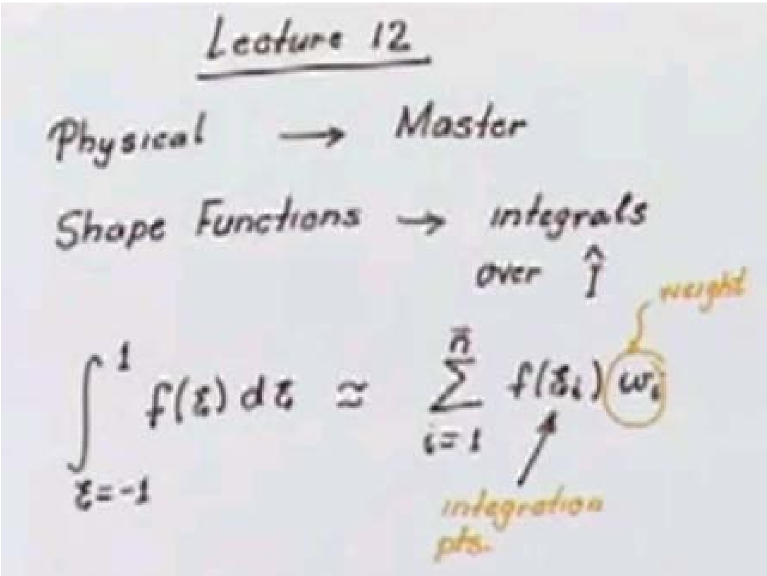
Finite Element Method

Prof C.S.Upadhyay
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Module – 4 Lecture - 2

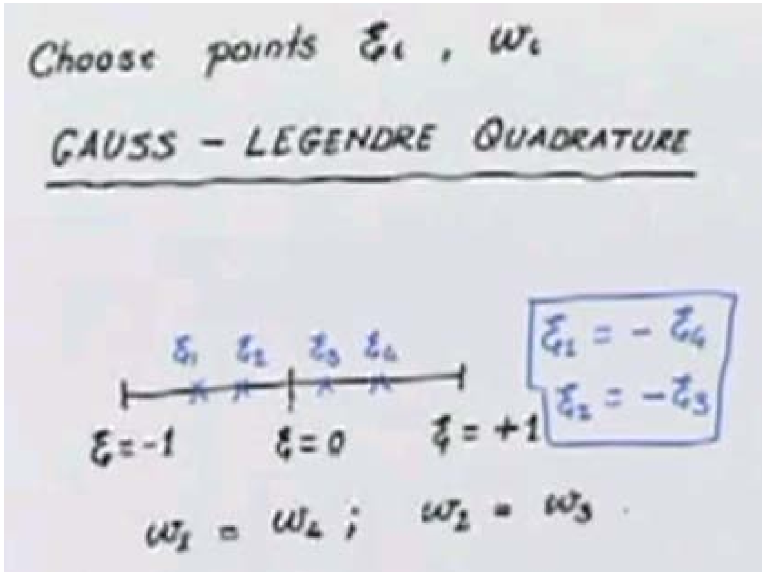
We are at the lecture 12; what we had discussed till now was how to do the element calculations in the master element. We had transformed our equations from the physical domain to the master domain and in the master domain we had said that we will be defining the shape functions for the element and transform the integrals that we had over this domain.

(Refer slide time: 00:23)



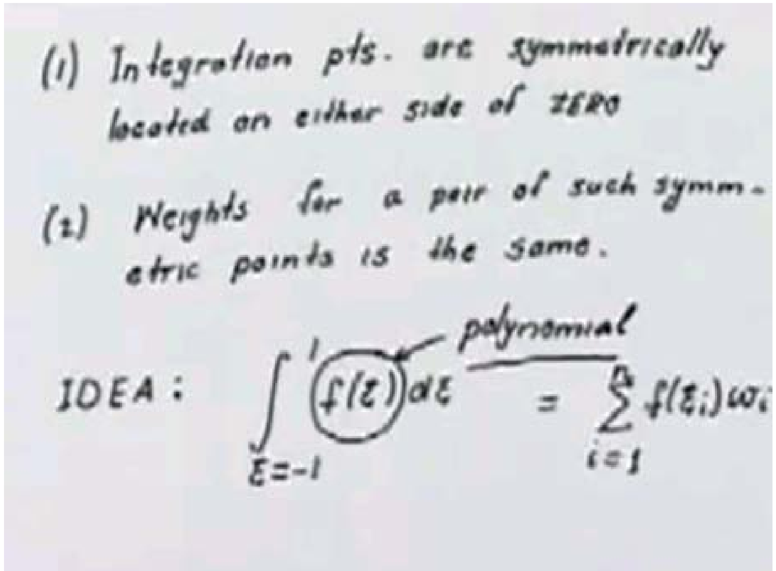
The question was how to convert these integrals into computer implementable form, because for us to do the integration of polynomials and other functions by hand is easy, but how will the computer do that? Computer generally what does it do? It integrates some quantity by replacing it with a sum. A sum over some points 1 to n bar let us say; so sum of the function psi, the f at the point psi_i into the so-called weights w_i. These points psi_i are called integration points and w_i is the corresponding weight.

