second row to the third row and the sixth row, because these are the only coefficients or the basis functions, which are active in this element.

With that understanding, for the element, there are only three active basis functions. Even the u_{FE} of the element is in terms of these three coefficients. The element stiffness matrix will be a 3 by 3 stiffness matrix as far as the piece-wise linear approximation is concerned. This 3 by 3 stiffness matrix for the element is what we would like to obtain. Once I obtain this 3 by 3 stiffness matrix, then we do the usual process of assembly to get my global system. What does this 3 by 3 stiffness matrix correspond to?

(Refer Slide Time: 27:58)

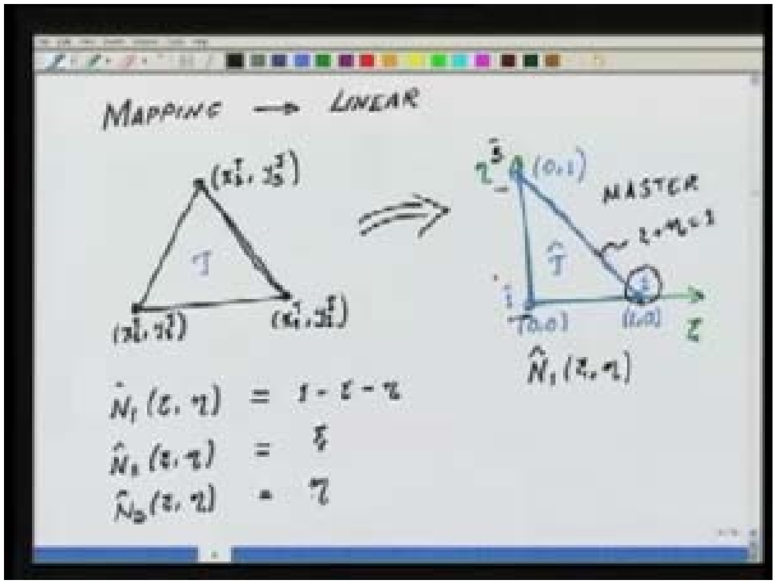


Just like we have done earlier, let us talk of the integral over the element del u_{FE} over the element tau del x (now here instead of the v, we are taking the basis functions which are non-zero in this element, which are nothing but) del N_i in the element tau over del x plus del u_{FE} in the element tau del y del N_i in the element tau del y. These are essentially corresponding to i=1, 2, 3. We will get the three equations from the element. In the three equations, when I substitute for the u_{FE} tau, I will get the three columns, which are there in terms of u_1 tau, u_2 tau and u_3 tau. If I do this, I will get the stiffness matrix entry in the element tau by expanding u_{FE} tau and is given as integral over tau del N_j tau del x del N_i tau del x plus del N_j tau del y del N_i tau del y. So this is going to be my ijth term of the element stiffness matrix, where, ij is equal to 1, 2 and 3.

Everything that we do is exactly the way we had done for one-D. Only thing is, the number of entries increases here and we are dealing with partial derivatives of the quantities of interest. If I have to do this for the standard model problem that we had taken, that is, the generalized one, then we will have to do it with the $a_{11,}$ a_{12} and a_{22} sitting there. The stiffness matrix entries for the element can now be computed in terms of the derivatives of the element shape functions N_i and N_j, where N_i and N_j have been obtained from our interpolation that we have done earlier. Now, if I take the piece-wise linears, we see we see that these derivatives are going to be constant. As far as the piece-wise linears are concerned and the model problem that we have taken, the integrant here is going to be a constant integrant. This constant we can easily obtain corresponding to each i and j. The integral of the constant against the area will be equal to the area times this constant.

