Finite Element Method

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Module - 6 Lecture – 2

Till now, we have looked at the one-dimensional problem where we have solved for a bar.

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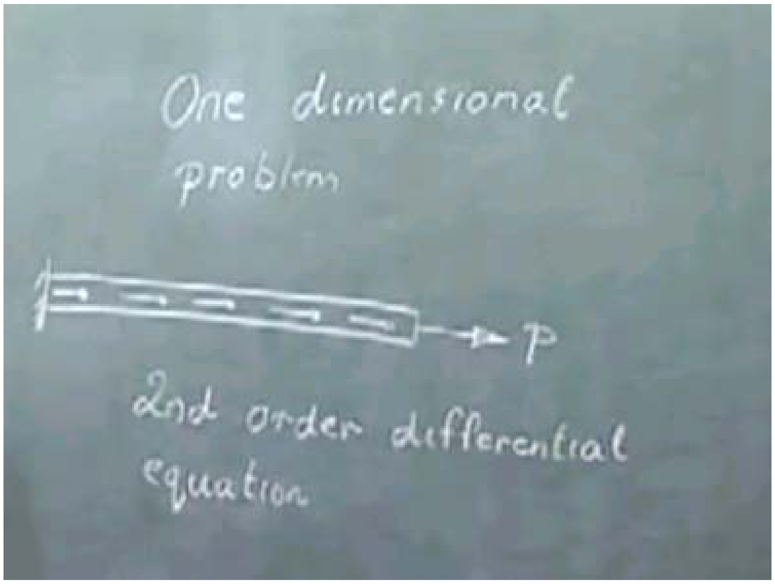

So a bar was essentially a thin member subjected to an end axial force or distributed axial force or even a point load. When we looked at the response of this bar, we came up with a second order differential equation. We have done a detail analysis of this problem and we have gone through the various steps of creating the weak formulation or the variation formulation which was used to design the finite element method; we discussed in detail how to construct the basis functions for this approximation. Now, let us change the problem a little bit. So, what we have now going to do is look at a one-dimensional problem, but of a different type.

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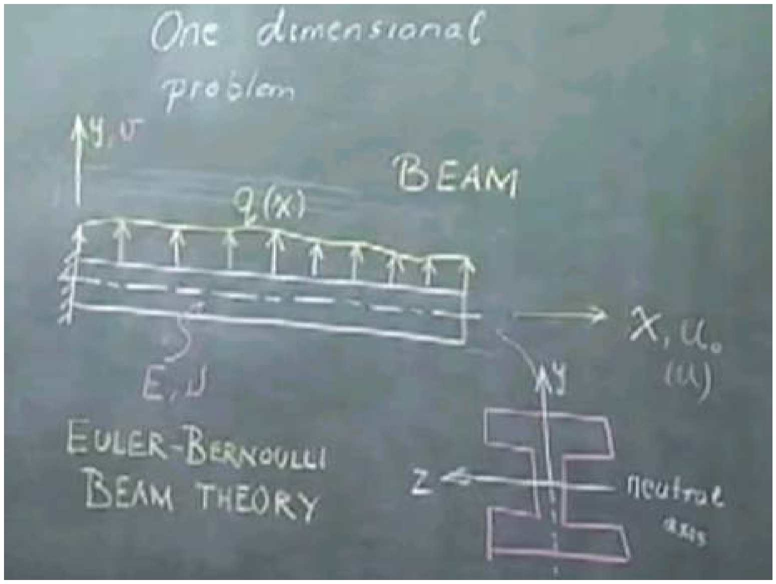

We have, let us say a support like this and here is our member again with an axis of symmetry, you will see what we mean by this and this member is subjected to some distributed transverse load. This load has an intensity of q of x, where this is my x direction and this is my y direction.

The material has Young's modulus E and Poisson ratio mu. Now, the cross section of this member we are going to take as symmetric section; so the cross section if I take this and I look at it I will have something, such that we have this y-axis going along the line of symmetry of the cross section here. And the z-axis is like this and this is called the neutral axis. This we all know is a beam and what we have going to do is look at the Euler-Bernoulli beam theory. I assume that Euler-Bernoulli beam theory is known to everybody according to which, I will have this space u_0 in the direction of x and u_0 and u and v in the direction of y.

We have taking the cross section to the symmetric; this is a represented symmetric cross section and this is a member. Now by the Euler-Bernoulli beam theory, if I solve this problem, I will get certain differential equations.

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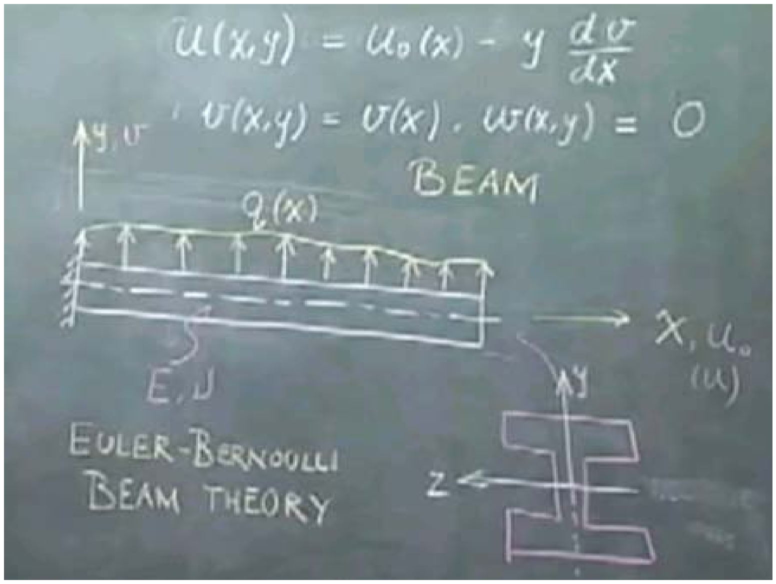

So what is assumption with the Euler-Bernoulli beam theory? That my u, axial displacement of the function of x and y is equal to u_0 is the function of x minus y dv dx; v as a function of x and y is equal to, it is only a function of x and w as a function of x and y that is displacement in the z direction is 0. This is based on the assumption that thin sections remain plain and the lines perpendicular to the neutral axis remain perpendicular even after the deformation. Let us now look at the equation of motion or the static equilibrium equation corresponding to this problem.