(Refer slide time: 02:49)



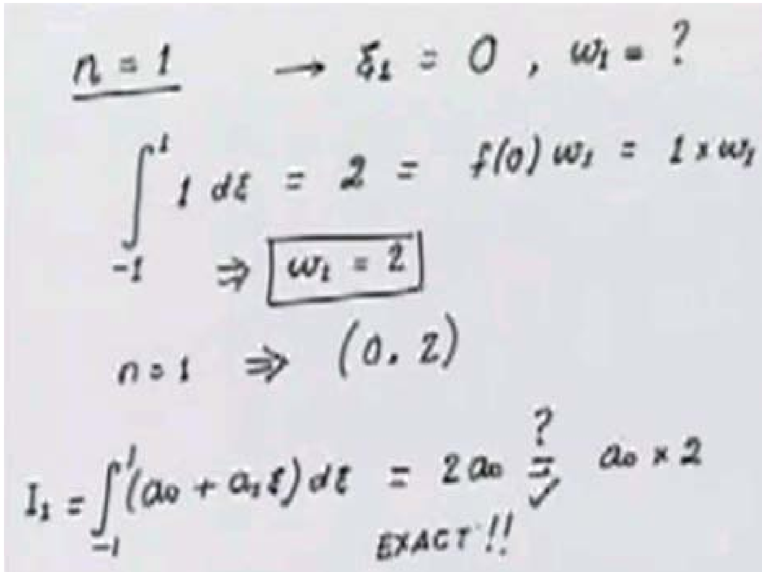
The question is fine, how do we choose points psi_i and the corresponding weights w_i? What we use here is the so called Gauss Legendre Quadrature integration rule; what is the basic idea? I have now my master element spanning from psi equal to minus 1 to psi equal to plus 1. The centre of it is the point psi equal to 0. What I am going to do is I am going to choose points which are symmetrically placed on either side of 0, so I can choose this point and the corresponding point here (Refer Slide Time: 03:54). These are going to be our choices of integration points such that if I call this point as psi_1 this point as psi_2, this is psi_3, psi_4 then psi_1 is equal to minus psi_4, psi_2 is equal to minus psi_3. Further, for this set of symmetric located points we will have the same weights. So w_i will be equal to w_4 in the example that we have taken and w_2 will be equal to w_3.

(Refer slide time: 04:55)



We will say that this, the feature of this method is (1) the integration points are symmetrically located on either side of zero. (2) The weights for a pair of such symmetric points is the same. How do you design the location and the weights of the points? The basic idea again is that, we would like our choice of points and weights to be such that for a polynomial of a particular order the integral comes out to be exact. If my integrant here (Refer slide time: 06:35) is a polynomial, if this is a polynomial then for a particular order of the polynomial this integral should be exact; that is the sum should be exact, therefore psi. Let us now go ahead and construct these sets of points, take the first example.

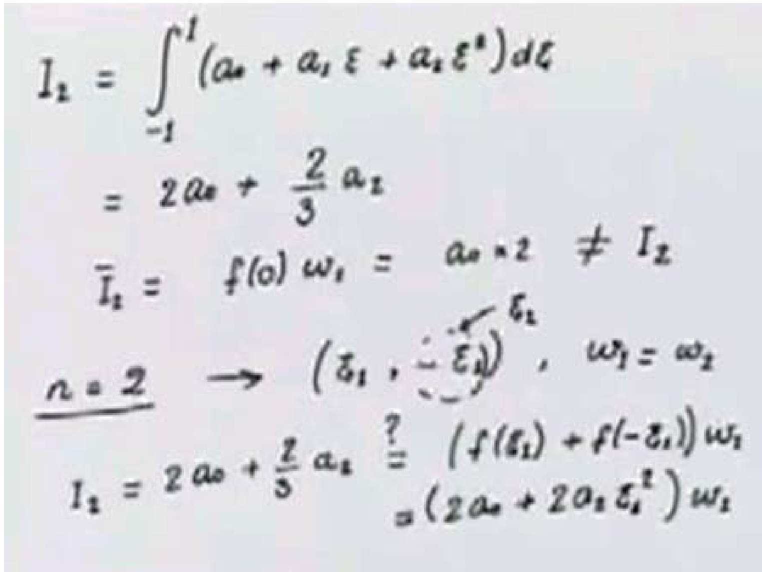
(Refer slide time: 07:13)



First example is let us take n is equal to 1. If I take n is equal to 1 then I notice that the point psi$_1$ can be nowhere else, but at the 0 location because it is a symmetrically placed point and the corresponding weight will be w_1. Now w_1 is what we have to determine. Let us take first a constant; I integrate the constant from minus 1 to 1; let us take the constant as 1. This integral will be equal to 2 (Refer slide time: 07:50). Now this I would expect one point rule to at least integrate the constants correctly. It is going to give me from the one point rule F at 0 into w_1 which is equal to 1 into $w_{1;}$ implies w_1 is equal to 1. This is going to be our one point rule. So one point rule for n equal to 1 implies the following pair at the point is 0 and the weight is 2. For which higher polynomials does this one point rule do an exact job? Let us look at the next higher polynomial. If I take fpsi to be linear in psi, it is a_0 plus a_1 psi d psi (Refer slide time: 08:55). If I do this integral, I am going to call this integral as $I_{1,}$ is equal to this. This integral will turn out to be 2 a_0 (Refer slide time: 09:08).

The second part will drop off because if you see this a_1 psi is an odd function and the integral of an odd function over this interval is going to be 0. My question is, is it equal to what I get out of the one point rule? Out of the one point rule f at the point psi equal to 0 is a_0 into the weight at the point psi equal to 1 which is 2; this integral is exact. The one point rule not only integrates the constants exactly it also integrates the linears exactly.

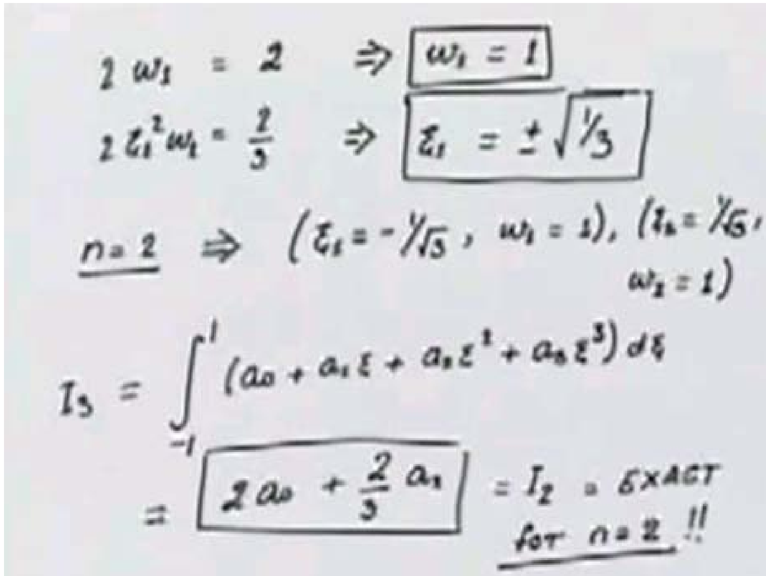
(Refer slide time: 09:58)



