Finite Element Method

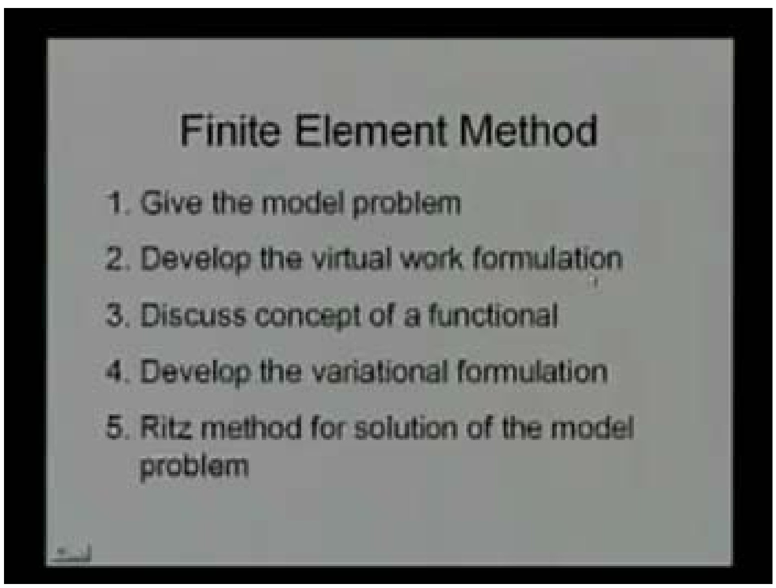
Prof C. S. Upadhyay
Department of Mechanical Engineering

Indian Institute of Technology, Kanpur

Module 1 Lecture 2

In the previous lecture we had talked about the objectives of the finite element method where it is used in an engineering analysis and what are the basics steps involved in the typical finite element analysis. In this lecture, we are going to develop our understanding of the method a little further by looking at a typical model problem.

(Refer Slide Time: 00:45)

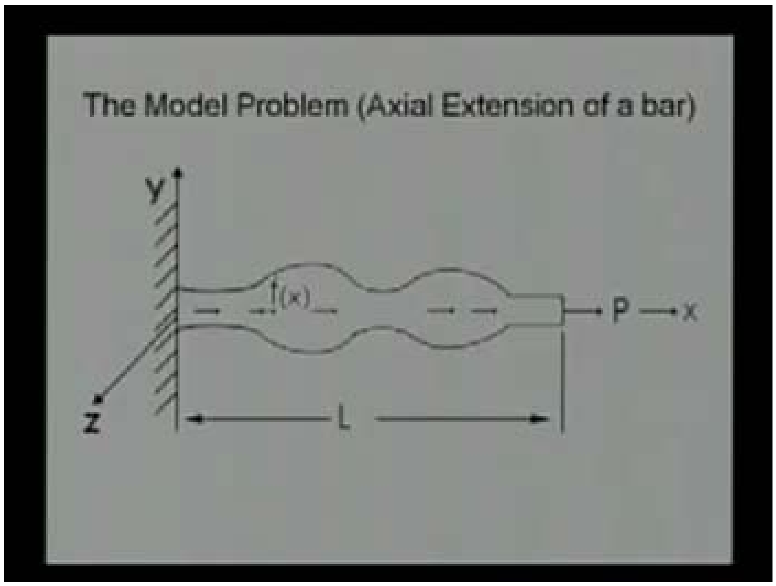


For that model problem we are going to develop virtual work formulation. After the virtual work formulation has been developed we will also discuss another concept called concept of a functional. Using the functional we are going to develop something called the variational formulation and we will show that for at least the model problem of interest both the virtual work formulation and variational formulations are the same. Finally, we are going to take the variation formulation and with respect to this formulation, we are going to develop a method called the

Ritz method. We will develop the variational formulation which we are going to apply to the model problem that we will consider and we will use a method called the Ritz method to obtain a solution to this model boundary value problem.

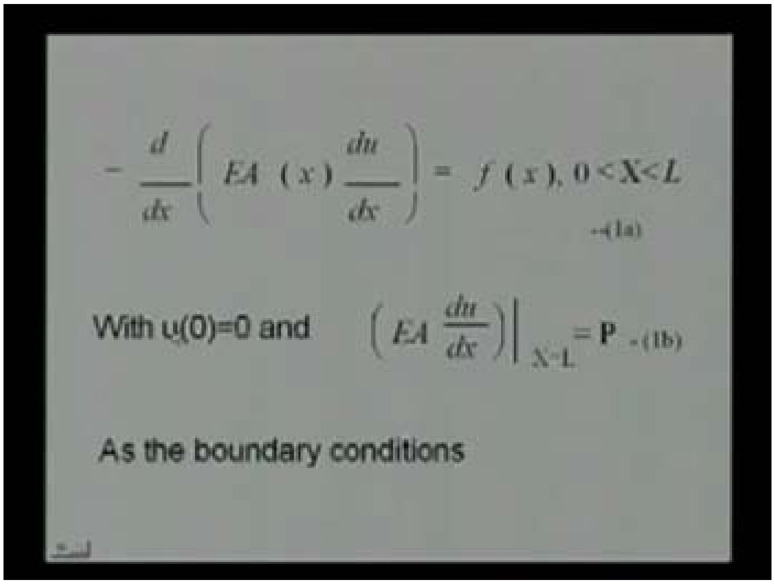
The Ritz method is going to be used here because it is a precursor of the finite element method. Let us take the model problem that we had discussed in the previous lecture.

(Refer Slide Time: 01:55)



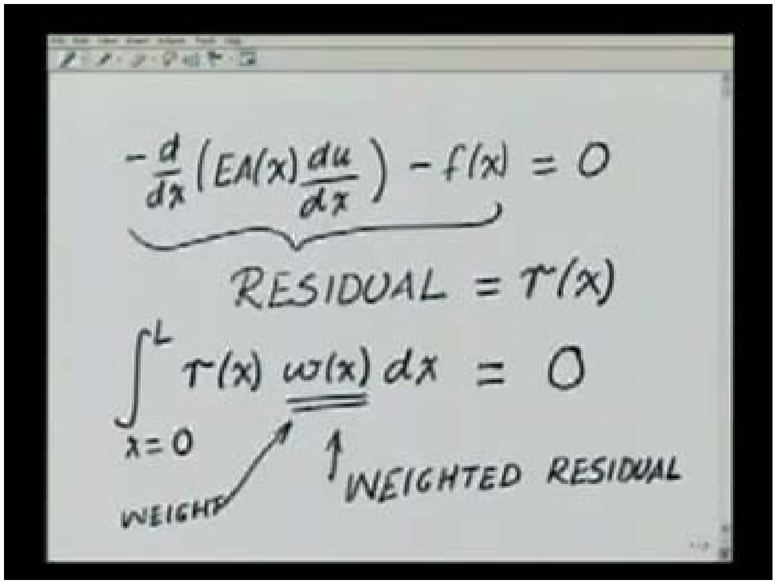
This is the extension of the axial bar. If we look, the problem has been complicated a little bit where in we have added a variable cross section to the bar. The bar has a variable cross section; it is loaded by the distributed force f(x) and an end load P at the point x is equal to L. At the point x is equal to 0 the bar is fixed; that is the displacement at this point is set to 0.