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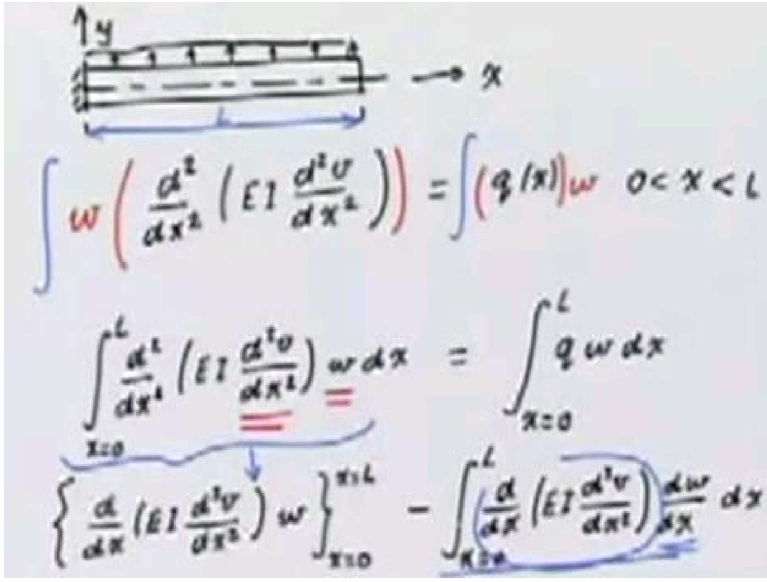

Let me again draw my beam here; this is my beam; this is the neutral axis of the beam, which is what we called x-axis here. Essentially the neutral axis is the one which remains unchanged in length, which does not deform, while everything above and below deforms. Given this, if I want to write the equation of equilibrium for the bending of the beam using the Euler-Bernoulli beam theory of the beam subjected to some distributed load, the equation of equilibrium is given by d two square dx square of EI d two v dx square is equal to q of x for x lying from 0 to L. As we have done earlier, let us look at the weighted residual formulation. We will take this expression, multiply it with the weight function w, and integrate it from 0 to L. This is what we have done earlier. So let us do the same thing here. I will have d two dx squared EI d two v dx squared into wdx, this is equal to integral x is equal to 0 to L q w dx. This is the first step that we do. Next what we have done earlier we will extrapolate those steps here also.

If you see now that here as far as w is concerned we have the 0^{th} order derivative of w; that is, w sitting by itself and as far as v is concerned, I have the fourth derivative of v. So what we have done earlier, we had said that you want to weaken the requirement of smoothness on v; because, here if I take it as such, the fourth derivative of v has to be defined. So we would like to weaken that requirement and transfer derivatives from v to w. How do we do that? We do that by integration by parts; let us do it once.

If I do integration by parts once, I will get d dx of EI d two v dx squared into w evaluated from at x is equal to 0 and at x is equal to L minus integral into EI d two v dx squared into dw dx dx; the right-hand side remains same. I am simply writing this part after integration by parts. I get one boundary term out and I have a part which is remaining in the integral, as an integral over the length. If I look at this part, see here I have transferred one derivative of w, but here I still have a requirement of the third derivative of the two exists. That is again lop-sided. I would like to transfer now again a derivative from the v to the w. If I want to do that - we will see why you want to do that - if I want to transfer this derivative, I will have to do integration by parts once again.

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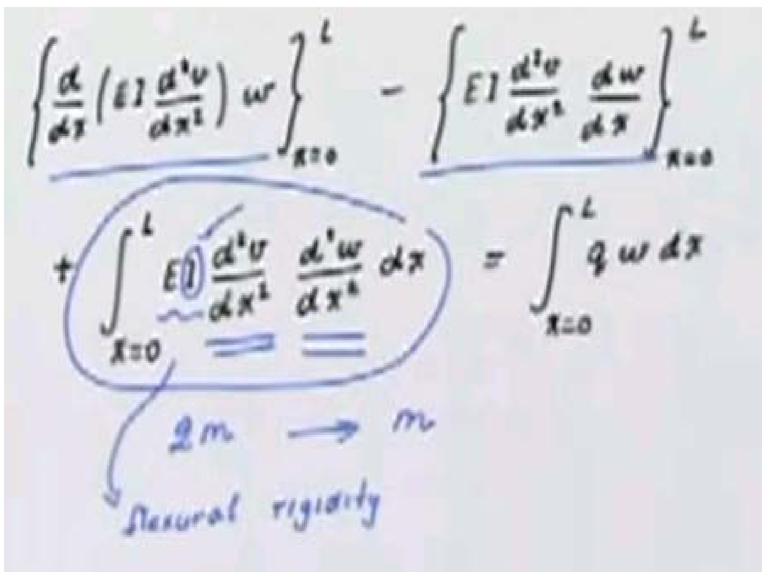

If I do integration by parts once again, after the second integration by parts I will get the following: d dx of EI d two v dx squared whole thing into w to L minus integral of EI d two v dx squared into dw dx whole thing evaluated from 0 to L plus integral over 0 to L EI d two v; this expression is equal to integral x is equal to 0 to L q w dx.

After doing integration by parts twice, what do I have if I look at this expression? Here I have the second derivative of v and the second derivative of w together in the expression. Now, if I say - why not do integration by parts once again? Then what am I doing? I am transferring another derivative from v to w. So, I will have a third derivative of w and the first derivative of v, which is not useful to us, because, then I am raising the smoothness requirement from w while lowering the smoothness requirement on v.

We would like to have the smoothness requirements on both v and w of the same type, because, here we are using a Galerkin approximation that is w and v have the same representation. We really do not want it as it is not because of the Galerkin, but by construction, we do not want to make the smoothness requirement lop-sided. So what we will see here is that if here it is a fourth order differential equation that we have taken, we have done integration by parts twice. If it is, in general, a 2mth order differential equation then I will do an integration by parts m times to get d mv dx to the power of m dm w dx to the power of m in the representation. It is the standard rule that you should follow.

Now here by doing the integration by parts twice, I have EI d two v dx squared d two w dx squared plus these boundary terms is equal to the work done by the distributed external force. All of you must know that I is the moment of inertia about the z-axis and EI is nothing but the flexural rigidity. So the weak formulation essentially is given by integral of EI - this part d two v dx squared d two w dx squared dx is equal to integral of q w dx minus this part minus d dx of EI d two v dx squared with w evaluated at x is equal to 0 and L plus this part which is EI d two v dx squared into d w dx evaluated at x is equal to 0 and L.