Let us become a little bit more greedy, and ask whether this integral can also be obtained exactly using the one point rule. This integral (Refer Slide Time: 10:05) if you see it will be equal to by what we have done $2a_0$ plus this part will drop off; this part is an even part it will give me a value 2 by 3 a_2. If I look at the one point rule, the one point rule gives what? One point rule will give me this integral equal to f at 0 into w_1 which is equal to a_0 into 2 which is not equal to I_2. The one point rule is only exact for all polynomials which are linear, for higher polynomials it is not exact and it will not give you the exact value of the integral. So for this; let us go to the next step. We go to the two point rule. If I take the two point rule what do I know? The points are psi_1 and minus psi_1 and the corresponding weight for the two pair, because these are symmetric points, so both will have the same weight. This is my point (Refer Slide Time: 11:35) if I want to call it as psi_2 is equal to minus and the weight of w_1 is equal to w_2.

What we want is this two point rule to be atleast exact for the quadratic. Because the one point rule could not do it so I expect the two point rule to do it. Let us take I_2 which we got as $2a_0$ plus two third a_2. This will be the function, the integrant evaluated at psi_1 plus the integral evaluated at minus psi_1 into weight 1. This is equal to fx psi_1 plus fx minus psi_1 will be equal to if you work it out 2 a_0 plus 2 a_2 psi_1 squared.

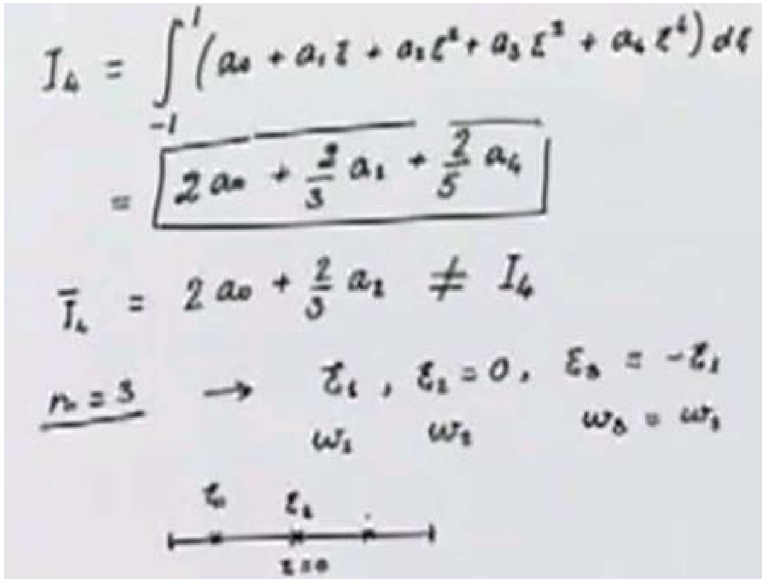
(Refer slide time: 13:08)



If we compare the two the coefficient of a_0 and a_2 I get this whole thing into w_1. If I compare the coefficients then I will get $2w_1$ is equal to 2 and if I look at the second part it will be 2 psi_1 squared w_1 is equal to 2 by 3. From here w_1 is equal to 1 and from here by substituting everything (Refer Slide Time: 13:58) I get psi_1 equal to plus-minus root of 1 by 3. So choosing these points psi_1 equal to plus-minus root of 1 by 3 and w_1 is equal to 1, the summation for the two point rule gives us an exact integral for I_2. Our two point rule n is equal to 2 will be equivalent to saying I have the point psi_1 equal to minus 1 by root 3 and the weight w_1 is equal to 1 and the point psi_2 equal to 1 by root 3 and the weight w_2 is equal to 1.

Let us see whether this two point rule can give us an exact integral for a cubic polynomial. What will the cubic be? This integral (Refer Slide Time: 15:00) if I do it will come out to be exactly the same as the integral I_2 or it will be given as $2a_0$ plus 2 by 3 a_2 and we will see by taking the two point rule, I will get the exact integral for n is equal to 2.

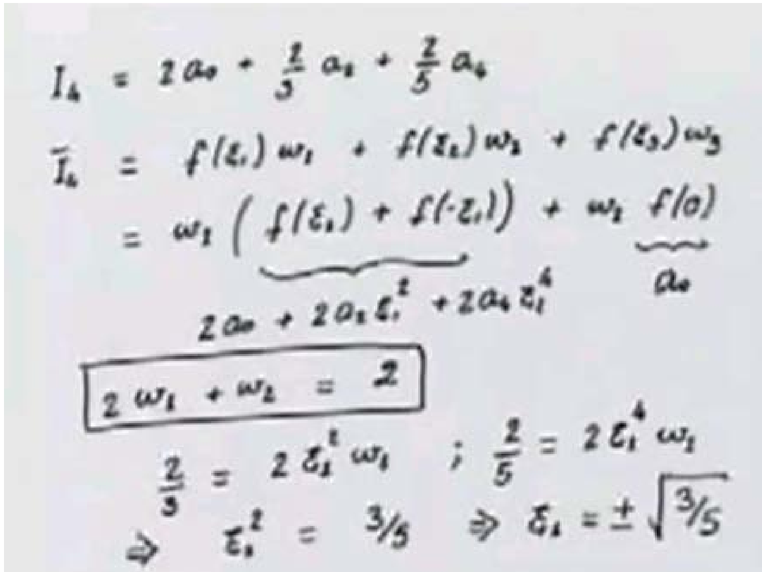
(Refer slide time: 16:10)



Let us ask if it will do the same job for I_4 which is the integral of a fourth order polynomial. It is a_0 plus a_1 psi plus a_2 psi square plus a_3 psi cube plus a_4 psi to the power 4 d psi. If I do this integral it will turn out to be $2a_0$ not plus 2 by 3 a_2 plus 2 by 5 a_4. And if I do the two point integration rule it will give me the two point integration rule it will give me 2 a_0 plus two third a_2 this one should check and this is not equal to I_4.