(Refer Slide Time: 02:34)



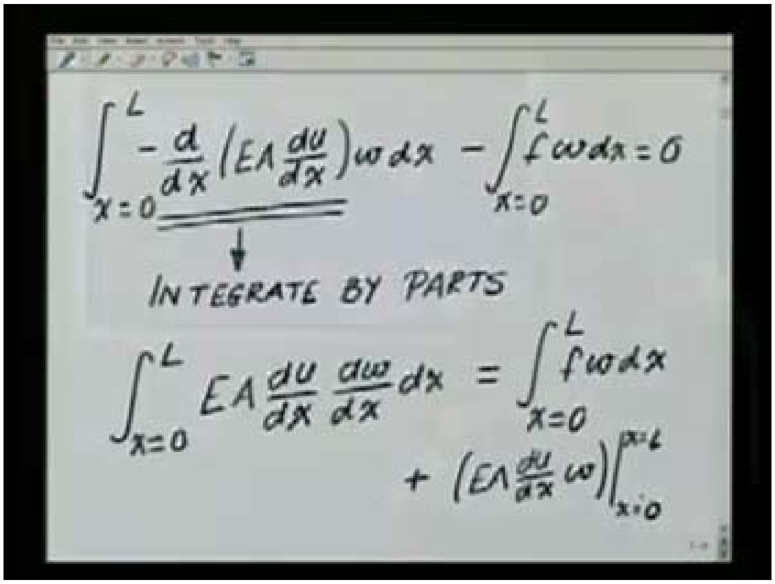
If I now write the model problem in a detailed form, the boundary value problem can be given as equation 1(a); where we have the differential equation given by – d dx of EA which is now a function of x du dx is equal to f of x for all points that lie in the interval 0 to L; u is the displacement of the bar. With the boundary conditions as I have already mentioned, u at the point x is equal to 0 is equal to 0 and the force at the point x is equal to L i EA du dx evaluated at the point x is equal to L is equal to the applied force at the end L equal to P.

(Refer Slide Time: 03:27)



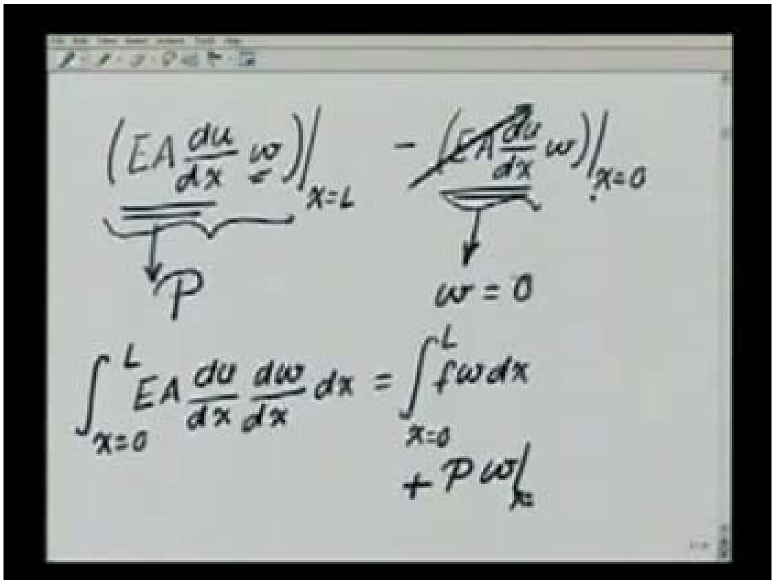
For the model problem given earlier let us now take the differential equation and move both terms to one side. That is, we will get – d dx (EA) which is the function of x du dx minus f(x) is equal to 0. This term that we have put on the left hand side is called the Residual. What we are going to do is, we are going to take this residual and multiply it by a weight function w(x) that is any function w(x) which is admissible. We will define admissibility later on. What do we call this residual? Let me call it by something r(x). Residual is a function of x given by r(x). I take r(x) multiply it with a function w(x) and then I integrate it over the interval. Obviously, this integral is going to be equal to the integral of the right hand side which is equal to 0. When I take this the integral of the residual multiplied by w(x) over the whole domain that is x is equal to 0 to L, this is called the Weighted Residual Formulation. The question is, what do we mean by this weight w(x)? This w(x) is given a name that is the weight function. This w(x) as we will see later on has to satisfy certain minimum smoothness conditions in the domain and certain other conditions on the boundary of the domain. Many people use this weighted residual formulation that we have written here to solve the problem. What we are going to do next is that we will take weighted residual and let us expand and write it again.

(Refer Slide Time: 06:02)



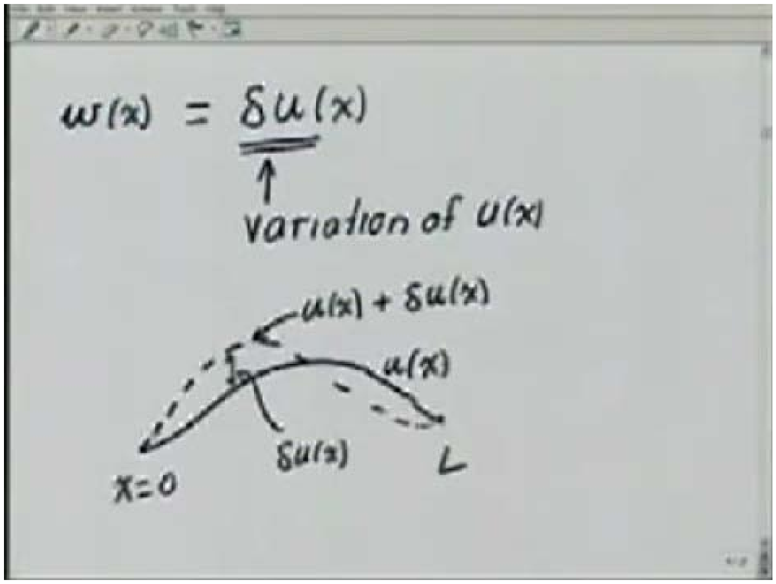
In the expanded form this becomes…. wdx minus integral of x is equal to 0 to L fw dx is equal to 0. We see that in this term second derivative of u is sitting. What we would like to do is pass one of the derivatives from u to w. That is, we are going to do integration by parts for this term. I integrate this term by parts to get EA du dx into dw dx and I will take the required terms on the right hand side. This will be equal to x is equal to 0 to L fw dx plus if I look at it, EA du dx into w evaluated at point x is equal to L minus the value evaluated at point x is equal to 0. What we have done, we have integrated this term by parts which gave us volume integral part that is the interval plus a part which is the boundary term.

(Refer Slide Time: 08:04)



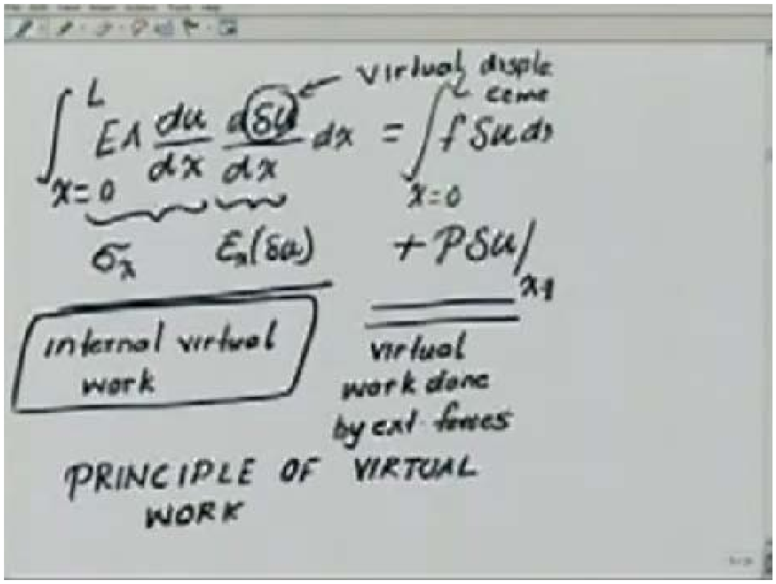
Next, what we are going to do is we will look at the boundary term which is EA du dx w evaluated at the point x equals to L minus EA du dx w evaluated at the point x equal to 0. If we look at this, this is the boundary part or the boundary term at the point x is equal to L which is the right extreme of the member that we have taken and this one is EA du dx w evaluated at the point x is equal to 0 which is the left extreme of the member that we have taken. In a model problem that we have taken, at this end EA du dx is the force given by the value P. While at x is equal to 0 EA du dx is an unknown term but at the end x is equal to 0, u is known. What we are going to do is at the end x is equal to L where the force is given we are going let w be free that is we do not put any constraint on w at this end. While at the point x is equal to 0 we are going to enforce the constraint that u is given equal to a value 0 by making w is equal to 0. If we make w is equal to 0, this term vanishes. What we are left with is the following formulation, integral x is equal to 0 to L EA du dx dw dx dx is equal to integral x is equal to 0 to L fw dx plus P w evaluated at point x equal to L. This is the formulation that we get. Once we have obtained this next what? If we stick to this formulation this is in a way enough to work with, but what we will like to do is set in our mechanics frame work. That is we are going to choose w to be not any function of interest, but a function called delta u of x.