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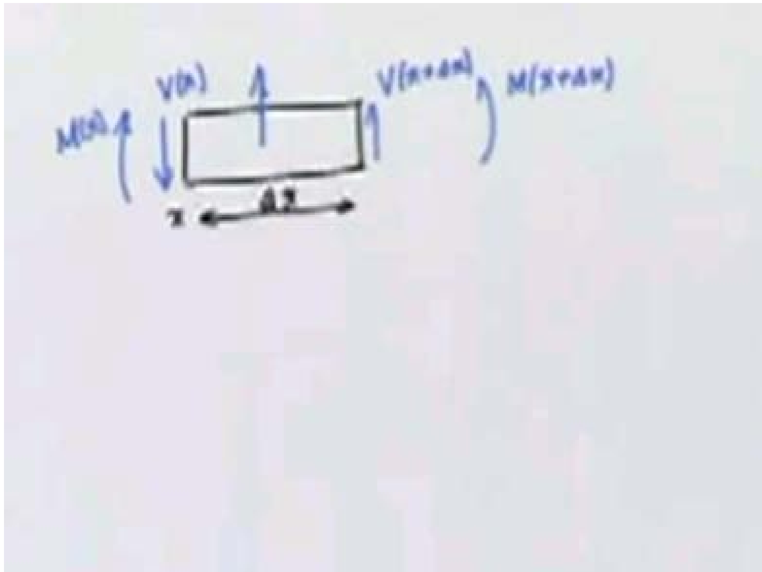

We know that this quantity, if I go by our equilibrium analysis on a small piece at a position x of size delta x. What do we know we have? Here we will have the shear force V of x, here we have the bending moment M of x, here we will have the shear force at V of x plus delta x, bending moment m at x plus delta x and the resultant force due to the distributed density is q x delta x. Actually what we derived came out of this - the equilibrium equation.

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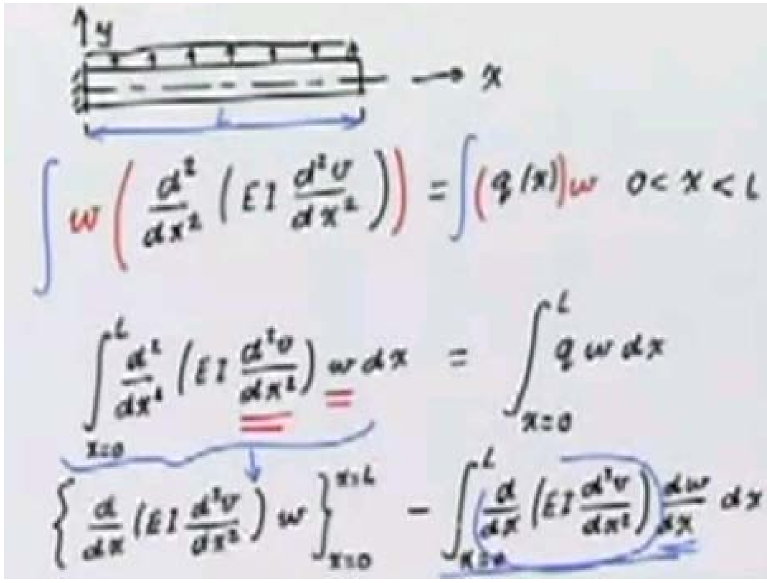

If I go back, you see that here in our equilibrium equation the u_0 does not play a role, because u_0 corresponds to an actual stretching and in this case - in the beam analysis - the way we are doing is the stretching and the bending modes get decoupled. So I will have the bending equation clearly in terms of v and the stretching equation only in terms of u_0. So I do not have to consider the stretching equation here. Since there are no loads in the axial direction, I would expect, not expect, I will get my stretching mode as this 0 displacement. We are ignoring that part here.

Let us come back to our problem here, as far as the weak formulation is concerned.

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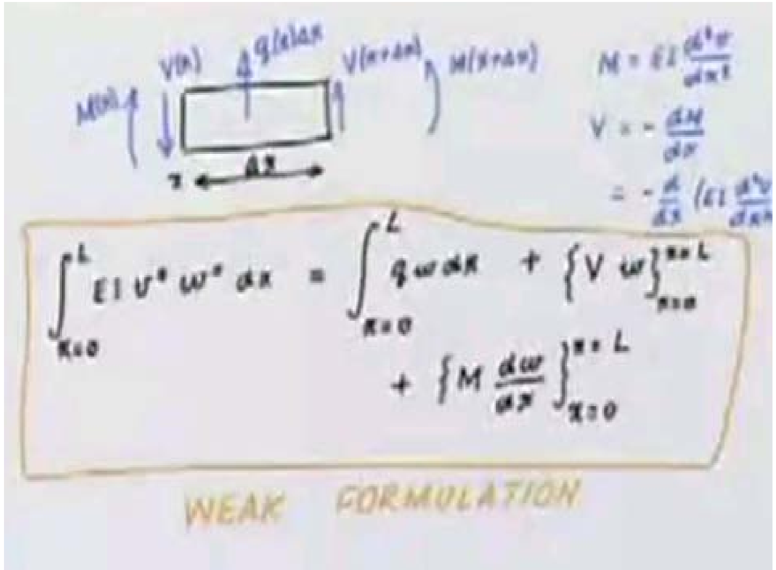

If I write the weak formulation now I will get for this one integral x is equal to 0 to L EI v double prime w double prime dx is equal to integral x is equal to 0 to L q w dx plus we had said minus of v dx of EI into w d two v dx squared, that minus of v dx of EI d two v dx squared is equal to the shear force v; because, if I go by what we have done here, M is equal to EI d two v dx squared and shear force v is equal to minus dm dx minus d dx of EI d two v dx squared.

What we will have here - the upshot is - v into w evaluated at the point x is equal to 0 and x is equal to L plus M into dw dx evaluated at the point x is equal to 0 and at the point x is equal to L. So essentially we have seen v w evaluated at L minus v w evaluated at 0 plus M dw dx evaluated at L minus M dw dx evaluated at 0. This whole thing is now my weak formulation (Refer Slide Time: 20:41 min). Once I have my weak formulation, you see that certain nice things are coming out of the weak formulation. What are the boundary conditions that are possible at the two ends of the member?