The two point integration rule can give us an exact integral for all polynomials up to a cubic. But for a fourth order it does not do the job. Then we naturally go to the three point rule. If I take the three point rule what will be the points? I will have psi_1, psi_2 is equal to 0, psi_3 is equal to minus psi_1, and here I have weight one, weight two, weight three is equal to weight one. If I look at this here is my psi is equal to 0 (Refer Slide Time: 17:50); this is my psi_2, this is my psi_1 and this is my psi_3 equal to minus psi_1.

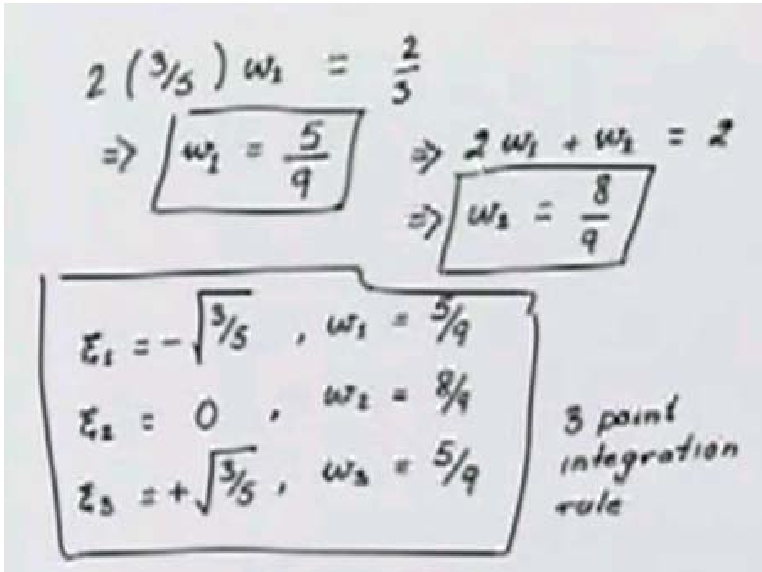
(Refer slide time: 18:18)



If I take this and it should give us an exact integral for the fourth order one; I_4 was equal to $2a_0$ plus two third a_2 plus two fifth a_4 what I want from the summation is equal to f at psi_1 into w_1 plus f at psi_2 into w_2 plus f at psi_3 into w_3. This (Refer Slide Time: 19:00) will be equal to w_1 f at $psi_{1,}$ plus f at minus $psi_{1,}$ plus w_2 into f at 0. f at psi_1 plus f at minus psi_1 you see that this is also a symmetric quantity. This will be equal to $2a_0$ plus $2a_2$ psi_1 squared plus $2a_4$ psi_1 to the power 4. This will be $2a_0$ (Refer Slide Time: 20:10).

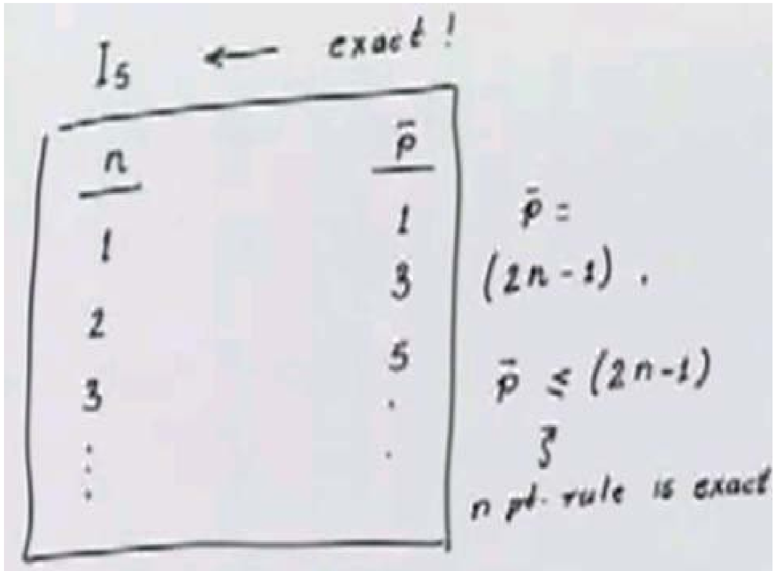
If I match the coefficients of a_0 a_2 and a_4 you see that by matching the coefficient of (Refer Slide Time: 20:32) this won't be equal to $2a_{0,}$ f_0 will be equal to a_0 by what we have done. If I match the coefficient I will get 2 w_1 plus w_2 is equal to 2, this is my first relation that comes out. Then coefficient a_2 I will get two third is equal to 2 psi_1 squared w_1 coefficient of a_4 will give me two fifth is equal to 2 psi_1 to the power of 4 w_1. If I take the ratio of these two things this will give me psi_1 squared, so divide this by this is equal to what will I get? psi_1 squared is equal to 3 by 5; implies psi_1 is equal to plus-minus root of 3 by 5 (Refer Slide Time: 22:00). See it is very easy to get these integration point locations though as the number of point's increases the task becomes more and more laborious.