(Refer Slide Time: 10:55)



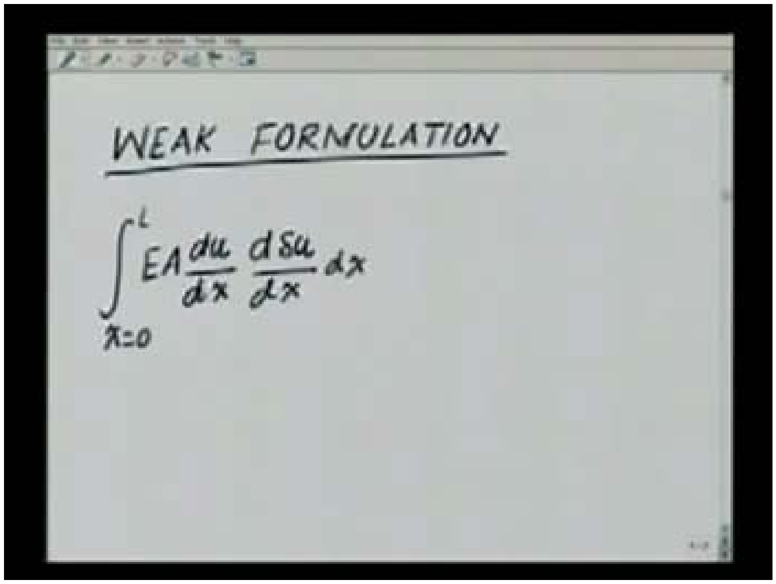
Delta u of x is said to be the variation of u. Let us imagine that I am giving you two points x is equal to 0 and L. In between this I have this function u of x and on top of this if I take a function which is close to this, cannot be very close it can be anything which is close to this function. We have to define what we mean by close. This function we are going to call as u(x) plus delta u(x), such that difference between these two functions is the function delta u(x). Whenever we are talking of these variations of u, it is as if I am taking the function in the neighborhood of the function u and taking the difference of those that we can call as the variation of the function u(x). If I take w(x) equal to variation of the function u(x) and put it back in our formulation that we have obtained by the integration of the parts, we will get f delta u dx plus P delta u evaluated at the point x equal to L.

(Refer Slide Time: 12:18)



This is something that we are familiar with. If you look at this part this is also equal to the stress in the bar $sigma_x$ due to the displacement of the bar. This part we can say is the strain due to delta u and the right hand side if we look at it is work done by the delta u against the distributed body force f and the end force P. What do we have here? We have something called the internal virtual work and this is virtual work done by the external forces. Why do I call it virtual work? It is because our equilibrium has been achieved under the action of the forces f and P and we have obtained a displacement for u, a corresponding strain Ex and a corresponding stress $sigma_x$. We imagine that from the equilibrium state we are going to perturb the system by a small amount delta u. This is the variation of u that we are talking about. This is called the virtual displacement delta u. The term that we have here on this side is the work done by this, the strain due to the virtual displacement against the stress that has been built up in the bar to counter the effect of the external forces. While on the right hand side it is the virtual work done by the virtual displacement delta u against the external forces that we have. If I take this equation, this is called Principle of Virtual Work.

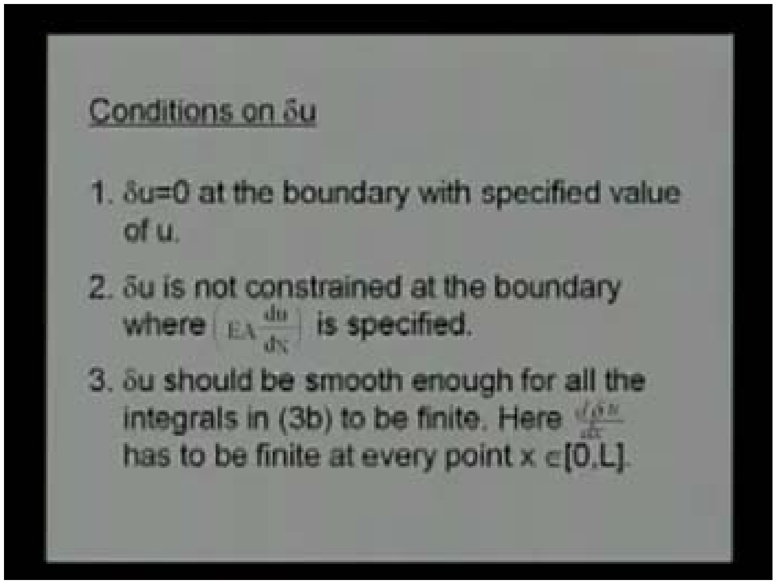
(Refer Slide Time: 15:36)



This whole formulation is also called by another name which is the Weak Formulation. Question is why do we call it a weak formulation? It is called weak formulation because if we look at first integral on the left hand side here, we have taken one of the derivatives from u and transferred it to delta u or w that we had taken earlier. What happens in the differential equation we require the second derivative of u to be defined, because it was the second order differential equation and at every point in the domain we had to have definition of second derivative of u. While in the initial weighted residual formulation instead of w I had put delta u. All that was required was delta u had to be defined in order to have this integral finite. But now if we look at this term we only need the first derivative of u to be defined. That is, instead of asking for the second derivative to be defined we are now requiring only the first derivative of u to be defined which is a weakening of the smoothness condition on u. That is all we need is a first derivative of u should be given. Similarly, we have now transferred the derivative from u to delta u. That is, now we want derivative of delta u to also be defined in the domain. That is why we call this weak formulation.

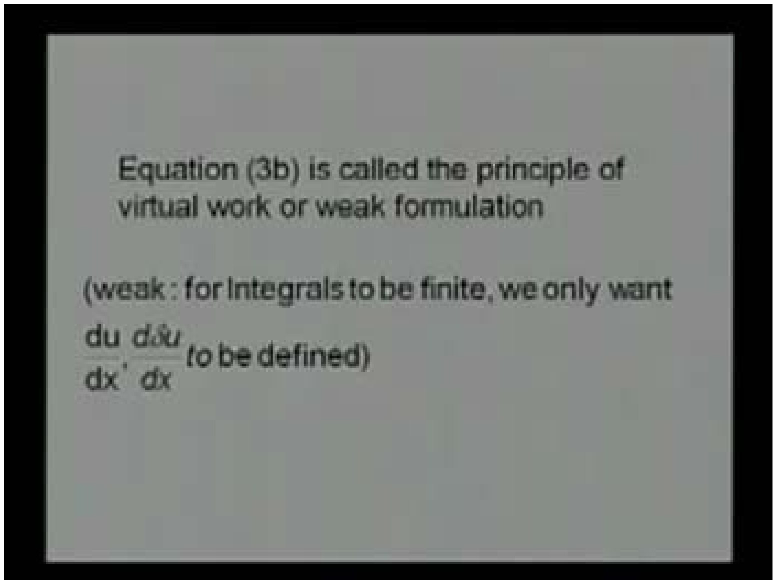
Next, let us now look at the properties of this variation delta u that we are going to use in our formulation in future. Let us look at the conditions that our delta u, or the variation of u, or the virtual displacement has to satisfy.