As far as this part is concerned, finding the integral is not difficult at all and one can do it very easily and explicitly. However, let us go and build some of the other features, which are going to be important from the point of view of a computational implementation. Let us first talk of the geometry representation. The geometry is going to be represented by x being given in terms of, that is, in the interior of an element tau, if I want to find any point x, it will be given in terms of the coordinates of the end points of the element. It will be given in terms of, as a linear interpolation, x_1 tau N_1 tau plus x_2 tau N_2 tau plus x_3 tau N_3 tau. Similarly, y is equal to y_1 tau N_1 tau plus y_2 tau N_2 tau plus y_3 tau N_3 tau. Now, if you remember, from a computational point of view, it was much easier to work. We could do everything in the physical element, but we decided to map our physical domain physical element in a one-D case to a master element. Similarly, in the case of this two dimensional problem, let us map our physical element into a master element. Let us define the mapping as a linear map.

(Refer Slide Time: 33:55)

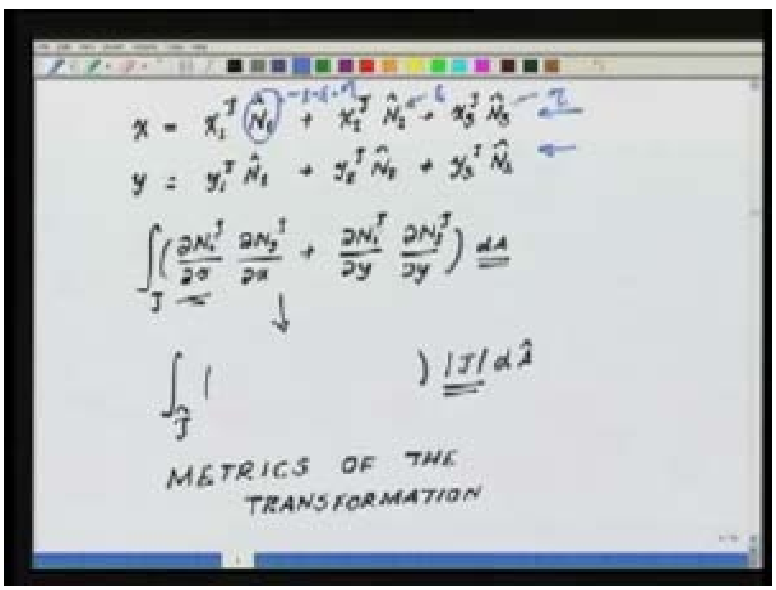


So what happens? I have my physical element, which has these nodes x_1 tau y_1 tau, x_2 tau y_2 tau, and x_3 tau y_3 tau. This, we are going to map to a master element given in terms of the coordinate system psi eta and defined as this or master element is going to be such that the first node has coordinates (0,0). The nodes of the master element I am numbering them with hat. The second node has coordinate (1, 0) and the third node has coordinate (0, 1). So what have I done? I have simply mapped my original physical triangle linearly to this master triangle. That is, the straight edges go to straight edges and the area maps linearly. I have done the linear map and now my next job is to define this map. How do I define this map? In order to define this map, let us take the linear shape functions and map them to the shape functions that we have, that is, the corresponding entities over the master element. I will call this master element tau hat. This is my element tau. So how do I do this mapping? A linear polynomial in the physical element goes to a linear polynomial in the master element. If a linear shape function was one at a physical node, it will be also one at the corresponding master node. If it was zero at a physical node, it will be zero at the corresponding master node and so on. If I have the N_1 tau, the N_1 tau in the master element will become N_1 hat. This N_1 hat will be a function of psi and eta.

Now, what is this N_1 as a function of psi and eta? N_1 hat as a function of psi and eta is quite easy to show, it is a function which is one here and zero along this line because remember that N_1 tau was zero along this line, so similarly N_1 hat has to be zero along this line. This line has an equation psi plus eta equal to one. If it is zero along this line, the function has to be one minus psi minus eta. Check that this will be one at this point where it is (0, 0) and along these two points - two and three.

This is how I have defined my N_1 hat. Similarly, I can define N_2 hat as a function of psi and eta is equal to psi. Again, the function has a value one at the second point and value zero at the first and the third point. This becomes psi, one can check. Similarly, N_3 hat as a function of psi and eta will be equal to eta. By doing this mapping, I have at least in principle constructed the corresponding map versions of the shape functions. How can I define the geometry mapping?