(Refer slide time: 22:25)



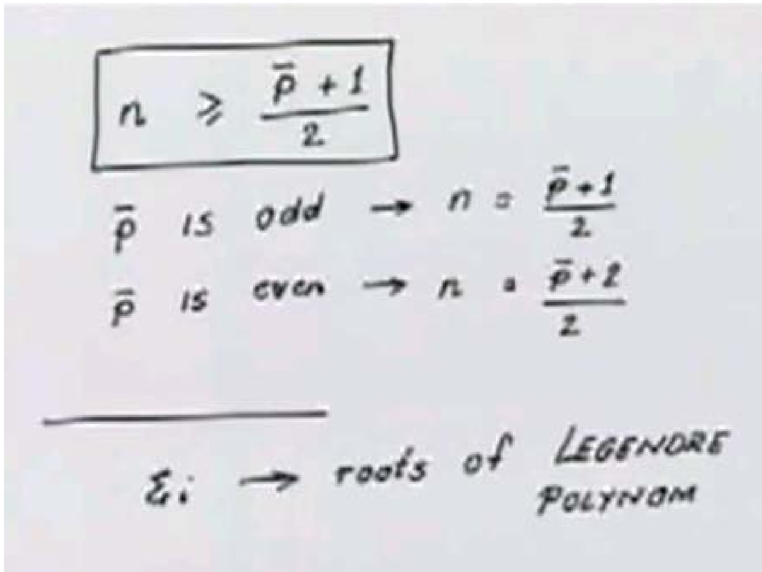
If I have that substituting this expression for psi_1 squared from our previous equation we will get that 2 into psi_1 squared w_1 equal to 2 by 3. So psi_1 squared into w_1 is equal to 2 by 3, implies w_1 is equal to 5 by 9 implies by our first equation. Since $2w_1$ plus w_2 is equal to 2 implies w_2 is equal to 8 by 9. We have found the locations of the points and the weights. So psi_1 is equal to minus root of 3 by 5, w_1 is equal to 5 by 9, psi_2 is equal to 0, w_2 is equal to 8 by 9, psi_3 is equal to plus root of 3 by 5, w_3 is equal to 5 by 9; (Refer Slide Time: 24:00); this is three point integration rule.

(Refer slide time: 24:48)



It turns out that if I go and check now for the fifth order polynomial, this was derived using the fourth order polynomial, if I find I_5 which will be the fifth order polynomial for this also the three point rule is exact. If I write it down, the order of the rule and the polynomial order P bar for which it is an exact; the one point rule was exact for the linear. The two point rule was exact for all polynomials which were at most cubic; the three point rule is exact for all polynomials which are at most fifth order. This way we can continue, can we come up with a generic rule? Yes, we can come up; you see that if I have n (Refer Slide Time: 25:36), if I have a polynomial which is 2n minus 1 P bar equal to 2n minus 1, then the n point rule does an exact job. If P bar is less than equal to 2n minus 1, then n point rule is exact. Out of this how do I choose the appropriate integration rule?

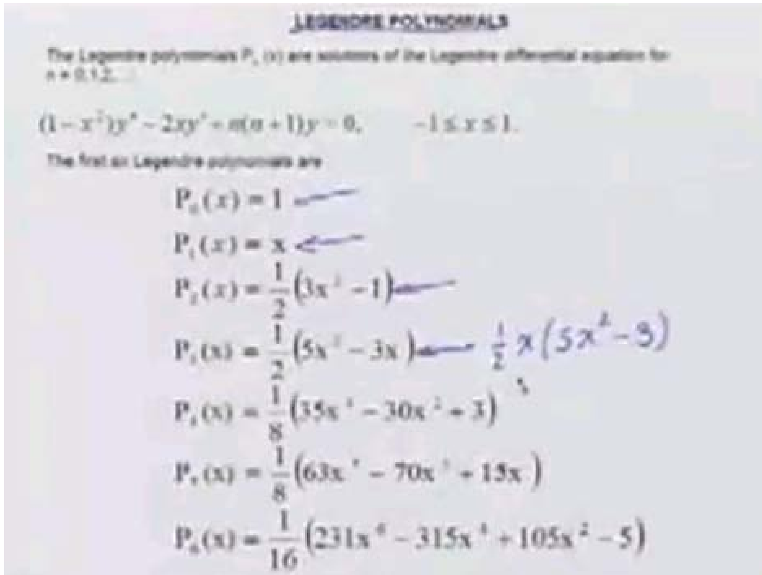
(Refer slide time: 26:36)



It is very easy now that you say that if I have a polynomial of order P then n should be greater than or equal to order P bar plus 1 by 2. If I use integration rule with number of points greater than or equal to P bar plus 1 by 2 then I know this integration is going to be give us an exact integral for that particular polynomial.

Let us say that if P bar is odd then n terms have to be P bar plus 1 by 2. So n is equal to P bar plus 1 by 2 will do the job. I am going to fix my polynomial integration rule to n equal to P bar plus 1 by 2. If P bar is even, that is the order of the integrant is in even, then I would choose n equal to P bar plus 2 by 2. That is it goes to the next higher integer point. So with this choice of integration points I can expect that whatever integrals I have to do, if they are polynomials I will get an exact value of the integral. This is very important from our numerical computational point of view. Question is, can we construct these integration points easily through some other means? It turns out that yes, these points psi_i are the roots of Legendre polynomials defined in the intervals minus 1 to plus 1.

(Refer slide time: 28:59)



Here we have an example of the Legendre polynomials that can be used to obtain the locations of the integration points. For example, if you look at the first polynomial it has its root nowhere in this interval from minus 1 to 1. Second one has the root at the point x plus 0 which for us is the point psi equal to 0. This is nothing but the point for the one point rule. If you look at the third one, the second polynomial which is a quadratic polynomial you see that the roots of this appear at plus-minus 1 by root 3. These are nothing but again the points - the coordinates of the integration points for the two point rule. The third one here can be written as half into x into 5x square minus 3. The roots are x is equal to 0 and x equal to plus-minus root of 3 by 5. This is exactly what we had derived using the long approach. Once I have these points then I can go back to what we have done there and obtained the weights for the integration rule. I can go on with the Legendre polynomials and I can get the integration rules.