(Refer Slide Time: 17:41)



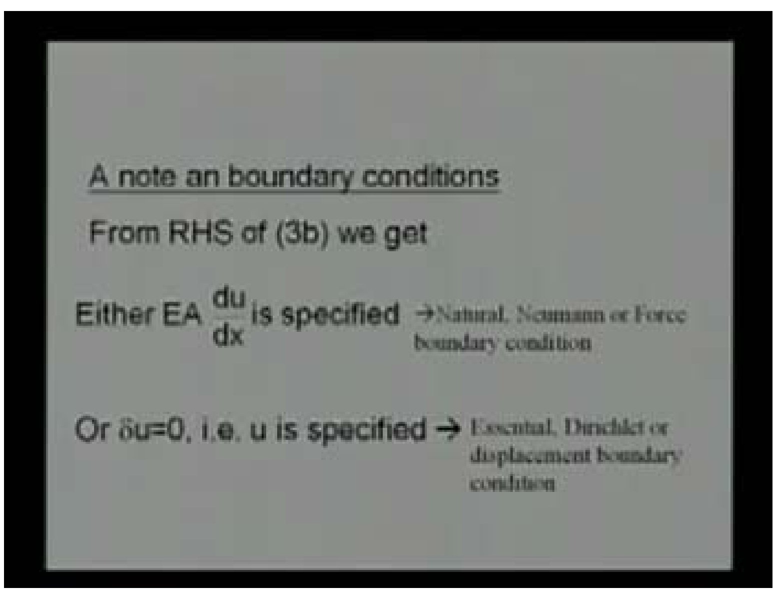
We see at the end x is equal to 0 for our problem we have u given; when u is given the variation of u is set to 0. Set a boundary where the displacement is specified and there variation delta u is set to 0. That is the first condition on delta u or on w that we have taken the weight function w. Second, what we have done is we have not put any condition on delta u at the end x equal to f. That is, at the boundary value the force is specified du dx for example, in our case at x is equal to L, will let delta u or w to be free. They can take any value that you wish. Third, as we have shown in the weak formulation or the principle of virtual work, all we need is that the derivative of delta u should be defined in the domain. So, when the derivative is defined the integral that we have in the variational or the virtual work formulation or the weak formulation are all finite.

(Refer Slide Time: 18:57)



We have already talked about this; the equation that we have written earlier is called the principle of virtual work or the weak formulation as we would like to call it. For engineers, especially people in mechanics, principle of virtual work is something that we know or we are aware of so we may go ahead with that formulation; but they both mean the same thing. Let us now concentrate on the equations that we have obtained earlier.

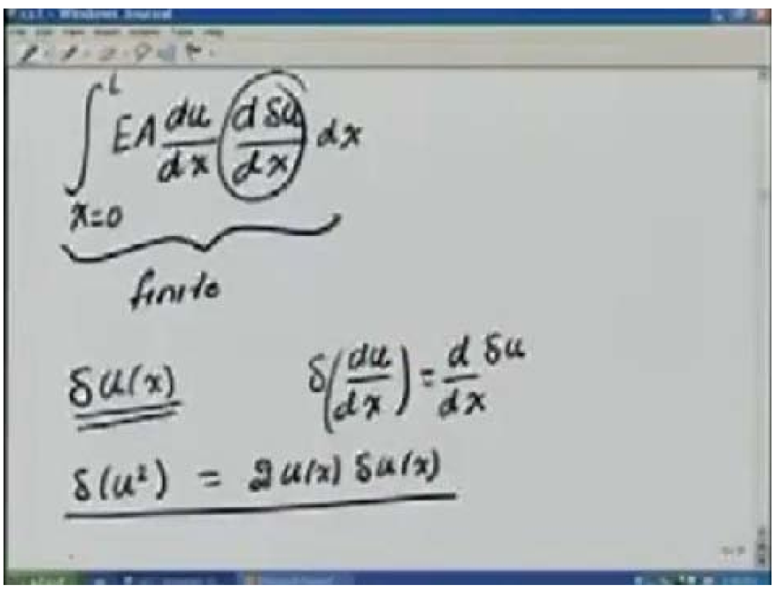
(Refer Slide Time: 19:31)



There we see that we have this pair EA du dx and delta u or w occurring together and we said that when the force is specified, we set delta w or delta u or w to be anything and when the displacement is specified we are going to set delta u or w equal to 0. If a force is specified at an end that boundary condition is called the Natural, Neumann or Force boundary condition. At the end where u is specified where we are going to fix delta u equal to 0 that end is called an Essential, Dirichlet or Displacement boundary condition. This we will generalize to two or three dimensions in the future. These definitions should be kept in mind that is, we are going to generally call them by the name Neumann or natural or essential or dirichlet.

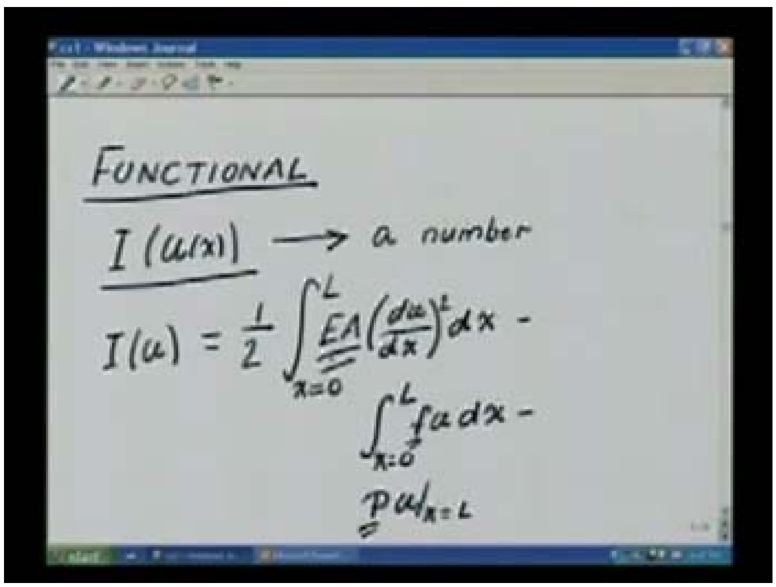
Why is the force boundary condition a natural boundary condition? If we have go back to our equation that we have written of weak formulation, we see that force appears naturally in that formulation. If it is on the right hand side of the equation it is called the natural boundary condition. The displacement being fixed at an end is not explicitly appearing in our weak formulation. It has to be enforced by forcing delta u equal to 0. That is why it is called the essential boundary condition that is it has to be forced in. This can now be rewritten whatever we have developed earlier using a different concept but we will get the same set of equation that we have obtained earlier.

(Refer Slide Time: 21:37)



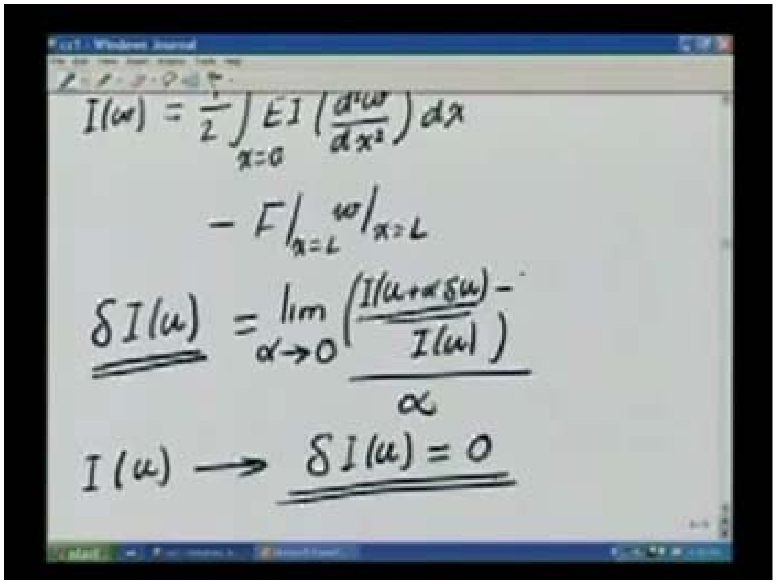
Next, we will go to a new concept from which we can develop our equations all over again. For that we will need to do a little bit of home work; we have to look at this variation of u(x) - certain more properties of it. For example, if I take variation of du dx instead of u of x this will be equal to derivative of the variation of u. Similarly if I take variation of u square it will be equal to 2u(x) into variation of u of x and so on. Variation essentially works in principle like the derivative but it is not the derivative because we are talking about pertaining to a function not taking differential of the derivative of a function at a point. Using these properties we are going to develop the next thing.

