Finite Element Method

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Module - 4 Lecture - 1

This is the eleventh lecture in the series. Till now, what we have looked at was developing the finite element method to solve a typical one dimensional problem and the typical one dimensional problem that we had taken was an axial bar subjected to some end load P and a distributed axial force F.

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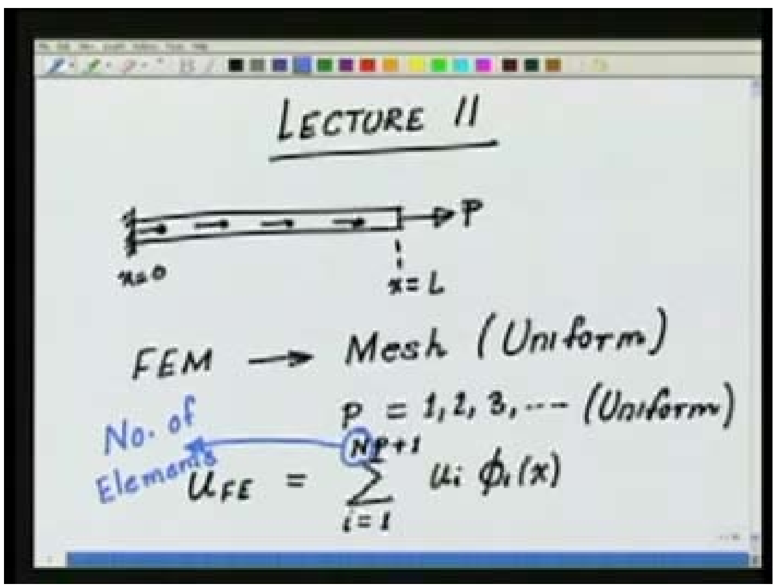

The bar is of length L and we have said that the bar could be constrained at the end x is equal to 0 and it could also have a uniformly distributed axial string that is it is lying on a elastic support which is going to apply a constraint to the motion or given a resistance to the motion. For this problem, we have developed a finite element method where our approach was to a make a mesh and for this we have made a uniform mesh and take approximations of order P. P could be 1, 2, 3 anything and this was uniform approximation; that is all elements have the same P order of approximation and what we had done is, we had highlighted the fact that the solution, the finite element solution is given as (Refer Slide Time: 01:57) summation of u_i phi_ix where this N is number of elements and P is the uniform order approximation

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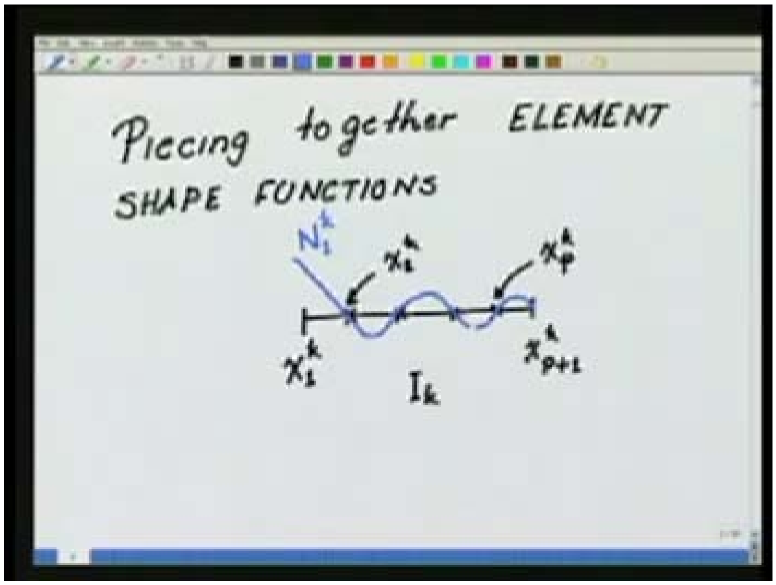

Then we had said that this phi_i(s) could be constructed by piecing together element shape functions. How I had been defining the element shape functions? If this is the generic element k so I will have this (Refer Slide Time: 03:20) as the first point, first node or this as the left end vertex or edge of the element as the node x_1 of k for the element, the last the other the right hand vertex is node $x_{p \text{ plus } 1}$k of the element and in between we have uniformly spaced p minus 1 points so this (Refer Slide Time: 03:20) will be point x_2k and this will be point x_pk. This is how we have defined the element and over the element we had then gone ahead and defined the shape function which was like this. This would have been our N_1k (Refer Slide Time: 04:15).

So, given a generic element we were able to define the shape functions of order P at the element level and we have said that the global basis functions are obtained by piecing them up and here we had seen the definition of the so-called edge functions and then the internal bubble functions and so on. What are the problems with this? Problems with using this kind of an approach, there are really no problems, but the basic difficulty that we are going to face is the following: that the shape functions as P increases, shape functions become more complex higher order polynomials and so does their derivatives.

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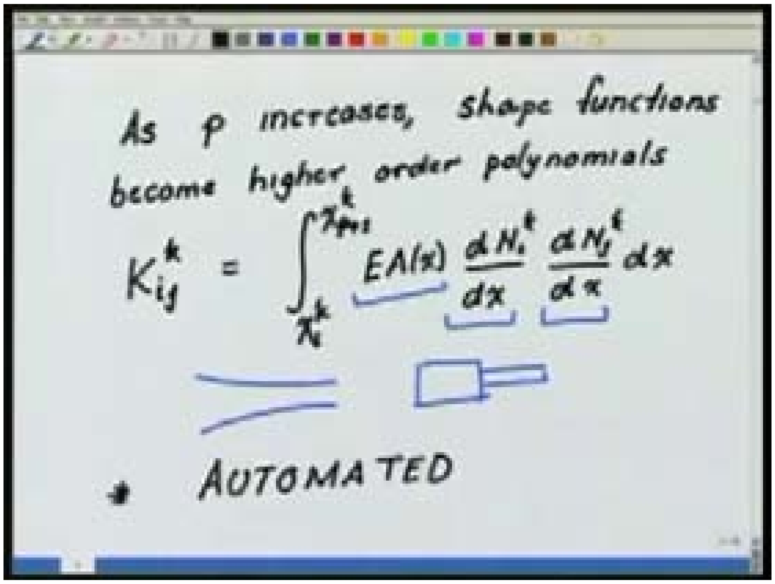