(Refer slide time: 30:33)

Table: Abscissas and weight factors for Gaussian integration

$\pm x_i$		w_i
	n = 2	
0.577350269189626		1.00000 00000 00000
	n = 3	
0.00000 00000 00000		0.88888 88888 88889
0.77459 66692 41483		0.55555 55555 55556
	n = 4	
0.33998 10435 84856		0.65214 51548 62546
0.86113 63115 94053		0.34785 48451 37454
	n = 5	
0.00000 00000 00000		0.56888 88888 88889
0.53846 93101 05683		0.47862 86704 99366
0.90617 98459 38664		0.23692 68850 56189
	n = 6	
0.23861 91860 83197		0.46791 39345 72691
0.66120 93864 66265		0.36076 15730 48139
0.93246 95142 03152		0.17132 44923 79170

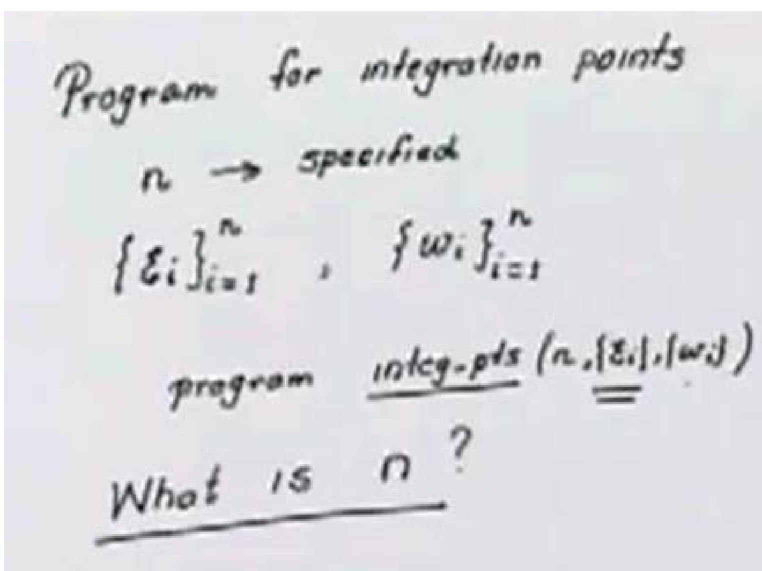
The location of the points corresponding to the integration rule of choice and it is given here in the form of a nice table, you see that if I have the one point rule there is nothing big that have to be done the location is psi is equal to 0 and the weight is 2. If I have two point rule here (Refer Slide Time: 30:51) I have two points with plus-minus X_i as my coordinates. This is the symmetric points and the weights are this. Here the efforts have been made; this has been borrowed from a book, to obtain the points with sufficient precision; if you see 1, 2, 3, 4, 5 so up to 15 decimal places this has been obtained. That is very important from the computational point of view because we would like to do our computation in double precision. If you want to do our computation in double precision we should ensure that these points the gauss points, or the integration points that we have and the corresponding rates are also obtained in double precision. Very nicely all these points are given here you see for the three point rule you have these things (Refer Slide Time: 31:38) and so on.

(Refer slide time: 31:51)

	n = 7	
0.00000 00000 00000		0.41795 91836 73469
0.40584 51513 77397		0.38183 00505 05119
0.74153 11855 99394		0.27970 53914 89277
0.94910 79123 42759		0.12948 49661 68870
	n = 8	
0.18343 46424 95650		0.36268 37833 78362
0.52553 24099 16329		0.31370 66458 77887
0.79666 64774 13627		0.22238 10344 53374
0.96028 98564 97536		0.10122 85362 90376
	n = 9	
0.00000 00000 00000		0.33023 93550 01260
0.32425 34234 03809		0.31234 70770 40003
0.61337 14327 00590		0.26061 06964 02935
0.83603 11073 26636		0.18064 81606 94857
0.96816 02395 07626		0.08127 43883 61574
	n = 10	
0.14887 43389 81631		0.29552 42247 14753
0.43339 53941 29247		0.26926 67193 09996
0.67940 95682 99024		0.21908 63625 15982
0.86506 33666 88985		0.14945 13491 50581
0.97390 65285 17172		0.06667 13443 08688

Here in this table, I think the points are given up to n is equal to 10 to the ten point rule. 1 2 3 4, up to the 10 point rule and in whatever we will do in this course and most practical applications we do not need more than a 10 point rule.

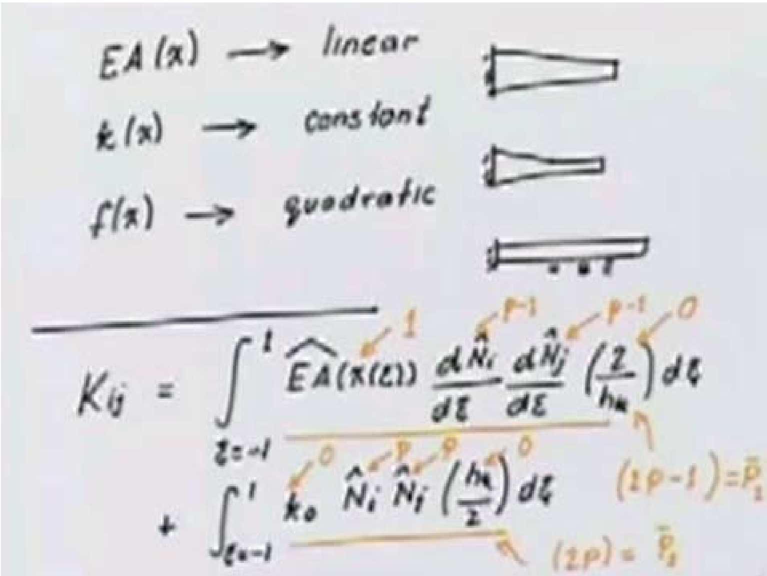
(Refer slide time: 32:20)



The question is, given this 10 point rule, you should be able to write a program where if I give you what is the n that I need n specified then you should return the corresponding points psi_i and the corresponding weights. Let us call this program or routine or the sub program whatever you want to call it as intake points where the input is n and the output is the values of the psi_i(s) and w_i(s).

Once we have this program where we input the n, I should be able to output what are the coordinates, all the integration points and corresponding weights. Then I should be able use this in the program. Remember we are going to use this in the finite element computations. More importantly, in the element stiffness matrix and the element load vector calculations; we have to first specify what is n. This question has to be first answered in order to obtain the n which will pass to this routine and this routine will return back the coordinates of the point's integration points and the corresponding weights. How are we going to determine this n?