(Refer Slide Time: 22:48)



We are going to define something called the functional. What is the functional? The functional is a function of the function. It will take, let me call, I as the functional. It will take a function u(x) and it will return to me a number which is I (u(x)). I give one function u(x) and get a number; I give another function u of x I will get another number. For example, I may define various functional like this, integral from x is equal to 0 to L EA du dx whole squared minus integral x is equal to 0 to L fu dx minus Pu at x is equal to L. This is a functional where P, f, EA are given to me as material data or input data to the boundary value problem that has to be supplied. If I give you the function u then this integral is going to return a number, this part is going to return another number, this part is going to return another number and the sum of it will be I(u) which is the number. Change the function u and put some other number w; then I will get another number. So this is called a Map of the function to real numbers. I can also define another functional I(u) by another example.

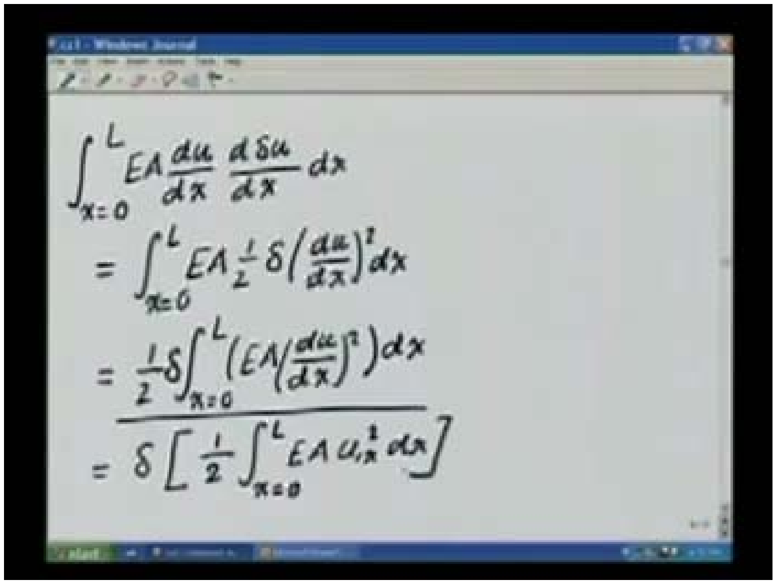
(Refer Slide Time: 24:56)



Let’s take this half of x is equal to 0 to L. We are not going to elaborate on it now but this is essentially related to our mechanics problems of interest, minus I may have F at the end x is equal to L, w at the end is equal to L. This can be another functional of interest. Given a functional we will also define the variation of the functional delta I(u). This variation will be given by definition as limit of alpha tending to 0 of I(u) plus alpha delta u minus I(u) whole thing divided by alpha. What do I have? Instead of u, I put u plus alpha delta u into my expression for the functional, evaluate that number, from that subtract the number corresponding to I(u), divide that by alpha, take the limit of alpha going to 0. This is called the variation of u. For the various boundary value problems, in many cases not always, we can define this functional I(u) and given this functional I(u) we will see that the variation of I(u) when set to 0 also gives us the solution to the boundary value problem.

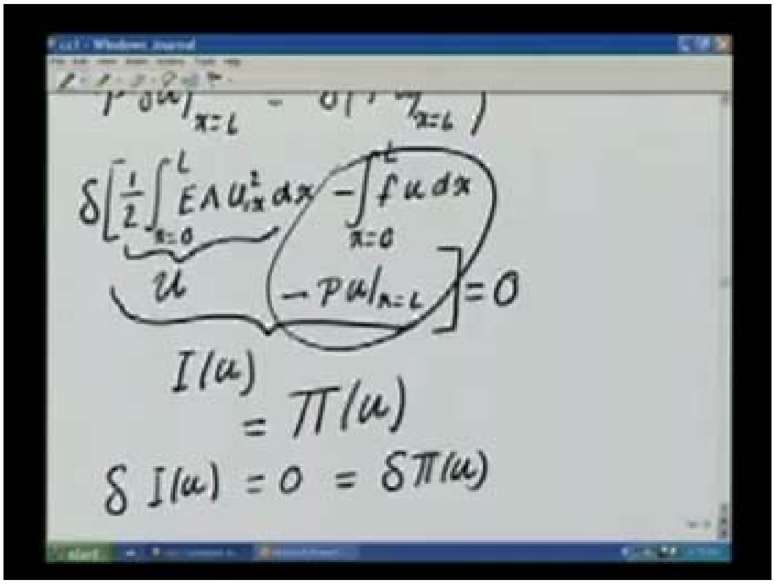
Let us now go back to our variational formulation or the weak formulation that we have defined earlier or the principle of virtual work for the model problem of interest and rewrite that in the functional I(u).

(Refer Slide Time: 27:18)



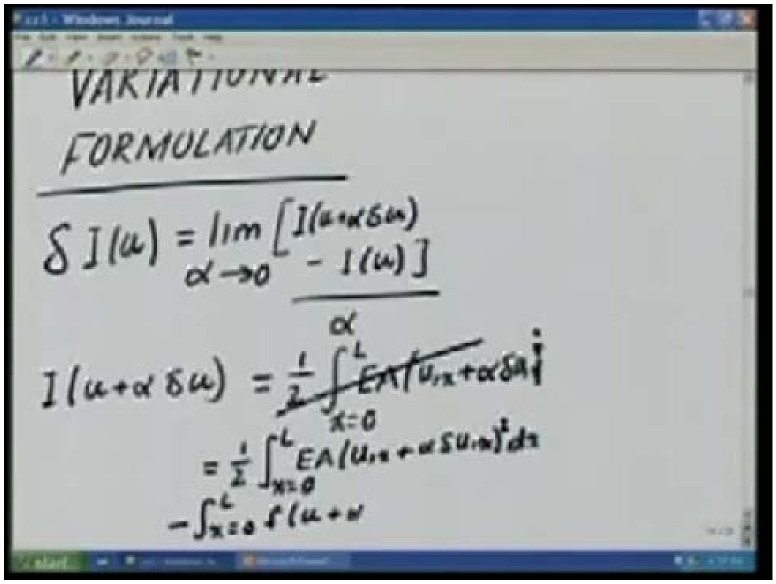
If we have that formulation, we have the integral x is equal to 0 to L EA du dx d delta u divided by dx into dx. By our definitions of the variation keeping EA as a given constant, we cannot vary EA because that is something that is supplied to us. We can write it as integral x is equal to 0 to L EA into half of delta of du dx whole square dx which we can write as half of integral 0 to L, we put delta of EA du dx whole square dx. The first term essentially can be written in a more compact way; integral delta of half of integral of x is equal to 0 to L EA, if I want to write it using simpler notation, EA derivative of u with respect to x squared dx. Similarly, the other term which we had integral of f delta u dx at x is equal to 0 to L can be written as delta of integral of x is equal to 0 to L f u dx.