When I have to evaluate what we had done after this, we went ahead and evaluated the stiffness entries at the element level, this we had done like this (Refer Slide Time: 05:50) we could have EA(x) dN_jk by dx so when we evaluate this integral to obtain the entries of the stiffness matrix, we find that this expression integrant the expression dN_jk by dx dN_jk by dx even EA need not be constant all the time; EA could be anything. It could be coming out of this kind of profile or it could be coming out of this kind of profile (Refer Slide Time: 06:30), anything. So EA also has a complicated expression may have problem of the interest. In all these cases to obtain each entry of the stiffness matrix we have do this integration manually. This integration may not be so easy to do. By hand we can do it but it will take some time. Also this integration cannot be automated which means that we cannot convert the procedure that we have followed till now to obtain the stiffness matrix entries. That cannot be converted into a computer program because the computer is not capable of doing these integrations in such a nice symbolic way as we can do. That is one problem.

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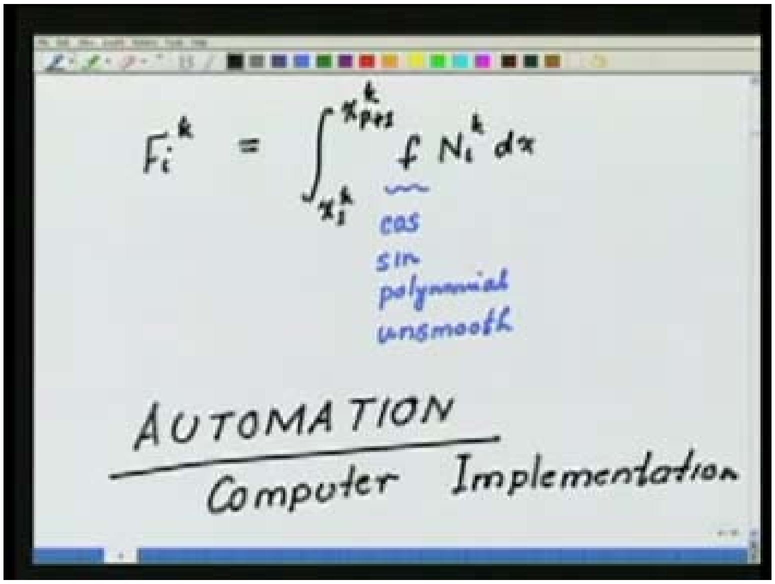

The other thing is if we take the load vector entries we see that here (Refer Slide Time: 07:45) the load vector has this distributed force term, it could be anything; it could be a cosine kind of a load, it could be a sine kind of a load, it could be polynomial load, it could be unsmooth, it could be anything. In each of these cases if I have to use a finite element method to get a solution I have to do these integrations specifically, for the kind of load which is applied on the member. This job will become very cumbersome. We do not want to do this job over and over again separately for different loadings, separately for different material constants etc., for different order of approximations P. We would like to bring it into a common platform which could be automated. So automation is something which we would like to aim for; that is computer implementation becomes mandatory.

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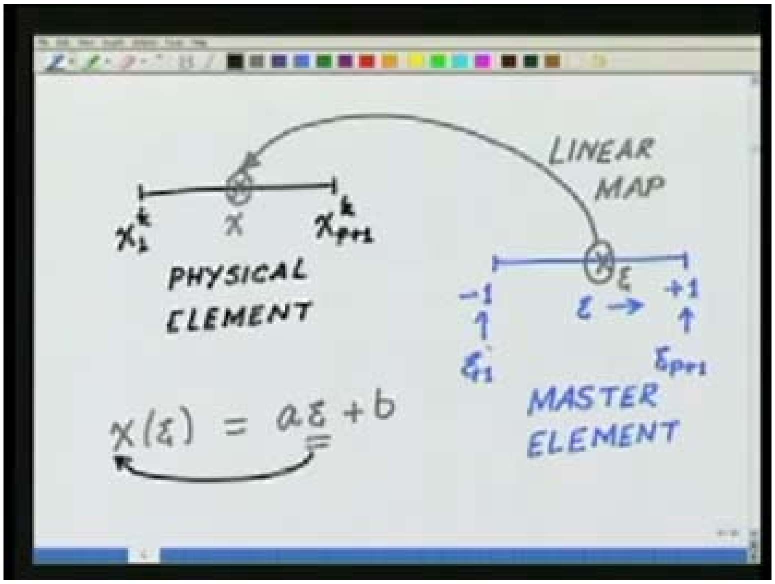

How do we go about doing it? The idea is simple. Let us take a generic element ik such that the first load of it is x_1k and the other one, the last node of the element is $x_{P \text{ plus } 1}$k. This we will call our Physical Element because this is the actual domain that is given to us. What we are going to do is, we need to say that this generic physical element comes from something called a master element. That is, we are going to define an element like this (Refer Slide Time: 10:10) so this I can say is the point psi_1, $\text{psi}_{P \text{ plus } 1}$. This master element again if we see has an interval which is of size 2 and it always goes from the point minus 1 to plus 1. This is called the master element and every point in the physical element, that is, if we take any point here x could be obtained from a corresponding point in the master element by a Linear Map or a fine map. We say that x is a function of psi and this is linear; that is it is given by a psi plus b; so given the point psi, I can find the point x. How do I define this linear map? This linear map, what does it do if I go back? It takes the point x_1 k - that is the point psi_1 to the point x_1k; the point $\text{psi}_{P \text{ plus } 1}$ to the point $x_{P \text{ plus } 1}$k. To define a linear we have to give its value at two points. So we are going to give its value at the point psi_1 and $\text{psi}_{P \text{ plus } 1}$.