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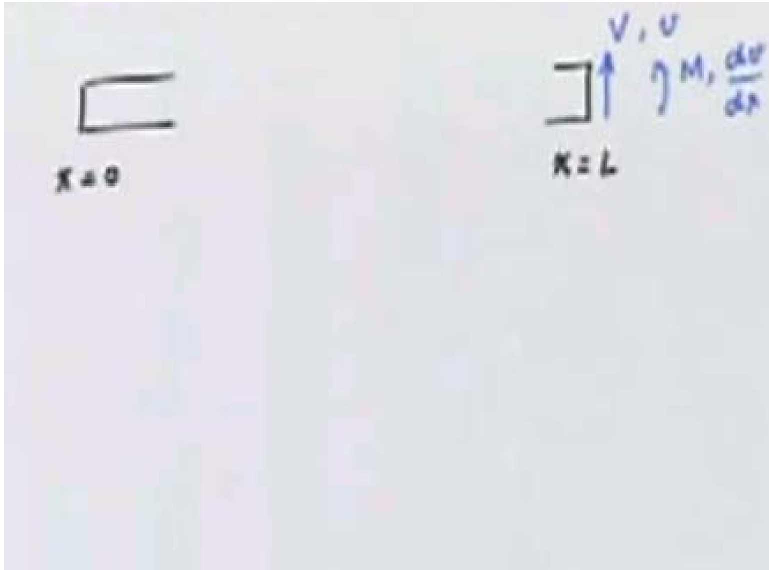

That is if I am talking of this end x is equal to 0 and the end x is equal to L. We can have, if you see that, let us take the end x is equal to L, you can have either the shear force specified here. That is I am telling you what is the size of the force that I am applying at this end or I have the displacement given. We have the shear force at the end or the v given and either I have the bending moment or I have the slope; dv dx is nothing but the slope; dv dx or what we called as the rotation.

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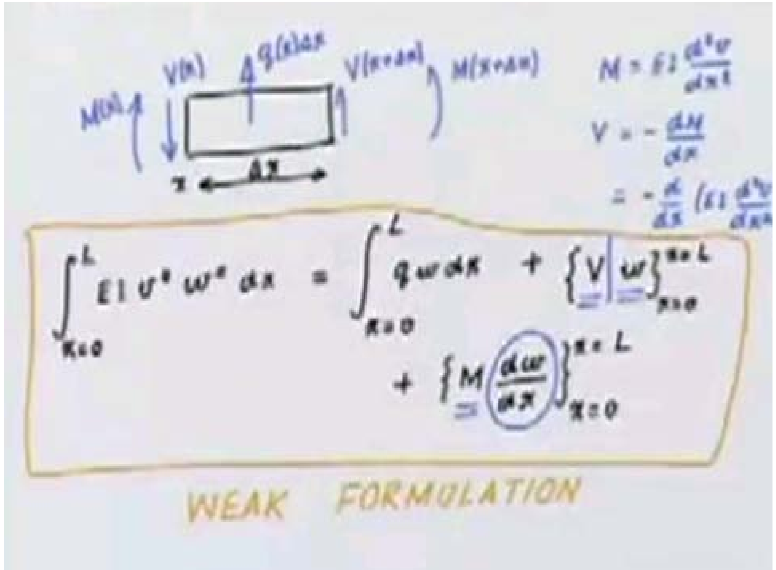

These two things are given which comes out if you see from your variation formulation, you have either the force at the end either the force is given or v is given, which means the w is constrained to be 0 there; just like we did in the earlier case. So, the end where v is given then w is allowed to be anything, but if small v is given, then w has to be constrained to be 0. Similarly, along with this I have to give a boundary condition on the end moment.

So either I specify what is the end moment M, in which case, dw dx is allowed to be free to be anything, or I say what is the rotation here? That is, what is dv dx here? In which case dw dx is forced be 0. You see we have through this we have naturally brought out what are the applicable boundary conditions for our problem of interest. Now, you can tell me what are these boundary conditions.

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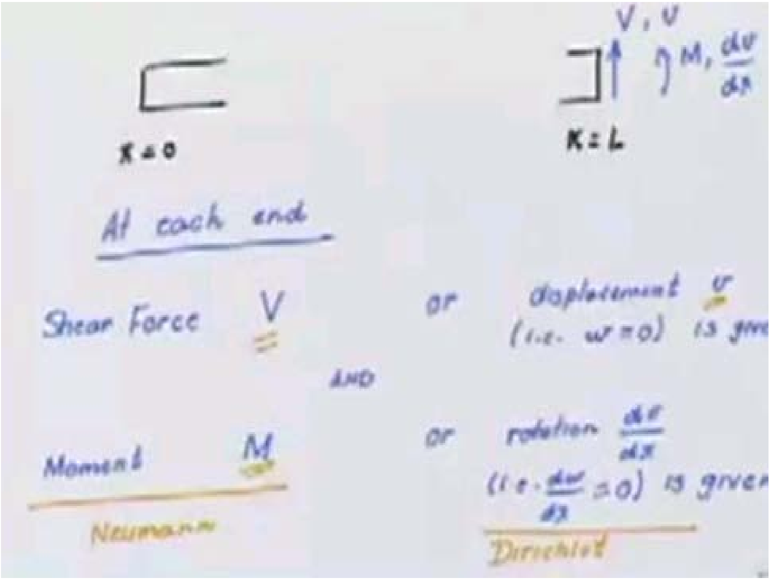

I have boundary conditions at each end of the following type that force V, which is shear force V or displacement V, which is equivalent to saying that i w is equal to 0 is given.

At each end, now I have to specify two boundary conditions and the bending moment M is given (Refer Slide Time: 24:21 min). So we have here our Neumann boundary conditions are given in terms of the specification of the shear force or the bending moment M. These are our so-called Neumann conditions and corresponding Dirichlet conditions or the displacement conditions or the specification of either the end displacement v or/and the end rotation dv dx; so this is Dirichlet. No other types of boundary conditions are possible for this beam model.

You see that as far as the shear force and the bending moment are concerned they naturally occur in the weak formulation. So they are called natural boundary conditions or the Neumann and these conditions of v or dv dx being specified has to be enforced through a constraint on the w. So they are said to be essential boundary conditions; that is they have to be enforced.