(Refer Slide Time: 29:21)



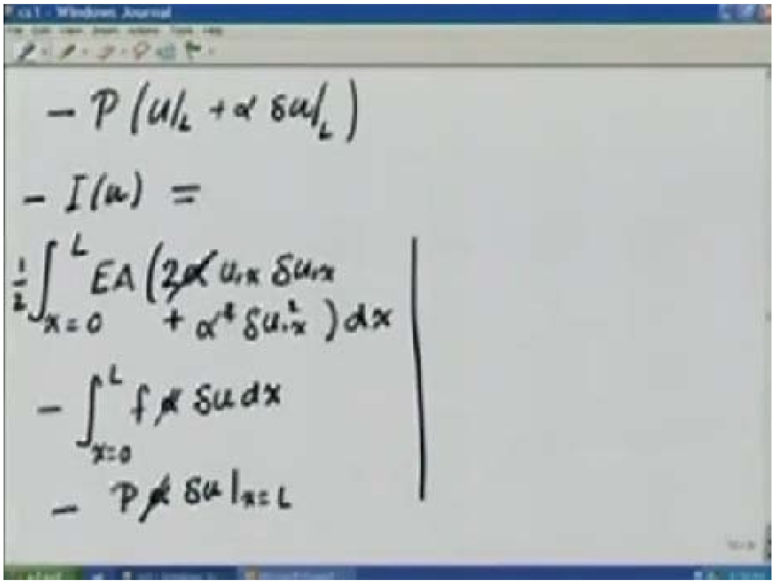
The term P delta u evaluated at the point x is equal to L is equal to delta of Pu evaluated at x is equal to L. By collecting all the terms together what we end up getting is delta of half of integral x is equal to 0 to L EA u comma x square dx minus integral x is equal to 0 to L f u dx minus P u at x is equal to L. This is equal to 0 from what we have set earlier. Then this term with in the square brackets is now our functional I(u). We said that the variation of I(u) is equal to 0 is also the weak formulation or the principle of virtual work we had given earlier. Can you tell us what is I(u) for the given problem? We have taken I(u), if you look at this term in I(u), corresponds to the strain energy u for the structure. The remaining part corresponds to the potential of the external forces. So what we have is this I(u) is nothing but the total potential energy pi of the structure. We know very well that the minimizer of the total potential energy is the function which solves our model problem. We can pose this again as, let us write it, in a concise way del I(u) is equal to 0 is also equal to del of the pi of u. Because the same equation can be obtained by taking a variation of the potential I(u) which in our case corresponds to the total potential energy associated with the system, this is also called a variational formulation.

(Refer Slide Time: 31:48)



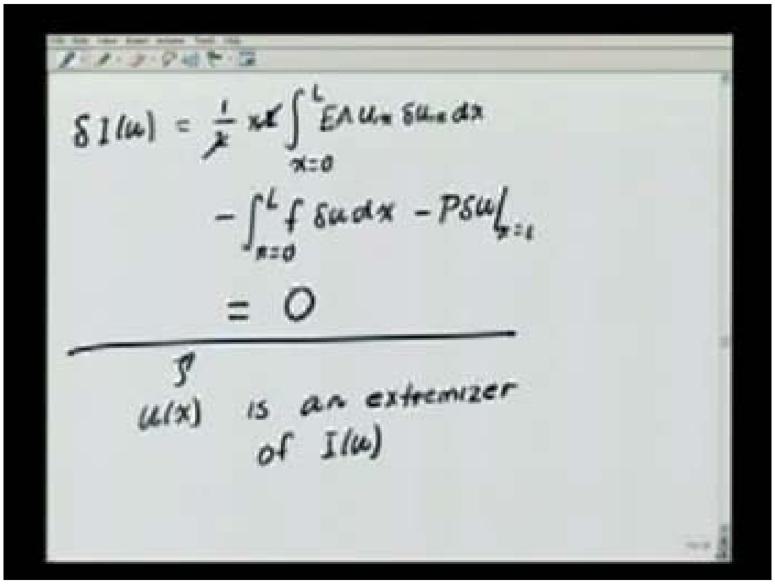
Let us check that we have written delta I(u) is given by definition as limit of alpha tending to 0 of I(u) plus alpha delta u minus I(u) whole thing divided by alpha. Let us see if it is really giving us that equation back because we simply rewrote everything in terms of potential and we said that yes it is equal to 0. So let us take that potential we have defined earlier I(u) and we have put u plus alpha delta u. What will we get? This will be equal to half integral x is equal to 0 to L EA u prime x plus alpha delta u prime x whole square. I will rewrite it here; it is equal to half of L equal to 0 to L EA u prime x plus alpha delta u prime x whole square dx and the other part will be minus integral x is equal to 0 to L f into u plus alpha delta u dx.

(Refer Slide Time: 33:40)



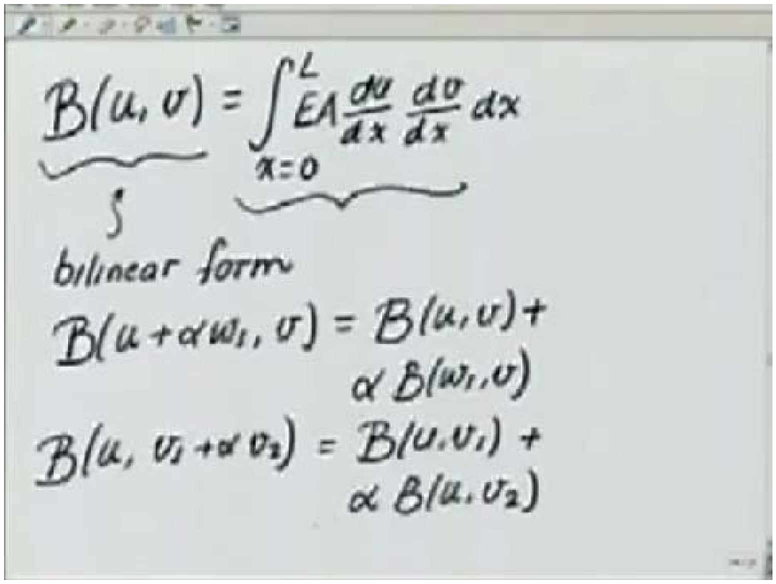
Finally, let us go to the next page; minus P u at L plus alpha delta u at L. From this if we subtract I(u) what do we get? I will get integral x is equal to 0 to L EA into alpha, half I will have in front, 2 alpha, u comma x delta u comma x plus alpha squared delta u comma x whole squared dx minus integral x is equal to 0 to L f alpha delta u dx minus P alpha delta u at x is equal to L. We said that we are going to divide this thing by alpha. When we divide this thing by alpha this term will go. We will be left with a single term alpha and here also this one will go and this one will go. What we have when we take the limit then? All the terms after division by alpha, I mean alpha setting in front, will go to 0; they will disappear. All the terms with no alpha in front will be what we are left with.

(Refer Slide Time: 35:19)



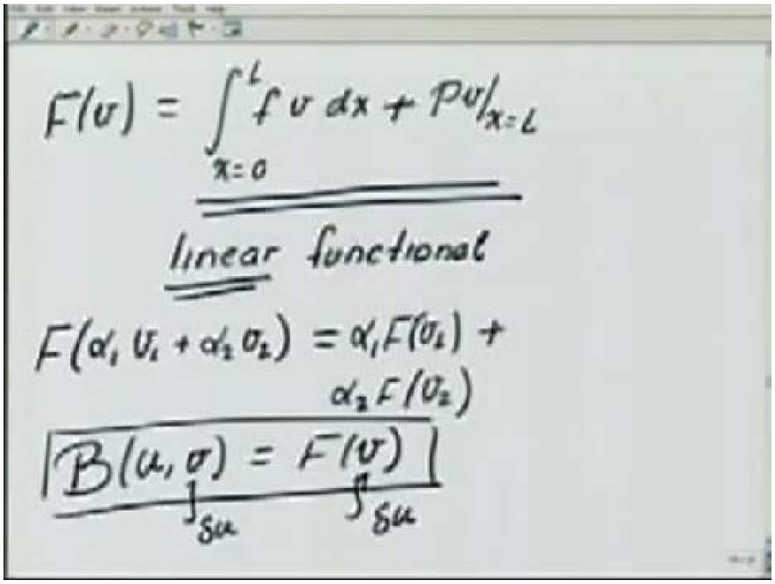
This will be equivalent to writing as delta I(u) is equal to half of equal to… 0 to L EA u comma x delta u comma x dx. The two and half will cancel; minus integral x is equal to 0 to L f delta u dx minus P delta u at x is equal to L and this has to be equal to 0. We see that we have obtained our principle of virtual work by this approach too. The variation of the functional has given us our principle of virtual work. What we have said is that we obtained in our formulation the variation of I(u) is equal to 0 or we say that the solution to the problem given by u of x to the model problem is an extremizer of I(u). Whether this extremization is minimization or the maximization or a point of inflection is not something that we are look at. As soon as u(x) is an extrimizer then the extrimizer solves the model problem of interest.