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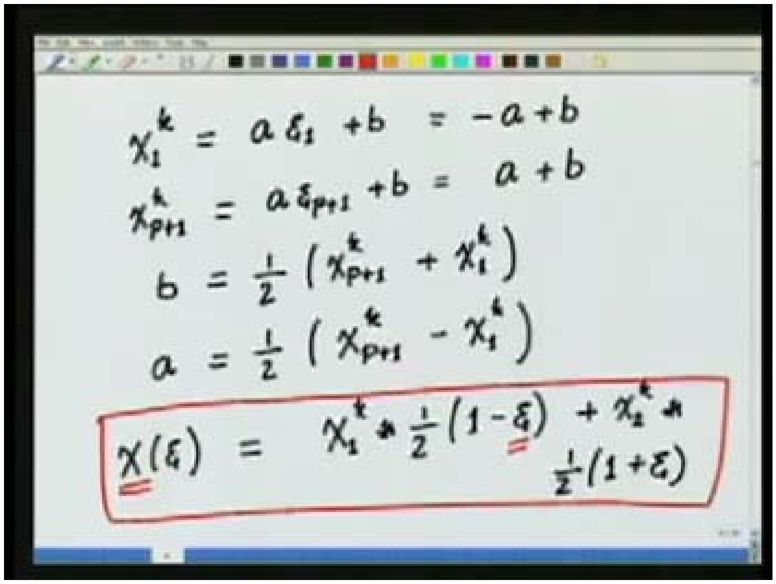

Let us go back and say that we know that x_1k is equal to a into psi_1 plus b which is equal to minus a plus b. Similarly, the point $x_{p\ plus\ 1}$k is equal to a $psi_{p\ plus\ 1}$ plus b this is equal to a plus b. From here I would like to find out what is a and what is b. So, b if we see it will come out I am simply adding up these two things as half of ($x_{p\ plus\ 1}$k plus x_1k) and a will come out to be if I go back and put it in there as ($x_{p\ plus\ 1}$k minus $x_1$of k). In this case we can define now x psi, I will put the a and b in the expression and I will collect terms and I am writing the final expression. It is going to be x_1k into half of 1 minus psi plus x_2k into half of 1 plus psi. This is mapping by which any point psi when it is given to us in the master element, we can find the corresponding point x in the physical element.

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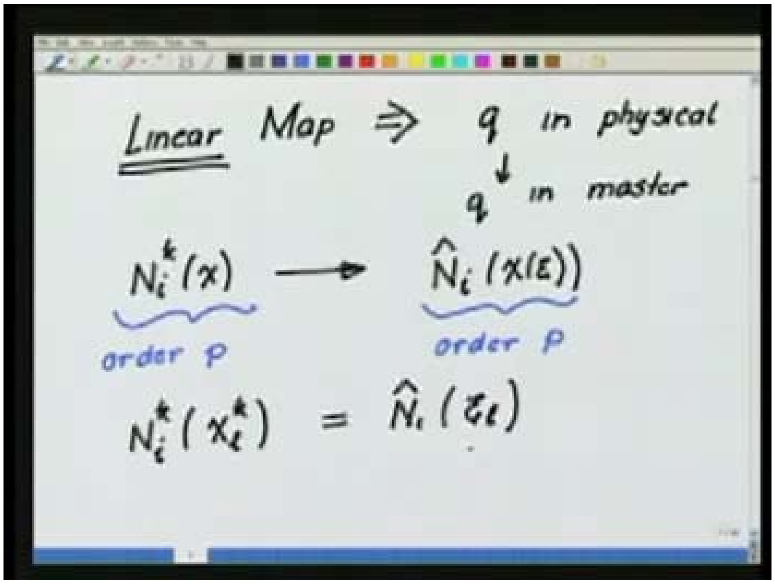

Since, this is a linear map it also implies a polynomial of order q in the physical element remains a polynomial of order q in the master element because the mapping is linear. Remember that, it is a linear map that is why it is true. If I have the shape functions defined in the physical element as we have done earlier N_ik(x) which is a polynomial of order P, will by employing this mapping, it should map to a polynomial of order P in the master domain. This is order P (Refer Slide Time: 15:50) and this is also order P. The shape function if I transform it using this mapping from the physical to the master element it remains a polynomial of the same order. Further, what do we know? The shape function, if it had a particular value at a given physical point x_i it also will have the same value at the given master point psi_i. So we say that the polynomial N_i at the point, we had given these points, x_l of element k it will be equal to N_i hat at the master point psi_i where x_lk is now obtained by the suitable linear map from the point psi_i.

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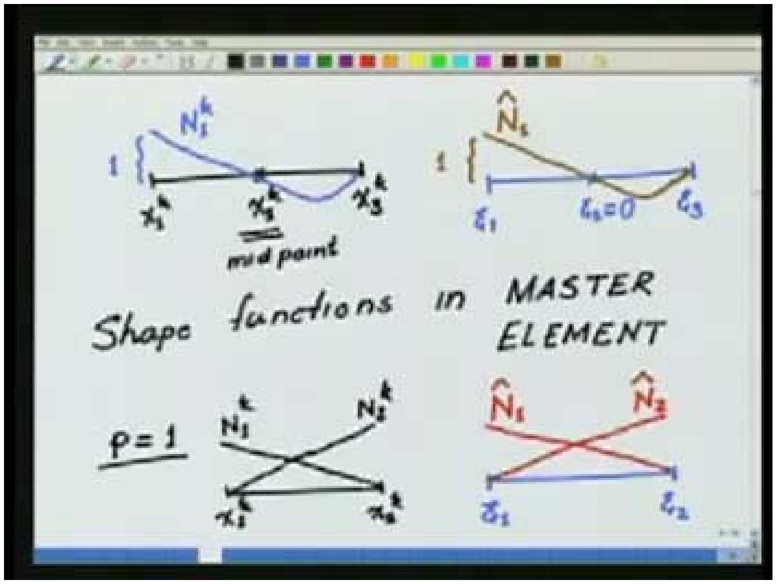

For example, if I had a quadratic element, this was my x_1k this is going to be x_2k this is going to be x_3k and in the master domain, the corresponding point will be psi_1 psi_2 psi_3. Now we know the way we have defined things earlier that this point x_1 of k is at the midpoint of the element. If I do the mapping this point psi_2 will be at the location 0 (Refer Slide Time: 17:52) which is the midpoint of the master domain and if I define the function, the shape function in the physical domain, let us say the first shape function like this (Refer Slide Time: 18:10) which had a value 1 at the point x_1 of k and 0 at the point x_2 of k and x_3 of k then in the master domain it will have the same feature that this value at the point psi_1 will be equal to 1, at the point psi_2 and the point psi_3 the values going to be 0.