(Refer slide time: 34:50)

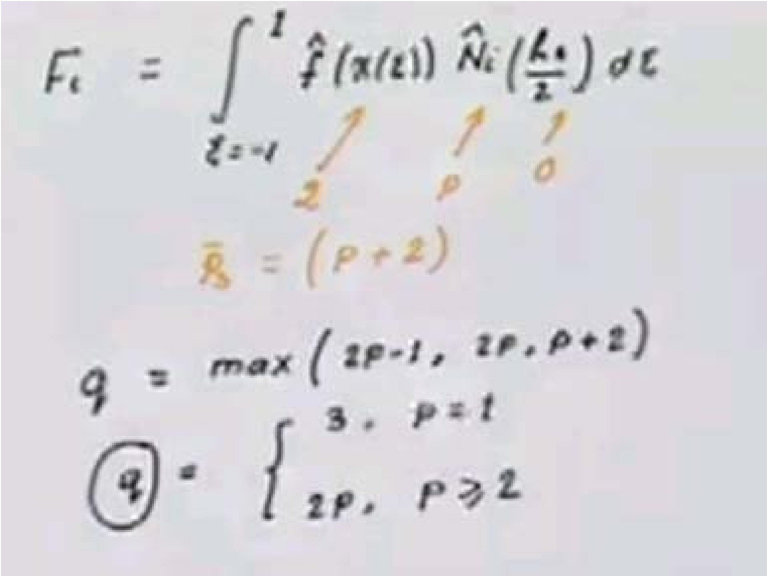


Let us say that the objective of our program is to handle a material parameter which is linear at most. That is it corresponds to tapering bar. I can have this kind of a situation which I would like to handle, a tapering bar.

We will say that our distributed spring support or the elastic support is constant (Refer Slide Time: 35:15). We do not want this to be anything more than a constant. And when we say constant and linear we mean that to be piecewise. That is I could have this kind of a tapered bar or I could have elastic support in only a particular region of my member (Refer Slide Time: 35:40). As long as it is satisfied piecewise we are happy. So k(x) has to be piecewise constant we are going to fix it which is what our code is capable of handling, the finite element program that we are going to develop. EA is linear and this distributed load f(x) you say it is almost quadratic.

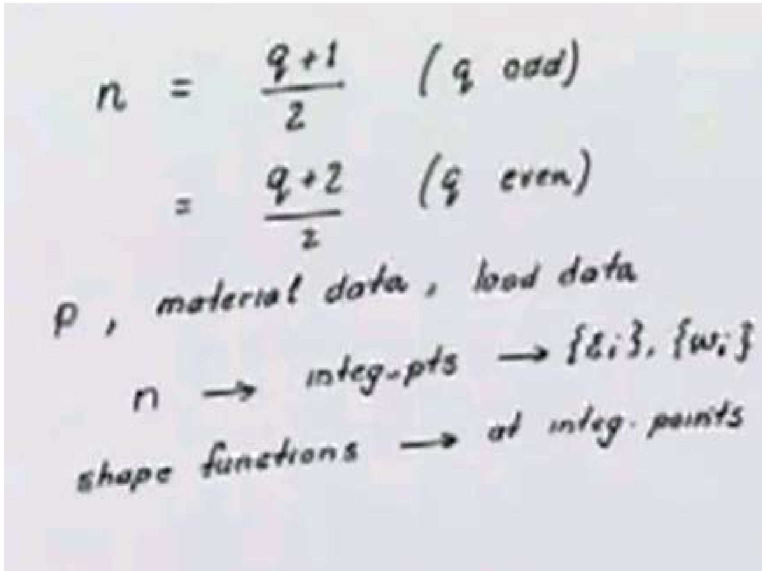
Again when you say quadratic it need not be quadratic in the whole domain, it may be quadratic in one piece, 0 in the other piece and linear in the other piece but not more than quadratic piecewise. If we have these things, then what is it that we have to do at the element level? If you remember that in the stiffness calculation at the element level we had these integrals (Refer Slide Time: 36:50) we had 2 by h of the element d psi plus integral psi equal to minus 1 to 1, if I am are putting k to be constant, it will k_0 into which a constant value N_i hat N_j hat into h of the element by 2 d psi. This is from what we have already developed. If I want to now fix the order of the integrants here what are the integrants? This is an integrant, this is an integrant. If you look at the order of the first one what is the order? The order of this one (Refer Slide Time: 38:00) is 1 it is linear, the order of this one is P minus 1, the order of this P minus 1, the order of this one is 0. The total order of this integrant is equal to, for this one, I will have the order of the integrant is 2P minus 1; I simply sum up this order. So 1 plus P, 1 is P 2P minus 1. If I look at it here, this one of order 0 this is of order P, this is of order P, this is of 0. Sum of the orders this integrant is of order 2P. This order I will call it as P_1 bar this I will call it as P_2 bar.

(Refer slide time: 39:10)



Again let us look at from the element k what is my load vector? Its load vector goes from minus 1 to plus 1 f hat function of psi N_i hat into h k by 2 d psi. By what we had done, this is of order at most 2 this is of order p this is of order 0. This one if I look at the order of the integrant here the order is P_3 bar, I will call it, is equal to P plus 2. For our integration rule the order of the integration rule to choose that I will have to take that the maximum of all this polynomial orders, because I would like to integrate the highest order polynomial exactly. What I will get is the order q is equal to max of 2P minus 1, 2P and P plus 2. You see that q is equal to for P is equal to 1. If I take this one is 1, this one is 2, this one is 3. So 3 for P equal to 1(Refer Slide Time: 40:40). If I take P is equal to 2 then this one is 3, this one is 4, this one is also 4. If I take P is equal to 3, then this one is certainly less than this one the second one, second one is 6 and this one is 5. This is equal to 3 for the choice of the variation of EA F and K that we have chosen and the order of the polynomials that we are picking for the approximations. The integration order, the maximum order of polynomial for which I would like to have exact integrals is given by either 3 when P is equal to 1 have or 2P when P is greater than or equal to 2.