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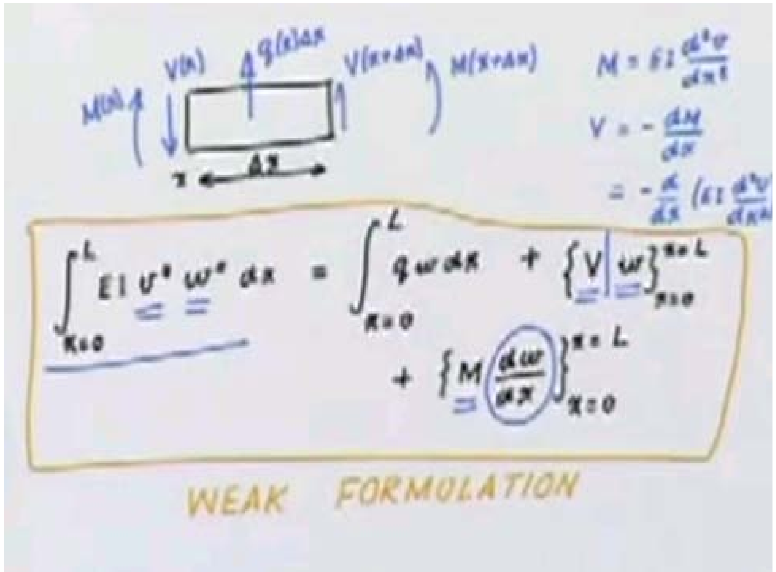

Let us go back to our weak formulation that we have here. You see as far as our terms are concerned, here we want our second derivative of v and second derivative of w to be defined, for this integral to be finite. If the second derivative of v and second derivative of w are not defined then this integral become infinite, they will not be defined. We need v double prime and w double prime to be defined. This is our so-called minimum smoothness requirement on the v and w.

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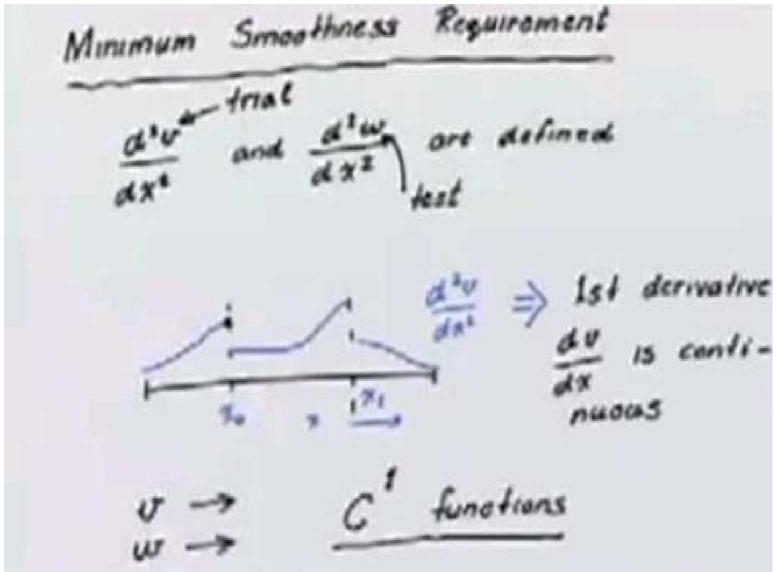

That d two v dx squared and d two w dx squared are defined. Remember that using the language that we had used earlier for the bar problem v is our so-called trail function; v is the trail function and w is our so-called test function. So for both the trial function and the test function we want the second derivatives to be defined. What does that mean? That means if I have, this is my domain, I can have a situation like this that the second derivative of this function can have so if I plot this d two v dx squared as a function of x, second derivatives can have jumps at points x_0 or x_1, at some finite number of points they can have jumps. That is the d two v dx squared and d two w dx squared by construction as far as the minimum smoothness requirement is concerned can be allowed to have jumps. If these two things will have jumps, the second derivative will have jump, it implies that the first derivative dv dx is continuous; that is, at these points, the first derivative will be continuous, but there will be a change in slope of the first derivative, which is essentially the jump in the second derivative.

If the first derivative of this function v as well as the w is continuous, obviously, the function v and w itself is also continuous. In fact, they are continuously differentiable. So we need to construct as far as minimum smoothness requirements is concerned v and w to have continuous first derivatives. Then we require our functions to have continuous first derivatives they are said to be C one functions. If you remember - why did we need this minimum smoothness requirement? To construct our basis functions for the approximation. As far as the approximation, yet we have not done any approximation, we have simply written the weak formulation. So, as far as our approximation is concerned for that the basis functions that we are going to construct for v and for w have to be C one function, that is they have to ensure that their first derivative is continuous.

The question is again - you should always ask this question - can we ever have a situation where the minimum smoothness requirement is indeed required or necessary?

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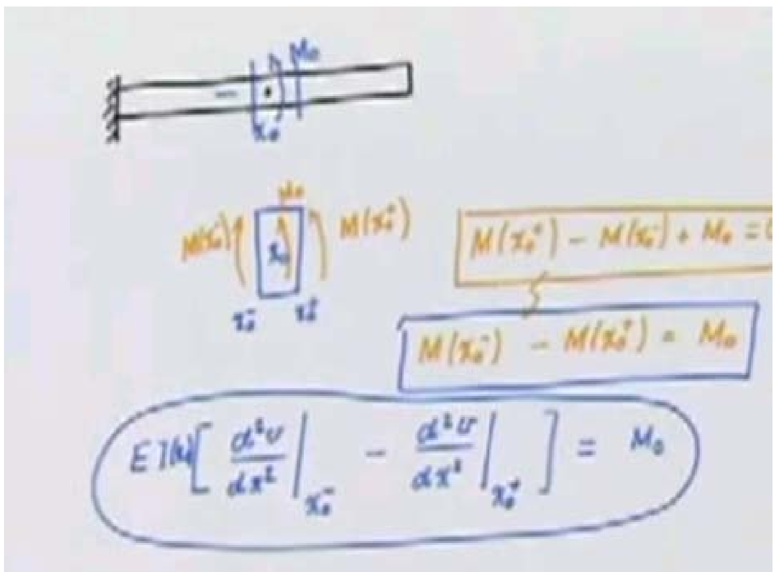

Let us take an example; because, you see that our construction of the basis functions is being done with respect to the minimum smoothness requirement. If I am interested in solving a set of problems for which this minimum smoothness requirement is never reached, then why do I have to construct basis functions like this? Let us see the situation - is it a practical situation?