(Refer Slide Time: 37:03)



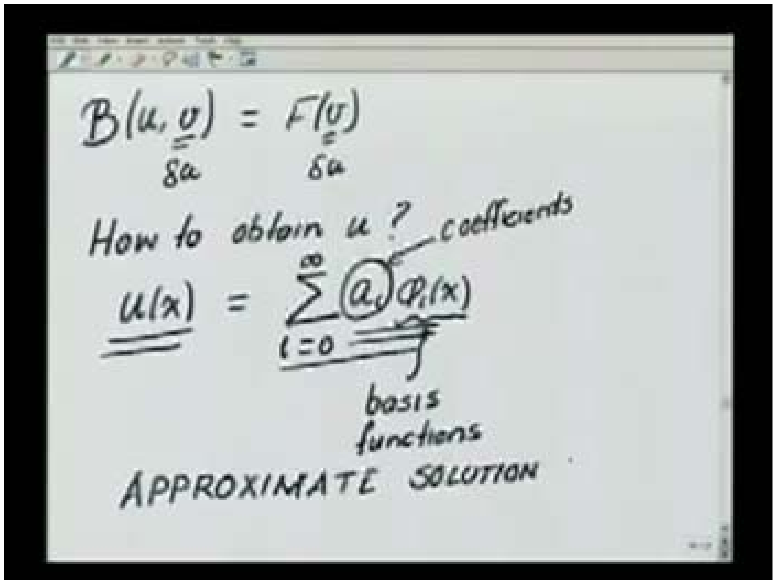
Next, let us introduce few notations. We are going to call it B (u, v) is equal to integral x is equal to 0 to L EA du dx dv dx dx. This term is given by this notation. This is called a bilinear form. Why is this bilinear? If I look at B u plus alpha w_1 v this will be equal to B (u, v) plus alpha B(w_1, v). It is linear in the function u. Similarly, if I put B u v_1 plus alpha v_2 this will be equal to B u v_1 plus alpha B u v_2 that is it is linear and also in v. That is why it is linear in both the functions which give us this number d (u, v) we call the bilinear.

(Refer Slide Time: 38:40)



Similarly, we can introduce another notation F(v) which is equal to f v dx for our model problem plus p v at x equal to L. This quantity is called a linear functional just like we have defined the functional earlier. What f does? It takes the function v and gives us the number F(v). Why it is linear? It is linear because if I take two functions alpha$_1$ v$_1$ plus alpha$_2$ v$_2$; F of that, if we put it here in the expression that we had taken earlier, it is equal to alpha$_1$ F(v$_1$) plus alpha$_2$ F(v$_2$). That is why it is called a linear functional. Now what happens when using this notation is that we have made our writing of these equations (40:04). Instead of v if we substitute as v with delta u then I will get Bu delta u is equal to F of delta u which is nothing but the principle of virtual work or the weak formulation or the variational formulation that we have obtained.

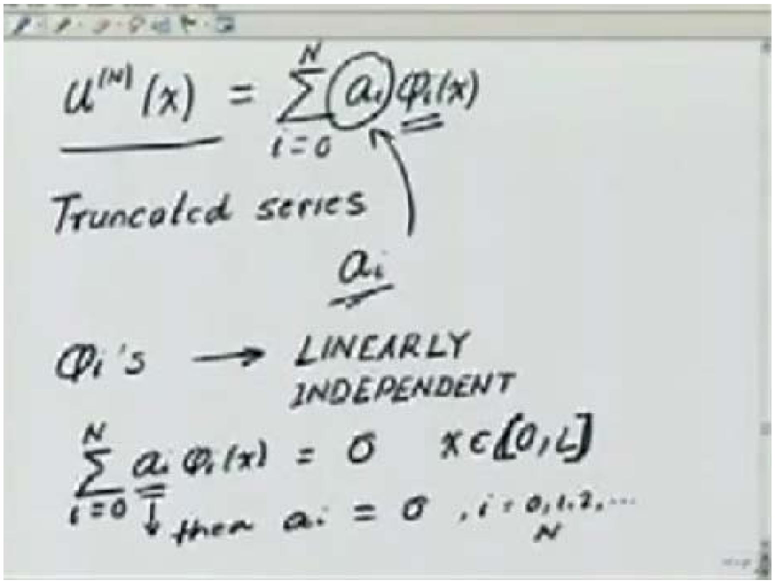
(Refer Slide Time: 40:34)



We have obtained this as our equation that we have to solve in order to get u. There should be various questions one should ask. Tell me what do we mean by delta u in this case and how are we going to get to u, how to obtain u as a function of x? Obviously, if we knew how to get a close function to the boundary value problem that we have considered earlier then we would not have had to come to this stage at all. What we would like to do is we would like to obtain a series solution to u. What we are going to do is we are going to represent u as a series in terms of… in this form. In this series what we would like to obtain is the coefficients of the series; provided we choose this function phi_i of x which you can call as basis functions. If we can choose these basis functions properly then the linear combination of the basis functions will form our solution u(x) and we will assume for the time being that the series is convergent. Implicitly we have assumed that this series converges to u(x) at every point. The job is to find the coefficient a_i. How can we use what we have done here to find the coefficient a_i? That is the first question.

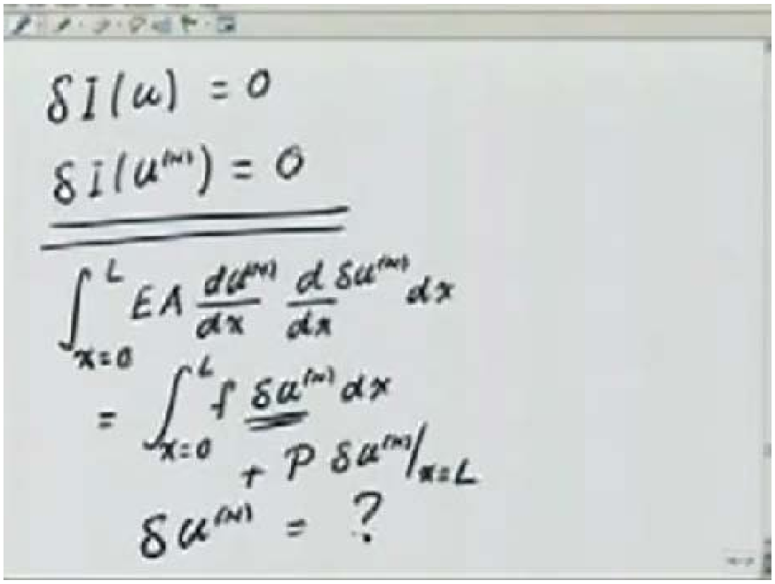
Secondly, we do not want to find in all cases all the infinite coefficients. We will be happy by getting something called the approximate solution.