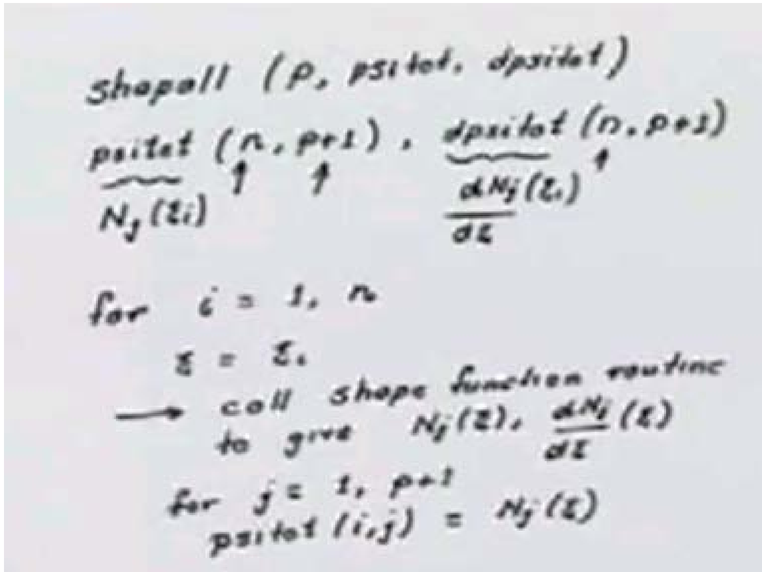
(Refer slide time: 41:47)



Then our n is equal to q plus 1 by 2 will do the job; this is true when q is odd and this is equal to q plus 2 by 2 when q is even. You see that in our approximations apriory, even before doing our computations we can fix what is the order of the integration we need because polynomial order of approximation P the material data that is EA and k and the load data are available to us a priory as input data; using those data I should be able to determine n.

The next task is if I know n, I know how to construct the integration points and the rates, given the n. I pass this n (Refer Slide Time: 43:09) to my routine integ- points and it will output to me the integration points psi_i and w_i. Once I have output these integration points in order to do smart computation; you remember that finally all this will be done by a computer. And the computer should not be subjected to repeated tasks over and over again. It is a good idea when I am using sufficiently large number of elements in the mesh and then to construct the shape functions at the integration points. How do we do this?

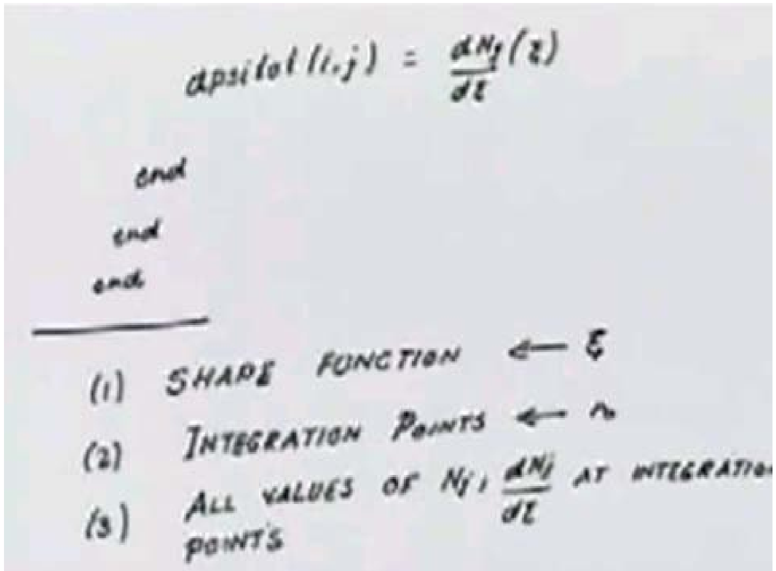
(Refer slide time: 44:15)



We can use another routine which we can call as Shapall which takes as input the P that we have and which outputs the following things (Refer Slide Time: 44:29). These arrays I am giving it some name P, psitot, dpsitot where pistot can be an array of size the number of integration points into P plus 1 and dpsitot is also size n and P plus 1. What am I am going to do here? For each integration points psi_i I am going to load the value of all the shape functions of order P obtained at these integration point.

Similarly, here (Refer Slide Time: 45:40) I am going to load at each integration point the values of the derivatives of shape functions with respect to psi at this integration point. How do I do it? You start loop over integration points. For each integration point you have already obtained or programmed, the routine which will give you the shape functions at a given point. So here (Refer Slide Time: 46:20) I will put psi equal to psi_i which is the integration point; call shape function routine to give N_j psi and dN_j d psi at the point psi. For j equal to 1 to P plus 1, I am going to do the following: psitot I, j is equal to N_j at this point psi.

(Refer slide time: 47:22)



Similarly, dpsitot i, j is equal to d N_j d psi at this point psi; then I can end the loops. This is a program which are a routine again which will take the integration points, the order of the shape functions that we want. Which order shape functions? It will go to my shape function routine return the value of the shape functions and the derivatives of the shape functions with respect to the master coordinate at each of the integration points and store it in a array where we give it some name. The first array is the array of the shape functions the psitot and the second one is the array of the derivatives dpsitot. Once we have loaded it then you see that all elements when you do the integration then we do not need to call the shape function routine over and over again before we do the integration. We can simply use these two arrays that we have created and these are not big, these are very small in size. We can simply use them over and over again in each of the element calculation.

In the next class or in the next lecture we are going to extend this further. We have if you remember; we have now developed capabilities to get the shape functions and the derivatives at any given point psi. Similarly, we can give you the integration points given the order of the rule n and thirdly given the order P we can get all values of shape functions at integration points. We are going to use this information in the next lecture to obtain the suitable routine which can do the element stiffness matrix and the load vector calculations. Once we have that routine then we would have done a significant bulk of our one D computational program finite element program that we want to write in one dimension. This we start off in the next lecture. Thank you.