Let us take a problem where, at this point, at some point x_0 in the domain I am applying a bending moment M_0. I am applying a point moment, a concentrated moment at the point x_0. Due to this point moment, there is a jump in the bending moment from this side to this side. That is if I take this block out x_0 this is I say x_0 minus x_0 plus; here I have applied M_0; this is M at x_0 minus; this is M at x_0 plus. I am shrinking the size of this piece, so that essentially I am at x_0. Then I will have M at x_0 plus minus M at x_0 minus plus M_0 is equal to 0. That is as I shrink the size of the piece to 0, I get that M at x_0 minus M at x_0 plus is equal to M_0.

Now, with this condition I have a jump in the bending moment at the point x_0, jump in the bending moment means what? That EI, let us say that the bar has uniform cross section, to be on the safe side, to be simple minded, and the Young's modulus is also the same.

We will have EI d two v dx squared evaluated from the x_0 minus; that is from this side of x_0 minus d two v dx squared at x_0 plus is equal to M_0. What I am getting? Because EI at this point x_0 is same for both sides. I will get the d two v dx squared at x_0 minus minus d two v dx squared at x_0 plus is equal to M_0 by EI. This is tells me that there is a jump in the second derivative at this point. You see that such solutions for which the second derivative has a jump are indeed feasible, very feasible, because we do have such problems where you have a concentrated moments applied at points.

I could have a concentrated shear force applied at a point; in that case what will I have the jump in the third derivatives is given by the shear force, but that is the smaller constraint that is the more forgiving constraint, that third derivative has to jump, but the minimum smoothness requirement is indeed reached when I have a bending moment applied at a point. This is highly feasible.

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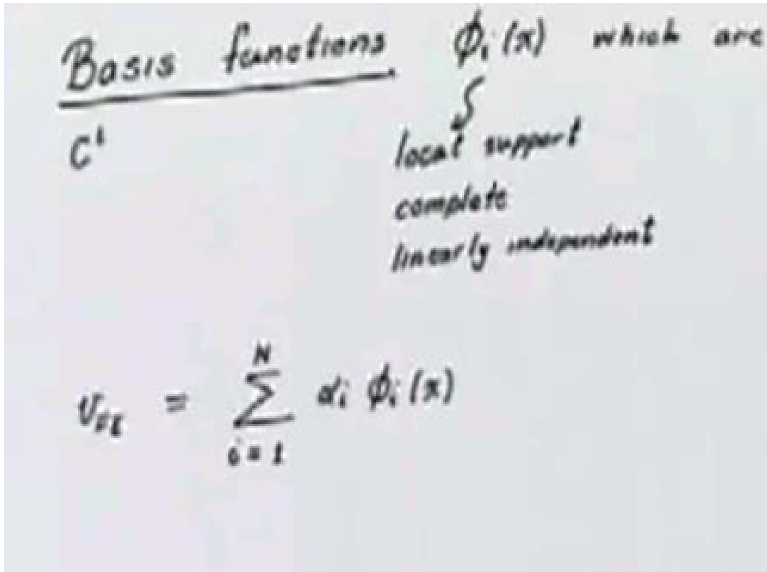

So from our approximation point of view we are going to construct basis functions phi_i of x which satisfy this C one continuity requirement. Remember when we are talking of basic functions we need a global definition of these functions.

But again from the philosophy that we have from the finite element computation that we have taken this phi_i of x have to have local support; what property they should have? They should have local support, they should be complete, that is again the appropriate order polynomials can be exactly represented and they should be linearly independent. If my basis functions by construction satisfy these properties, then we are in good shape. Let us now go and construct the basis functions. If I have this then, obviously, I will say the v finite element, that is the representation of the solution, will be equal to sum of i is equal to 1 to sum N $alpha_i$ phi_i of x; such that v finite element has continuous values and first derivatives at each point in the domain.

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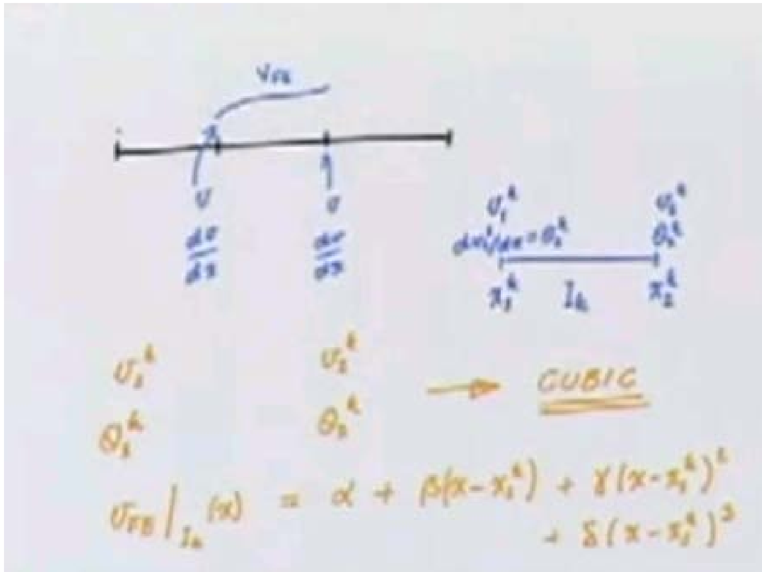

How are we going to construct these basis functions? To construct these basis functions, let us first take a partition of the mesh into some element. Let us say that we have made three uniform elements.

If I look at a particular element, let us say this element; for this, I would like this is my V_{FE} here, I would like the V_{FE} to have continuous values of the function and the derivative at these end points, because that where it is transitioning from one element to the other. What would I like to have continuous? So here, if at this point if I say I gave v and dv dx and at this point I give you v and dv dx which are unique at these points; similarly I do here and I do here and then I say that I would like to construct functions such that they take these values and give the function $V_{FE.}$ How am I going to construct those functions? If you look at this element, so let us take a generic element I_K. So here this is my point x_1 of k; this is my point x_2 of k. What I am saying? I am saying that here the v_1 of k is given and I will say dv_1 of k dx is given. This I will call as $theta_1$ of k. And similarly, here v_2 of k and $theta_2$ of k are given. So given these four values, what is the minimum order polynomial that we can fit? Given these values v_1 of k, $theta_1$ of k at the point x_1 of k, v_2 of k, $theta_2$ of k, we ask a question what is the minimum order polynomial which can interpolate these values in this element?