Since this N_1 hat is a polynomial of the same order as N_1k and so on, can we define the shape functions in the master elements, the shape functions defined in the master element. First, let us look at the linears. How do we define linears? Well, linears correspond to P equal to 1. Here we had the physical element (Refer Slide Time: 19:42) and this was N_1k, this was my N_2k, this was my point x_1k, and this is the point x_2k. Similarly, this is my master element; this is a point psi_1, this is a point psi_2 and this (Refer Slide Time: 20:20) is the function N_1 hat, this is the function N_2 hat.

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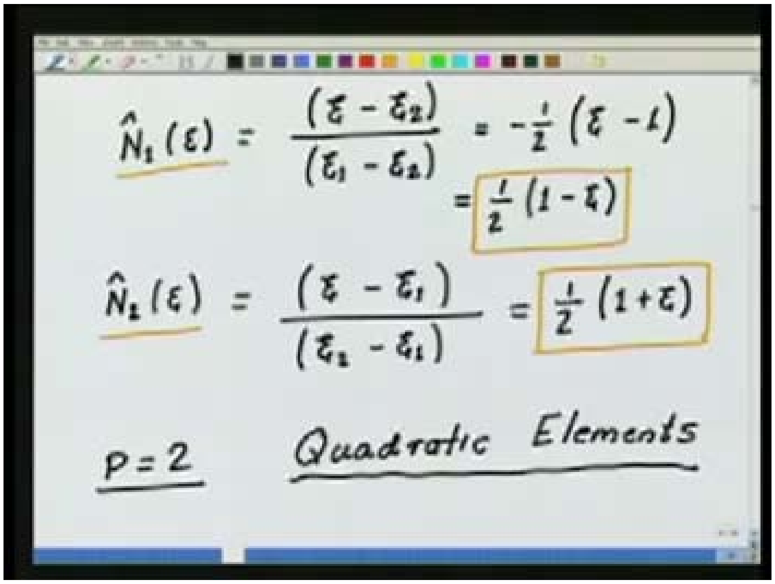

Tell me now, what is this function N_1 hat? As a function of psi what is it equal to? It is very easy we will follow the same principle that we followed in the physical element that is we will use the definition of Lagrange polynomials that we had given in the physical element. Now the definition will be applied to the master element. Here obviously, what we are going to say is that, it is a linear which has to vanish at the point psi_2 and it should have a value 1 at the point psi_1; so psi_1 minus psi_2. What is psi minus psi_2? It is going to be equal to minus of 2; so this is going to be minus 2 and psi_1 minus psi two. What is psi_2? - 1. It is going to be equal to minus of half into psi minus 1; this is going to be equal half of 1 minus psi. This is the definition of the first shape function (Refer Slide Time: 21:53), the linear shape function, first linear shape function.

Similarly, N_2 hat of psi is equal to psi, it is a linear which vanishes at the point $psi_{1\,;}$ psi minus $psi_{1,}$ and it has the value one at the point psi_2. psi_1 for us is equal to minus 1 and this is (Refer Slide Time: 22:30) equal to 2 so it becomes half of 1 plus psi. So the second shape function and first shape function can be defined using exactly the definition of the Lagrange polynomials, but applied to the master element and remember that here the values have to vanish at points which are appropriately obtained by mapping from the physical element. Similarly, if I want to go to P equal to 2 (Refer Slide Time: 23:03). For P equal to 2, how are we going to define this shape function in the physical element? I am not going to do it in the physical element; I will do it in the master element.

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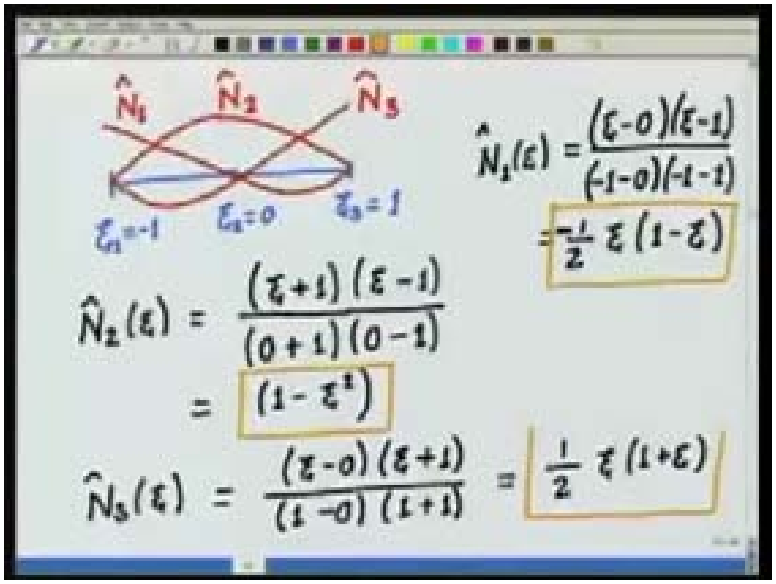

Here is psi$_1$ which is equal to minus 1, here is my psi$_2$ which is equal to 0 and here is my psi$_3$ which is equal to 1. The first shape function for the quadratic is going to be 1 at the point psi$_1$, 0 at the point psi$_2$ and the point psi$_3$. This is going to be my N$_1$ hat (Refer Slide Time: 24:02). Similarly, the second one is going to 1 at the point psi$_2$ and 0 at the points psi$_1$ and psi$_3$ and the third one is going to be 1 at the point psi$_3$, 0 at the points psi$_1$ and psi$_2$. Again we use the same definition of the Lagrange polynomials that we had defined earlier. I know that my N$_1$ hat as a function of psi will be equal to psi minus psi$_2$ which is 0 into psi minus psi$_3$ that is it has to vanish at point psi$_3$ which is 1 and here we will have psi$_1$ minus psi$_2$ so minus 1 minus 0 (Refer Slide Time: 25:00) here I will have psi$_1$ minus psi$_3$.