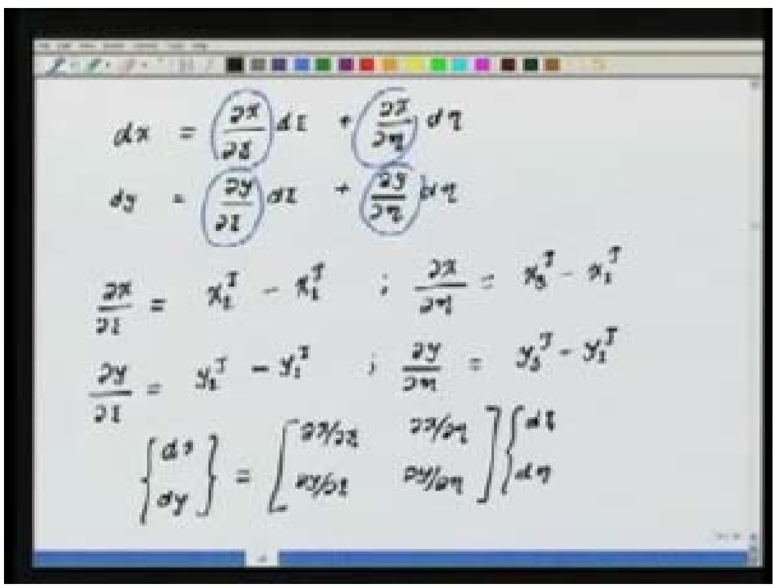
(Refer Slide Time: 38:34)



Geometry mapping becomes x is equal to x_1 of tau into N_1 hat plus x_2 of tau into N_2 hat plus x_3 of tau into N_3 hat and y becomes y_1 of tau into N_1 hat plus y_2 of tau into N_2 hat plus y_3 of tau into N_3 hat. This is how I can give the x and y at any point psi and eta in the master element, the corresponding value of x and y.

Once we have done this mapping, remember that we have to do master calculations. In the master element, if you remember, we had an integral of this type. We would like to convert it to an integral over the master elements. So from the physical I come to the master where I have the integral over the master element tau hat of this integrant. This integrant converted into an expression in terms of psi and eta into (note that the area in the physical element has to be mapped to the area in the master element) and that will be in terms of something called the Jacobian into dA hat. The question is, what is this and how do I use the derivatives in the master element to obtain the derivatives in the physical element? For that we will have to obtain the metrics of the transformation. So how do we obtain the matrix of the transformation?

(Refer Slide Time: 41:14)