There are four values given; the minimum order polynomial which will fit these four values is a cubic. The finite element solution what we are going to do is V_{FE} in the element I_k is a function of x would be represented as a cubic, as some alpha plus beta x, I will put it as x minus x_1 of k plus gamma x minus x_1 of k whole square plus delta x minus x_1 of k whole cube. This is my cubic. I have deliberately written it in terms of x minus x_1 of k to basically make life convenient for me; otherwise, this is still a cubic it does not matter whether I write it in term of x minus x_1 of k or in terms of x; only the coefficient should have different meanings. If I take this cubic, my idea is to now define a cubic in terms of these fours specified values, because, I would like to define this function piecewise ensuring continuity of the value of the function as well as the derivative at the inter element interface.

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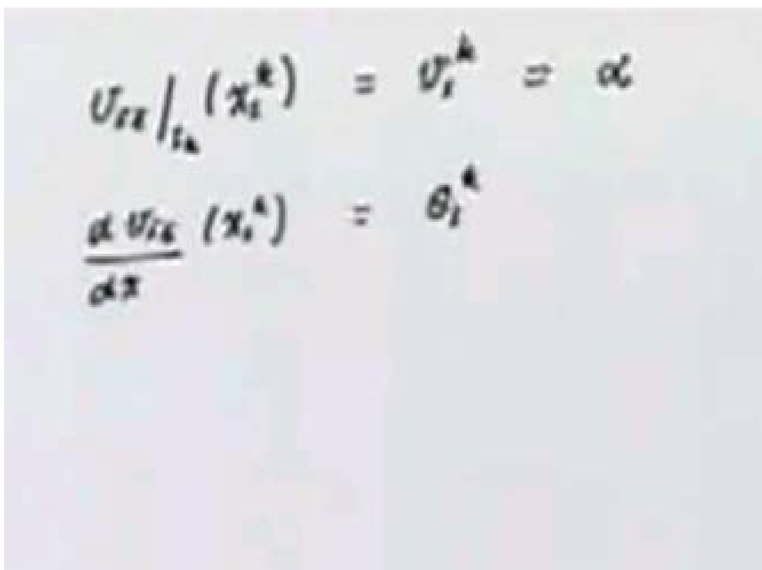

If I go ahead and put that V_{FE} in the element k at the point x_1 of k is equal to what we have given its v_1 of k which is equal to from or expression, this is the cubic, which is equal to alpha. Next, I say d V_{FE} dx from the element x_1 of k, I am not going to write that over and over again, at the point x_1 of k is equal to theta_1 of k.

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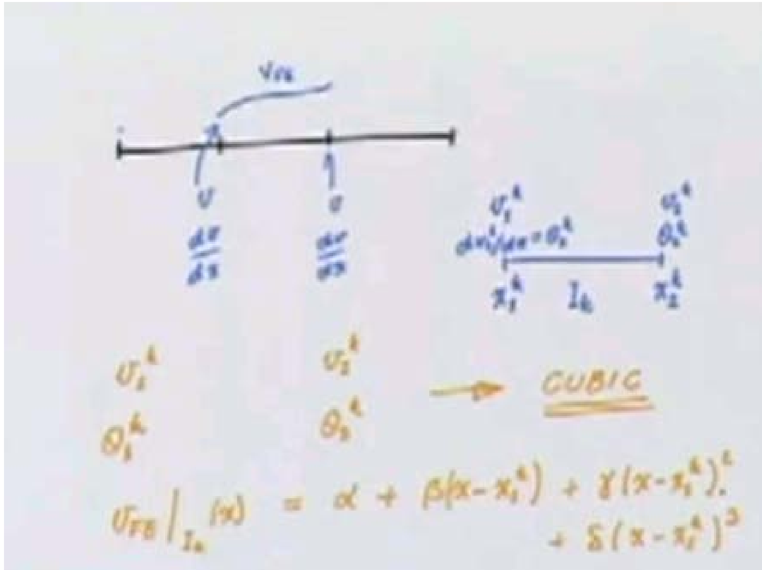

This if I look at what you have done, if I take the derivative of this it will become beta plus two gamma into x minus x_1 of k plus C delta into x minus x_1 of k whole square and the point x_1 of k we are left with beta.

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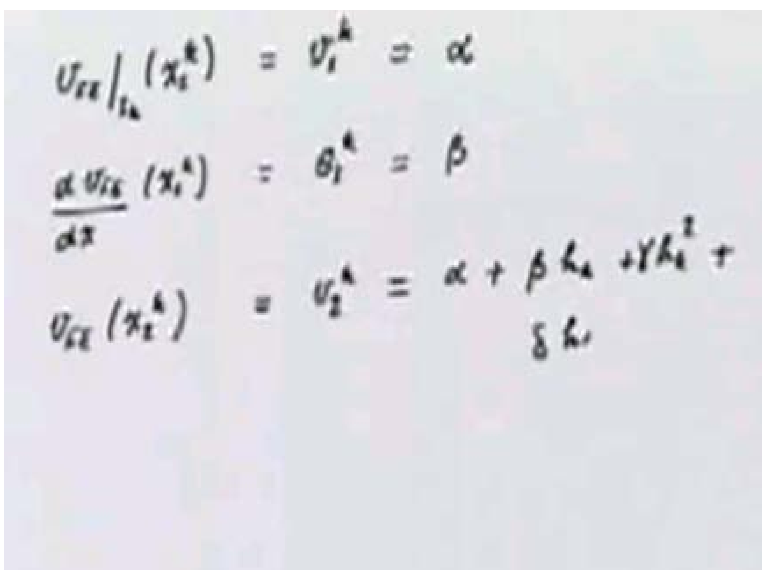

This will be equal to beta. Similarly, I go to the other end V_{FE} at the point x_2 of k is equal to v_2 of k; this is equal to alpha plus beta. Now x_2 of k minus x_1 of k is equal to h of k. So this is h of k plus gamma x_2 minus $x_1$of k whole square which is h of k whole squared gamma plus delta into h of k whole cube.