We see that this will become equal to this is minus 2, so it becomes equal to half of psi into 1 minus psi, minus half of psi into 1 minus psi, so this is what our definition of N$_1$ hat for P equal to 2 will be. Similarly, I can define N$_2$ hat this (Refer Slide Time: 25:52) has to vanish at the point N$_1$ and N$_3$ so its psi minus, this is going to be psi minus of minus 1 so psi plus one into psi minus 1 and it has to have the value 1 at the point psi$_2$ so 0 minus of minus 1 is 0 plus 1 so 0 minus 1 so this (Refer Slide Time: 26:30) is going to be equal to one minus psi square. Similarly, N_3 hat psi will be equal to, if I go ahead and do the same job it will be equal to, psi minus 0 into psi plus 1 divided by one minus 0 into 1 plus 1. This is going to be half of psi into 1 plus psi. The same job can be done for the cubics.

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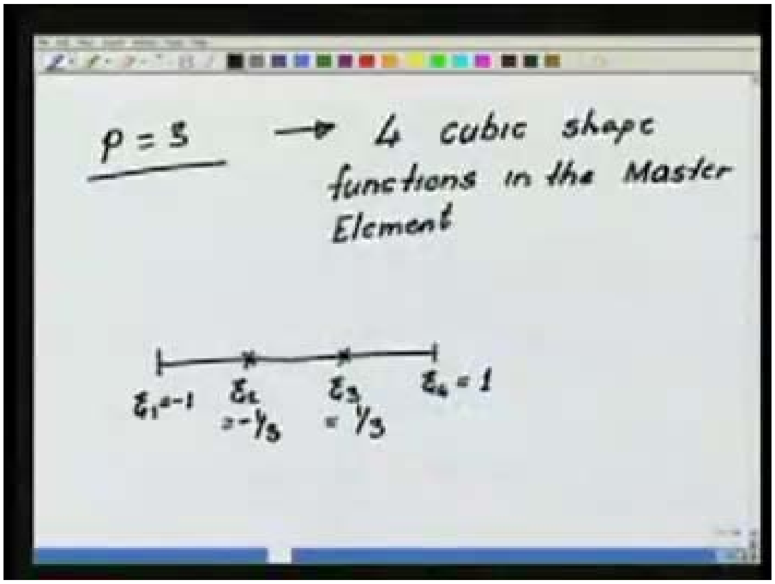

For the P equal to 3 also we can define the 4 cubic shape functions in the master element. The only thing is one should remember that in the master element the points psi_1 equal to minus 1 this is psi_2, this is psi_3, psi_4 is equal to plus 1. So we have the gap between these points is equal to 2 divided by 3. This is minus 1 (Refer Slide Time: 28:28), this point becomes minus one third, this point becomes plus one third. As we had given the algorithm earlier we can apply the algorithm here that the point for an element of order P, psi_1 is equal to minus 1 and $psi_{1 \text{ plus } l}$ equal to psi_1 plus 2 by P into l for l equal to 1, 2,… upto P actually, this 2 by P is nothing but the interval length between two consecutive points.

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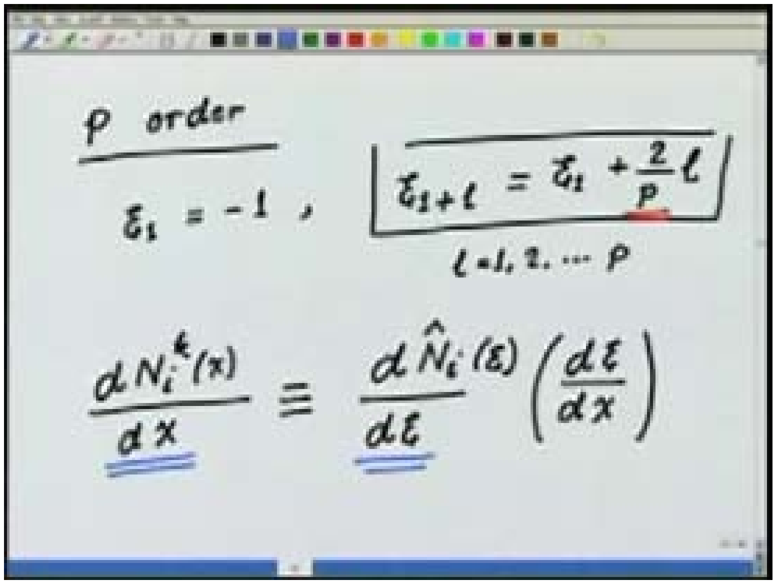

If I define these points psi k in such a way then I can define my Langrange polynomials which are nothing but the shape functions quiet easily. Before defining the shape functions, we should define these points psi k and then go ahead and define the shape functions.

If I have this definition of shape functions, how is it going to change things? What we know is that we were in interested in finding (Refer Slide Time: 30:10) for a given physical element, the derivative of shape function with respect to x because that is what went into our element calculations. So, this will now become equal to this (Refer Slide Time: 30:25). The derivative with respect to x will now be given in terms of the derivative of the shape function defined in the master element with respect to psi into the derivative of psi with respect to x. If we go back to our definition of element transformation (Refer Slide Time: 31:10), here if I apply the derivative if I try to find the derivative of this with respect to psi, what will I get?