(Refer Slide Time: 43:05)



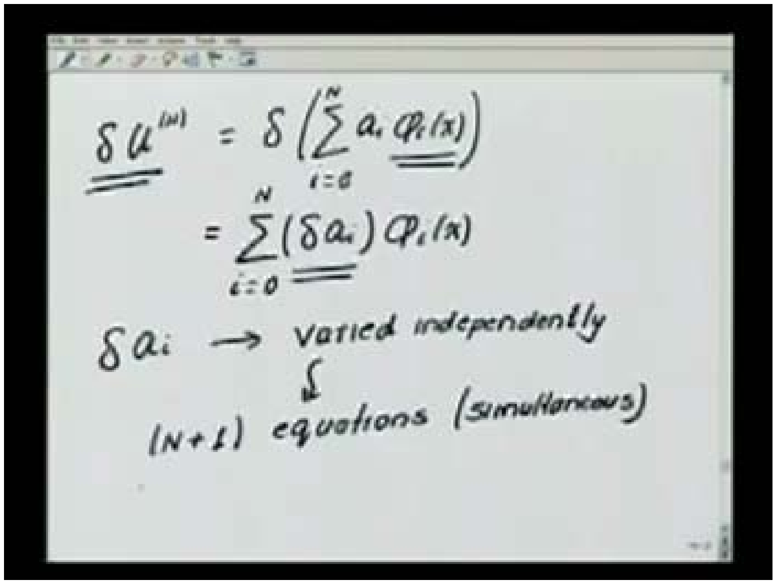
We will take instead of the full series a truncated series where I will have N terms u of N of x. This is now called truncated series. Again, in this truncated series if we can find all these coefficients a_i, if they are known then given this function phi_i, we have obtained our approximate solution for the truncated solution. The question is, how do we choose these phi_is? One thing that the phi_is have to satisfy is that the phi_is have to be Linearly Independent. Linearly independent in our case means, let us say that I take this finite series only and I put… equal to the linear combination is equal to 0 for all x which is in this range 0 to L, we can say including this end points. Then linear independence means that if this is what we required then the a_is will come out to be 0, all of them, then all the a_is will be equal to 0. If this happens then we say these function phi_is are linearly independent. In forming this series, we need these linearly independent terms. Then we would like to now put this in our principle or our virtual work or the variational formulation.

(Refer Slide Time: 45:22)



Let us go by the variational formulation that is instead of asking for delta I(u) equal to 0, we are now going to say delta I(u) of N is equal to 0. When we say delta $I(u)^{(N)}$ is equal to 0 then what do we get. This will be written as integral x is equal to 0 to L EA d u of N dx d delta u of N divided by dx dx is equal to integral x is equal to 0 to L f delta u of Ndx plus p delta u of N evaluated at the point x is equal to L. What we have done is we have instead of u we put u of N. We say that we are looking for the function u of N which is an extremizer of this I that we have defined for our model problem of interest.

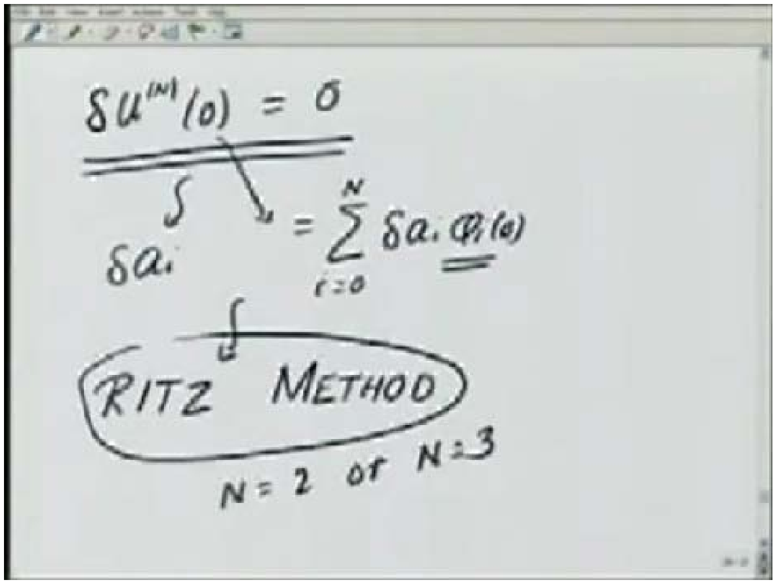
(Refer Slide Time: 46:58)



The question is what is delta u of N? Delta u of N for us will be equal to delta of sigma of i is equal to 0 to N a_i phi_i x. We have chosen what is our phi_i. That is each of these phi_i is an unknown function that we are going to use to construct the series. The delta of u of N will become sigma i is equal to 0 to N delta of a_i into phi_i of x. You see what has happened is that the variation of u of Nis given in terms of the linear combination of the variation of the a_is.

What we will do next is to find if it is given in terms of the variation of these a_is, each of this delta a_is can be varied independently. What does it mean? Because this delta u is something that is under our control; it is something that we are specifying, we can choose delta a_1 is equal to 1, let us say and all other delta a_is equal to 0; that will be 1 delta u. Similarly, I can choose delta a_2 is equal to 1 and everything else equal to 0; that will be another delta u of N. With this various choices of delta u of N, by fixing the values of delta a_i we should be able to construct various forms of this delta u of N. What we will do next is, vary this independently and end up getting variation (N plus 1) equations which are simultaneous equations in terms of the (N plus 1) variable a_i. This is another thing that we should note.

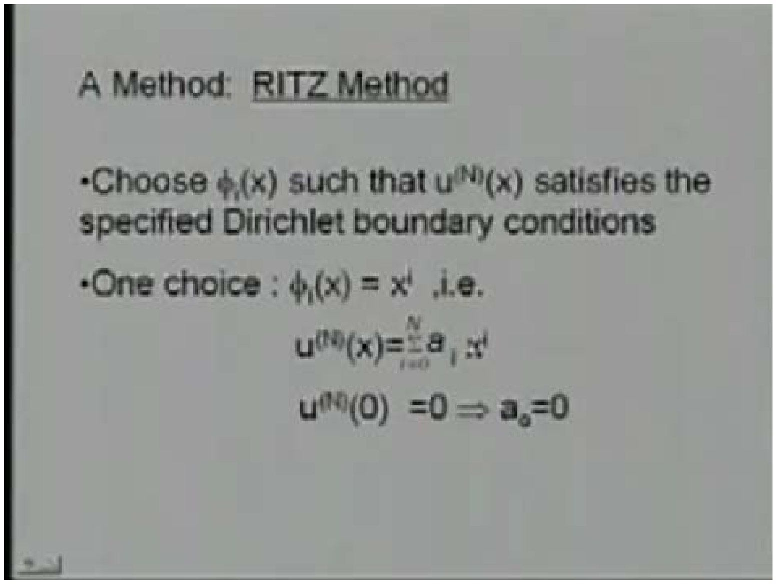
(Refer Slide Time: 49:26)



Let us take the model problem that we have posed. For that model problem we need delta u of N at 0 is equal to 0. This also we have to specify. This is another condition that is going to come on these delta a_is. Why will it be on delta a_is? This is because delta u of N equal to 0 is equal to summation delta a_i phi_i evaluated at 0, i is equal to 0 to n. We are going to look at this in a detailed way and we are going to pose something call the RITZ method. We are going to choose particular forms of this phi_is specifically for the model problem that we are concerned and we are going to basically formulate the problem in terms of the truncated series and solve it by minimizing the total potential energy with respect to the coefficients delta a_is.

What we will do in the next class is we will take N is equal to 2 or N is equal to 3 with some specific forms of f and P and solve the problems and see what are these coefficients.

(Refer Slide Time: 50:56)



In the next class, we are going to develop the RITZ method for the model problem that we have taken where the N term series that we are going to use for the approximation of u(x) will be given in terms of very specific functions. We have said that we would like our delta u of N as well as u of N to satisfy the zero condition at x is equal to 0. Let us take the phi$_i$s to be functions which are x of i that is x of 0 is equal to 1, x of 1, is x, x of 2, x of 3 and so on up to x of n.

Then u of Nx is the summation of a_i x of i. When we put u of N at 0 is equal to 0 then automatically a_0 comes out to be equal to the 0. Once we have a_0 is equal to 0 then our series will be written in terms of summation of i is equal to 1 to n a_i xi and what we need is to develop sufficient number of equations to obtains these a_is. That is we need N equations in terms of the N unknown a_is. Once we get them then we will get the solutions and through plots of solution for certain boundary value problems that will take as typical example we will show that this method does very well for certain classes of loading and also will bring out the drawbacks of this method which we are going to exploit to develop the finite element method as a tool to overcome these drawbacks.