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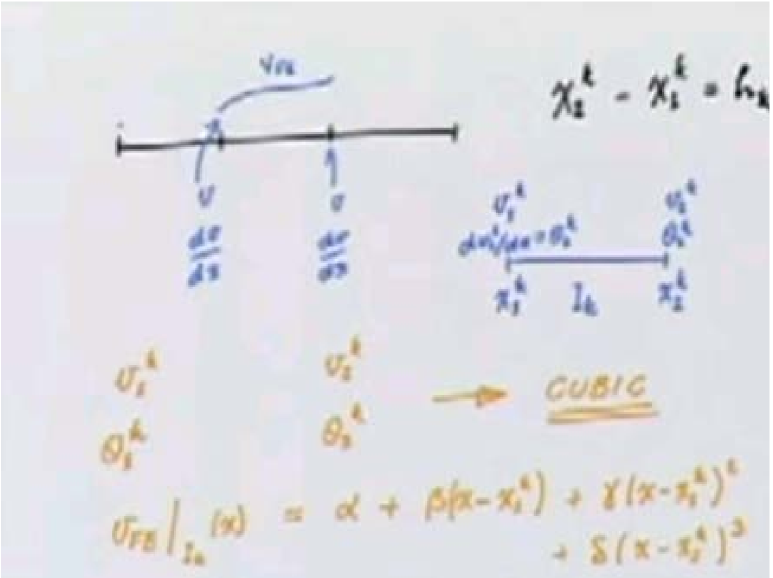

So here if I put x_2 of k minus x_1 of k is the size of the element h k. From this expression, that is what I am going to get.

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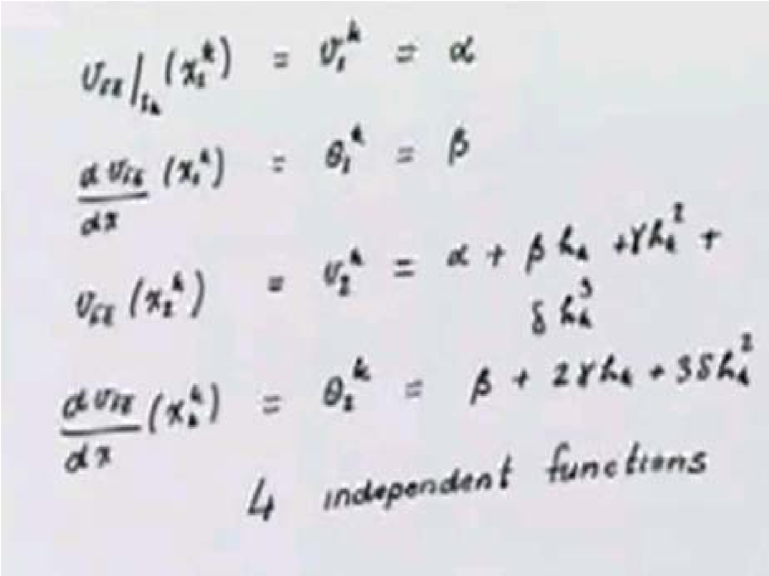

Similarly, I will put d V_{FE} dx at the point x_2 of k is equal to theta_2 of k this is equal to beta plus 2 gamma h_k plus 3 delta h_k square. You see that now the alpha and beta are easily obtained; they are nothing but v_1 of k theta_1 of k. Now, in terms of v_2 of k and theta_2 of k and the v_1 of k and theta_1 of k, I can obtain gamma and delta.

Once I have obtained these coefficients of the cubic in terms of these four given quantities, then now I can write my cubic expression in terms of v_1 of k into something plus v_2 of k into something plus theta_1 of k into something plus theta_2 of k into something; that gives me how many functions? In a cubic, for example, how many independent basis functions will we have? As we have done for the linear we have two; for the quadratic we have three; again this is the cubic, we will have four independent functions.

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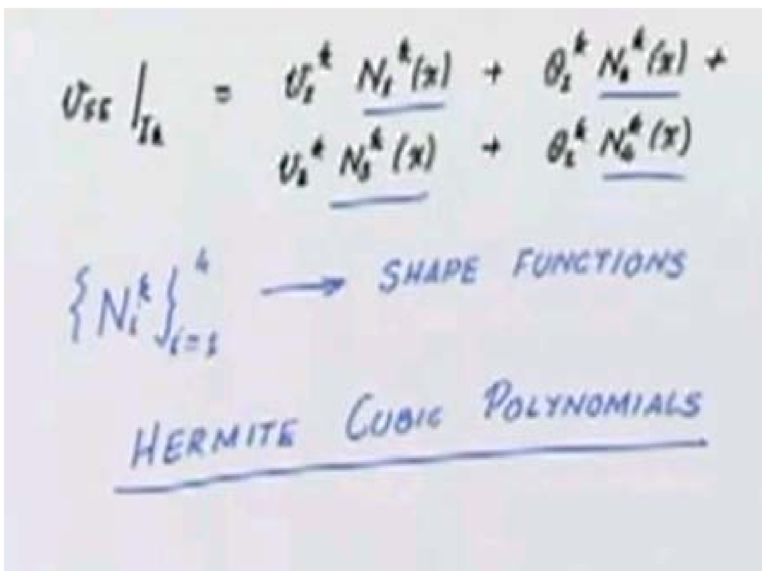

These we will define in detail in future, but it is quite easy to now do the algebra here and derive this functions. What will we have? If I write it V_{FE} in the element Ik is equal to V_1 of k and I will call this function as N_1 of k as a function of x plus theta_1 of k is a function N_2 of k plus the function V_2 of k plus theta_2 of element k into N_4 of k x. These four are going to be our independent functions.

Once I define these functions, where have I defined them? I have defined them in the elements; so these functions N_I of k for I equal to 1 to 4 are the shape functions. Now, by piecing together our shape functions, we can construct our global basis functions.

In the next lecture, we are going to define these shape functions that we have constructed at the element level; we will give the expression for those in the physical element as well as in the master element, because here also we will do the same thing that we did for the second order differential equation.

We will take out physical domain to the master domain, physical element to the master element, we will convert all our expressions, all our integrals from integrals over the physical element to integrals over the master element, because ultimately we will have to do the numerical integration.

From that point of view, we are going to do that conversion, define the shape functions in the master element and see there is a curious change from what we have done earlier. And we are after that going to say how to do the integration; the points will remain the same, essentially what is the order of integration rule that I have to take and so on. And we look at some basic properties of the solution using these functions.

If you see your N_I of k, these are cubic functions, they are given a name - these are called Hermite cubic polynomials and remember they are very different from the cubic Legrangial polynomials that we had defined in the element as our element shape functions for cubic approximation for the second order differential equation. This we have to keep in mind. We will look at these things in the next class in detail, look at pictures of these and then will go from there.