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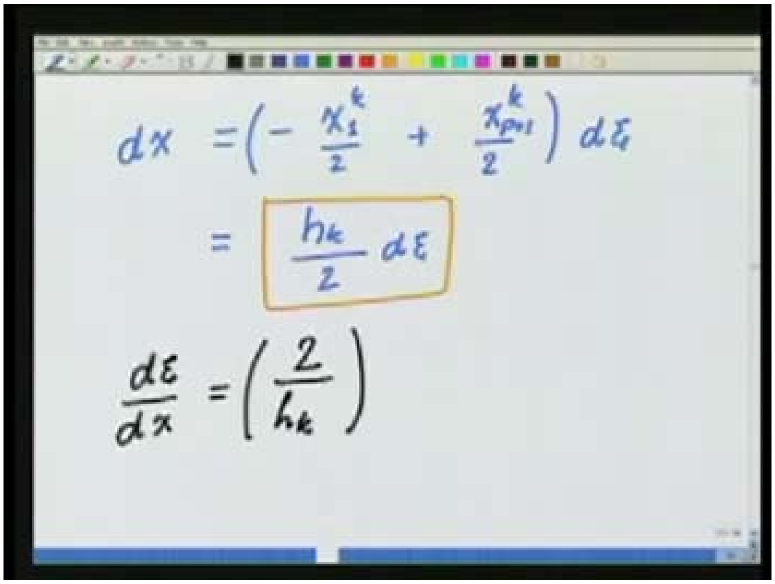

I will get dx from the expression that I had written earlier is equal to minus x_1 of k by 2 plus $x_{P\ plus\ 1}$k by 2 into d psi. This is equal to, what is $x_{P\ plus\ 1}$ minus x_1 for the element k? It is nothing but the size of the element k so I can write it as h_k by 2 in to d psi. dx becomes h_k by 2 d psi. The derivative can I substitute? So, d psi dx becomes equal to 2 by h of k.

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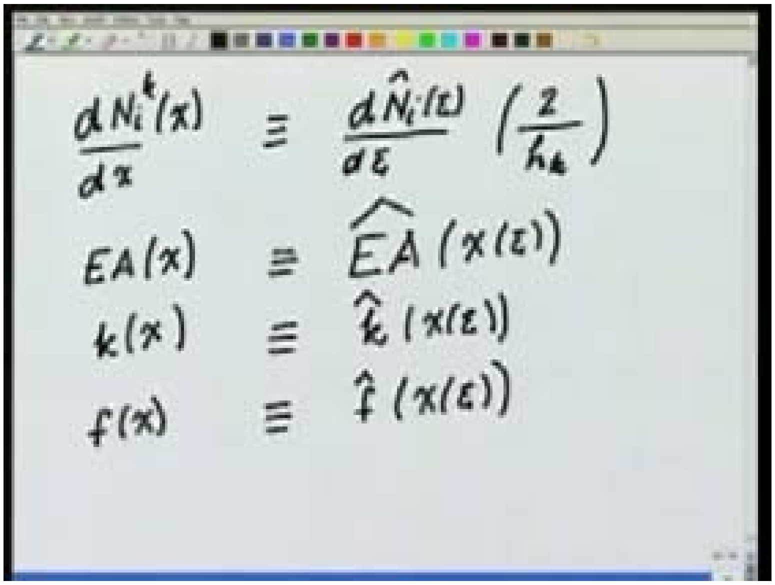

We will have dN_ik dx as a function of x is equal to dN_i hat d psi into 2 by h_k. Once I have done this transformation then, we have to also do the further transformation that EA of x because the material constants could be different. EA as a function of x is equivalent to EA hat x of psi. What does this mean? I can write by using the transformation properly that the function EA as a function of x can be written as a function of psi given by EA hat. Similarly, if I had the distributed springs k(x) that would be equivalent to k hat is a function of psi and f of x which is the distributed load it will be equivalent to f hat which is a function of psi. This we can do and take all our material constants, the loading data from the physical element to the master element. Let us use these transformations, now redefine the integrals over the master element. Remember that, we have done all this to bring the integrations from the physical element to the master element.

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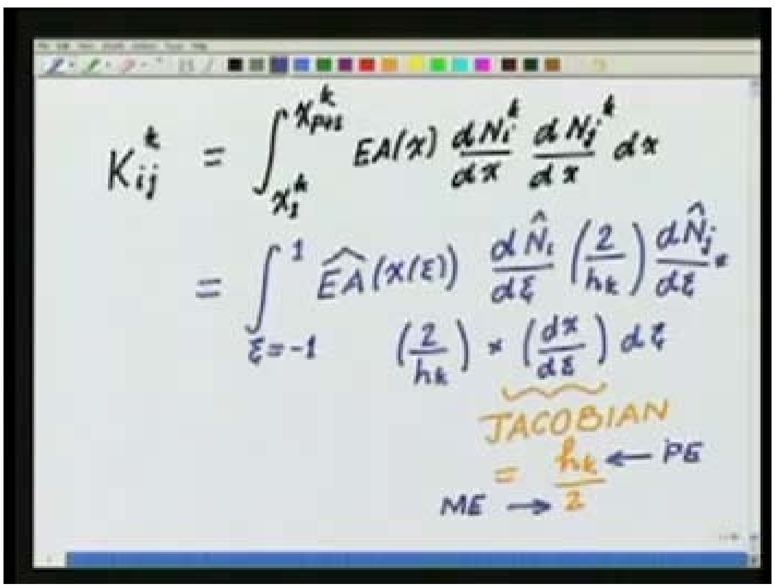