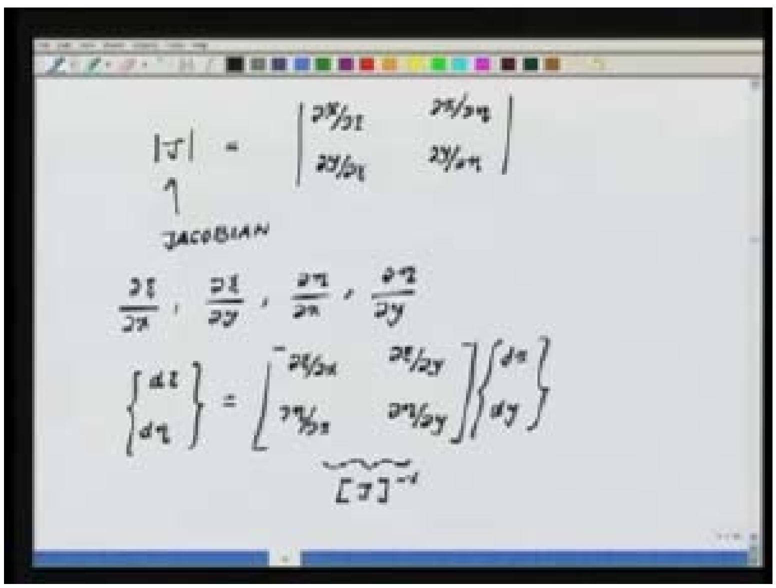


If I say that I want dx, dx will be equal to del x, because x is now a function of psi and eta, del x del psi d psi plus del x del eta d eta. Similarly, dy is equal to del y del psi d psi plus del y del eta d eta. This is because x and y are functions of psi and eta. So, change in x is again in terms of change in psi and eta. Now we have to obtain these quantities. Where are these quantities going to come from? These are going to come from the definitions of x and y that we have given in terms of the three nodal xs and ys in the physical element and the definition of the shape functions in the master elements. One can show from the expression we have there that del x del psi is equal to x_2 of tau minus

x_1 of tau. Let us go back and see. Here, if you remember, N_1 hat was one minus psi minus eta, N_2 hat is psi, N_3 hat is eta.

If I take this definition and go back, dx d psi will be equal to x_2 minus x_1. Similarly, del x del eta is equal to x_3 tau minus x_1 tau and del y del psi is y_2 tau minus y_1 tau del y del eta is equal to y_3 tau minus y_1 tau. Let us now rearrange this thing in a matrix form. So I can write it as, dx dy is equal to del x del psi , del x del eta , del y del psi, del y del eta, d psi d eta. I have simply rewritten this expression in a matrix form. Once I have this expression in the matrix form, then I can talk of the Jacobian.

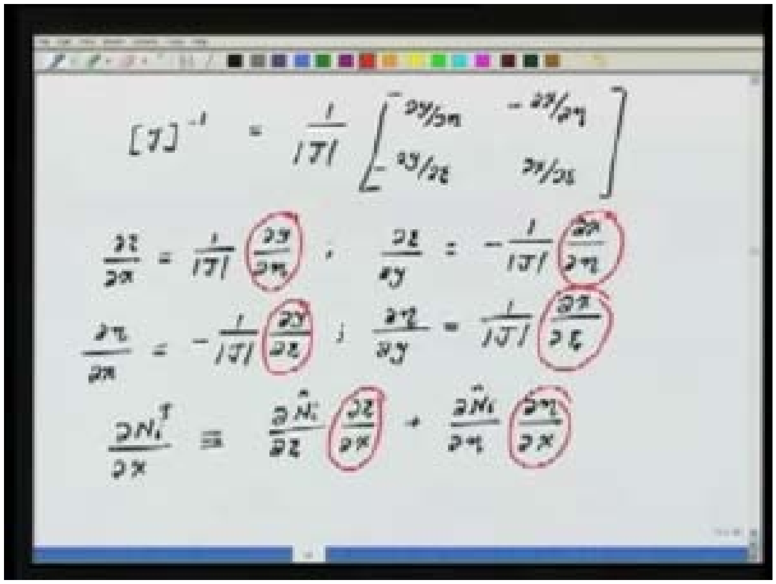
(Refer Slide Time: 44:34)



The Jacobian is nothing but the determinant of this expression of this matrix that we have written. And we would like to obtain expressions for del psi del x, del psi del y, del eta del x, del eta del y. That is, since we can write x and y in terms of psi and eta, we can also write psi and eta in terms of x and y. So we want metrics of the inverse transformation. How do I do it? By exactly following what we have done there, we can get d psi d eta is equal to del psi del x, del psi del y, del eta del x, del eta del y dx dy and this matrix is nothing but the inverse of what I will call as the Jacobian matrix here. We take the inverse of this, bring it on this side and we will get d psi d eta in terms of dx dy.

So, all we have to do is, given this matrix J, we need to find the inverse of that matrix. Now what is going to be the inverse of this matrix, say J inverse?