Our earlier K_{ij} for the element k was integral $x_{1;}$ I am going to drop the superscripts may be added, I will added x_1 to $x_{p\ plus\ 1}$k EA of x d N_ik dx d N_jk dx. This we are going to evaluate in the master domain. Obviously, when I transform the transformation will take us from minus 1 to plus 1 instead of x_1 k to $x_{p\ plus\ 1}$ k; integration domain will be from minus 1 to plus 1 all the time. EA will be replaced by EA hat which is a function of psi. d N_ik dx will be replaced by d N_i hat by d psi into d psi dx and what is d psi dx? 2 by h_k , dN_j dx will be replaced by dN_j hat d psi into d psi dx which is 2 by h_k and dx will be replaced by dx d psi; I will explicitly put it as dx d psi d psi. What is dx d psi? dx d psi is called the Jacobian (Refer Slide Time: 37:00). Physically, what does the Jacobian of the transformation mean? It means that the ratio of the length of the original element to the new element. The ratio of the length if we see that, if I go back to the expression that we had derived for the dx in terms of d psi, this quantity will be equal to h_k by 2 (Refer Slide Time: 37:30) where if we see this is the length of the master element and this is the length of the physical element. The ratio of the original element, length of the original element to the length of the current transformed element is called the Jacobian.

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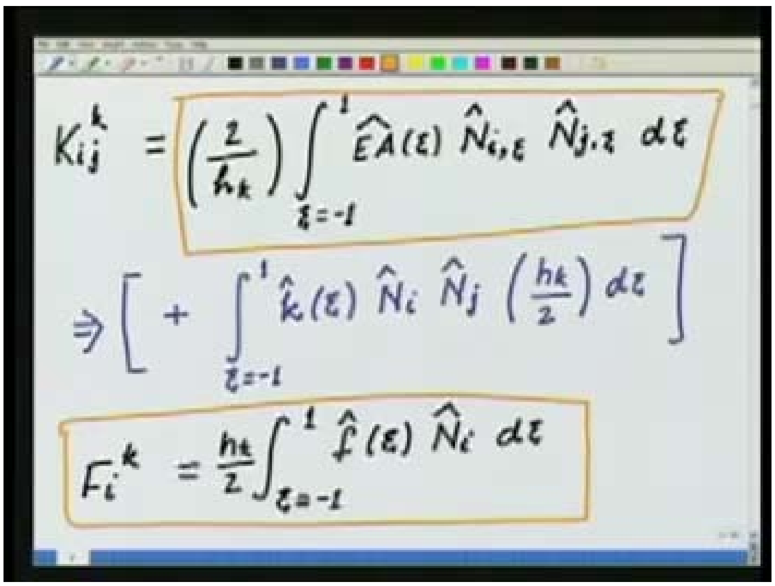

If I club everything together, I will get this K_{ij}k is equal to by doing suitable cancellations 2 by h_k integral psi is equal to minus 1 to plus 1 EA hat, N_i derivative of N_i hat with respect to psi derivative of N_j hat with respect to psi d psi. If I have the elastic support also present then it is not difficult, in the case of elastic support I will add this extra bit. Integral psi equal to minus 1 to plus 1 we will have k hat of psi into N_i hat into N_j hat, k dx d psi will be h_k by 2 d psi. If I have the elastic support, this extra term also has to be added to the element stiffness matrix entry.

Similarly, the load vector for the element will now look as integral psi equal to minus 1 to 1 f hat of psi into N_i hat d psi and here I will have the effect of the Jacobian which is h_k by 2. The simple transformations this we have all done in our calculus courses. The advantage is that from the integration over the physical domain now we have brought all the integrals over the master domain. For any element this is how we are going to do the integration.

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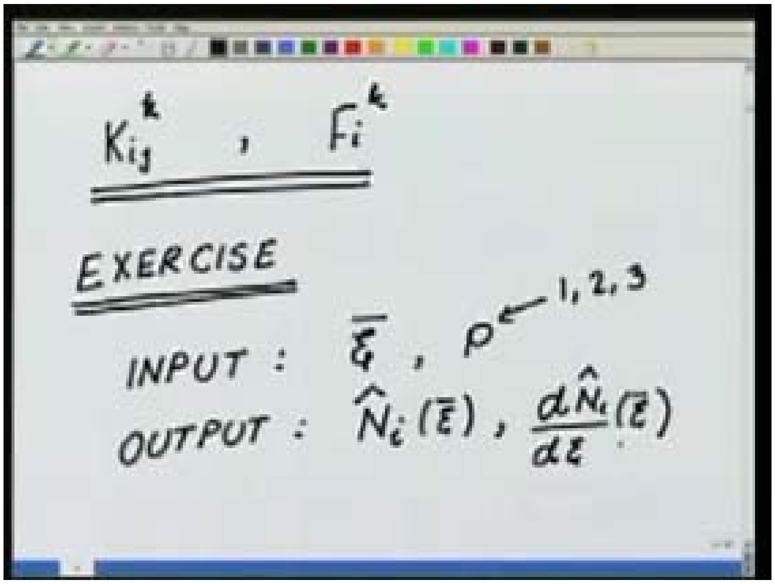

Once I have the entries of the element stiffness matrix and the element load vector, then I can go and follow the same procedure of assembly that we had followed earlier, then apply boundary conditions; nothing else changes. It is only in the integrations that we have come from the physical element now to the master element. So before we go ahead let us pause and look at the motivation for doing this. The motivation, we remember is that we want to go for a computer implementation, that is automation of this whole process of doing the element calculations; doing the assembly, applying the boundary conditions and so on. With that in mind we have defined, we have converted the integrals from the physical element to the master element.

The following exercise (Refer Slide Time: 41:45) should be done at this stage, now that we are conversant with what a master element is. I give the following inputs, the point in the master element, a particular point psi bar, I will call it psi bar this is the point lying in the master element which I give and I give order of approximation P. P let us take is either 1 or 2 or 3. Now given this input what I want as output is the values of the shape functions N_i hat at the point psi bar and the values of the derivatives of the shape functions at the point psi bar. One has to write a program or a sub routine which will take the position psi bar in the master element and the order of approximation P as an input and returns as an output dN_I that is the value of shape function in the master element and the value of the derivative of the shape functions in the master element (Refer Slide Time: 43:10). Getting this expression out of this is a little cumbersome but it is easily doable. This should be done and out of this the sub routine that comes now will able to give us the values of the shape functions and the derivatives at any point in the master domain.