(Refer Slide Time: 46:58)

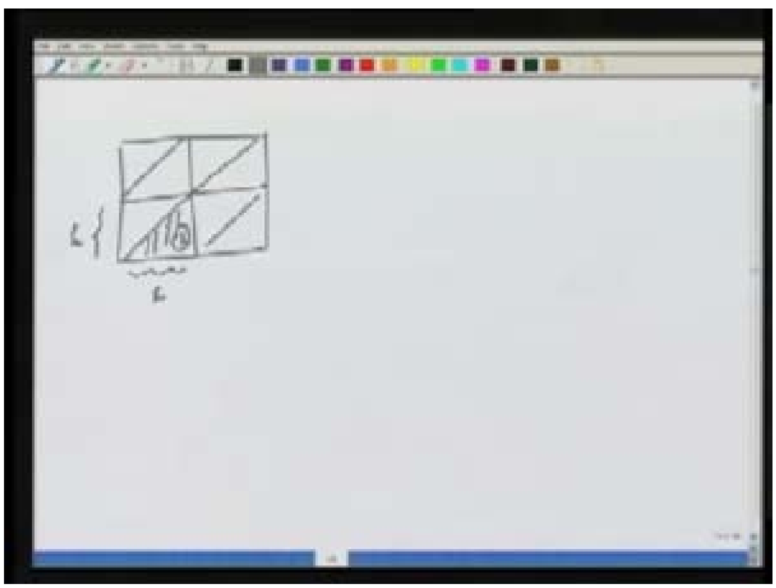


J inverse is equal to one by Jacobian, we interchange the diagonal entries and so I will get del y del eta, del x del psi and take the negative of the signs of the half diagonal entries and so minus del x del eta , minus del y del psi. This is my J inverse and the J inverse is now going to give me my del psi del x and so on. Del psi del x is equal to one by Jacobian into del y del eta, del psi del y is equal to minus one by Jacobian into del x del eta. del eta del x is going to be minus one by Jacobian into del y del psi and similarly, del eta del y is equal to one by Jacobian into del x del psi. Obtain the matrix of the inverse transformation. Why do I need it? Because, if you remember, these quantities are equivalent to saying del N_i hat del psi del psi del x plus del N_i hat del eta del eta del x. Because I can rewrite my N_i tau as a function of psi and eta, that is, N_i hat as a function of psi and eta, then the x derivative of N_i tau is given by this chain rule that del N_i del psi del psi del x del N_i del eta del eta del x. We now need to obtain these quantities. But these quantities are quite easy because we have already obtained each one of these quantities from the definition of x and y.

Given these quantities, I can go ahead and now define what is del psi del x, del eta del x. In this case, you notice that for this triangle, for linear mapping, del y del eta del x del eta del y del psi del x del psi are all constants. Why? Because x and y was linear in terms of psi and eta. Take the derivative of the linear and you will get constant. Because they are all constant, the Jacobian is also a constant. This is true only for the linear maps of triangles. We will see that when we go to quadrilaterals, Jacobian will no longer be a constant. So, given these entities now we can plug them in and obtain all the quantities.

For the element, I have to input the three nodal coordinates the x_1 x_2 x_3, y_1 y_2 y_3 and I can construct the metrics of the transformation. Let us now take a very simple example of the same domain that we had taken. Let us assume that my elements have edges of size h.

(Refer Slide Time: 51:12)



Given these elements of size h, I would like to construct, let us say, for the first element in my domain, the entries of the stiffness matrix using linear approximation.

In the next class, we are going to construct the stiffness matrix of the elements and then we will talk a little bit about the assembly for this special problem. Another important thing that we are going to talk about in the next class is impassing, how to apply the natural boundary conditions, that is, the force boundary conditions. We will also finally discuss how to apply the displacement boundary conditions. Once we do that, then we are not going to be happy with linear approximations in the null element. We would like to extend it to higher order approximations, quadratic, cubic, fourth order and we will see how we can construct the basis functions or the shape functions in an element, corresponding to any order of approximation that we desire and then how to put it in the framework of whatever we have developed here as far as the element calculations are concerned.