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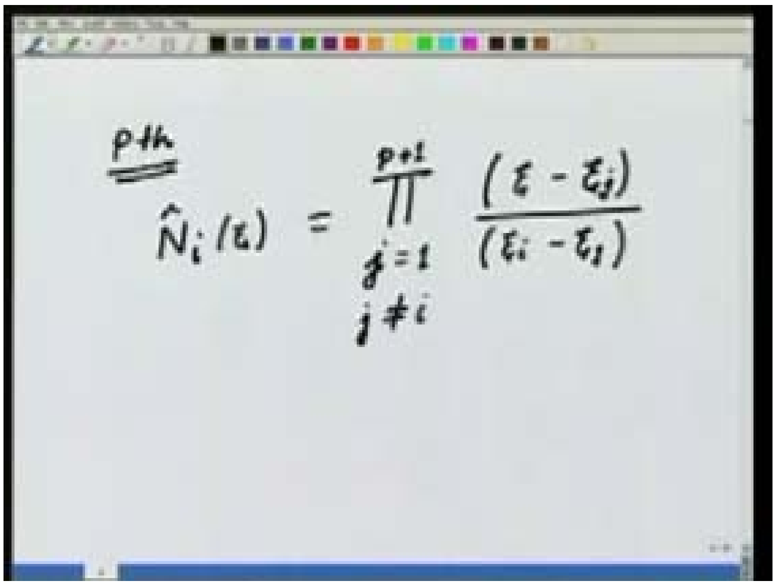

Let me remind you that for Pth order element, the shape functions by definition will be what we had done earlier; j equal to 1 to P plus 1 (Refer Slide Time: 44:00). From this finding the derivative is easy but here we want to do this exercise only for P equal to 1, 2 or 3.

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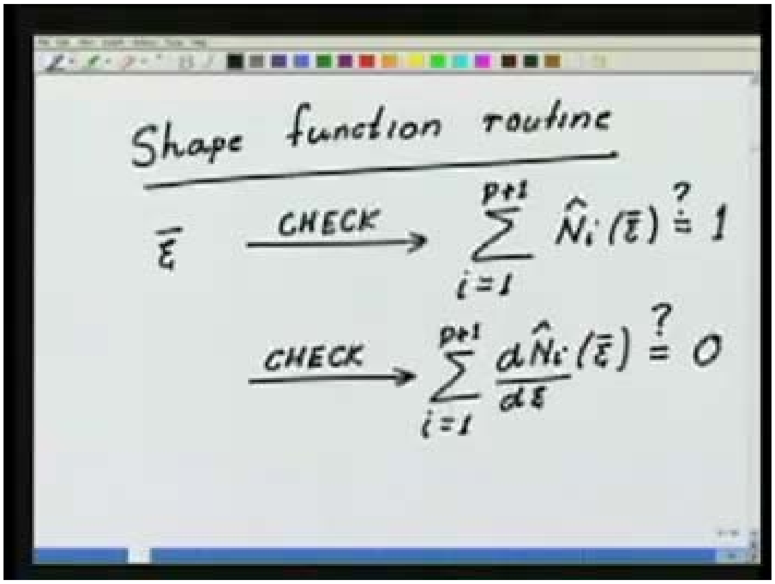

Once we have the shape functions, we have the shape functions routine. What are the checks that one can do is that for the given psi bar check whether the sum of all the shape functions is equal to 1 or not. The second check that we have to do, at least these checks can be done, this could be incorporated in the program that sum of the derivatives of the shape functions at the point psi bar is equal to 0 or not. If it is not 0 and if this is not 1 then there is something wrong in the program so one should go and check, whether the program is doing, what is wrong? Where is the bug? And the bug has to be fixed.

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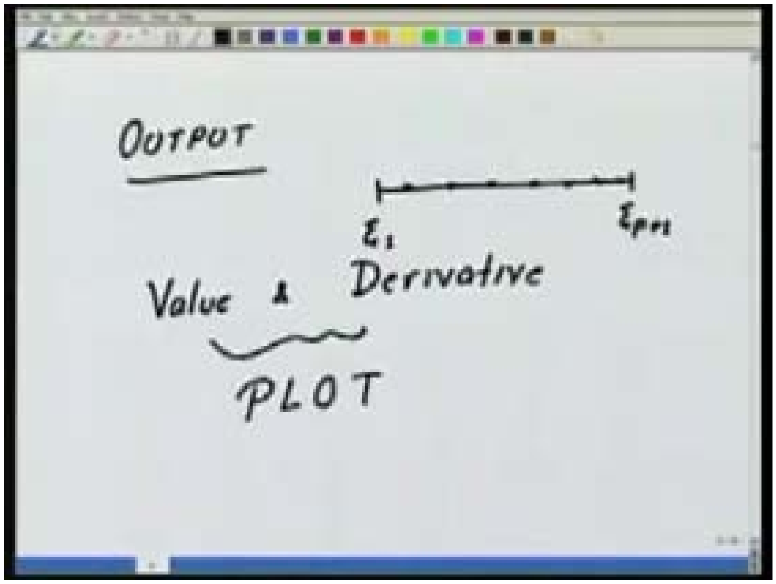

One should also take this program and output at suitably located points in this interval so I can take smaller set of points and at each of these points, output the value of the shape function each of the shape function and the derivatives and then plot it so output value and derivative and plot it using a suitable plotting program. See whether, these functions actually look the way we have been drawing them. That is they have a value 1 at a particular point, the shape functions and they vanish at the other psi x. These things one should do before using these programs. Once this is clear then it will also gives us graphical image of how the shape function look. Then what we are going to do is, this integral that we have defined over the (so now I am calling) generic integrant.

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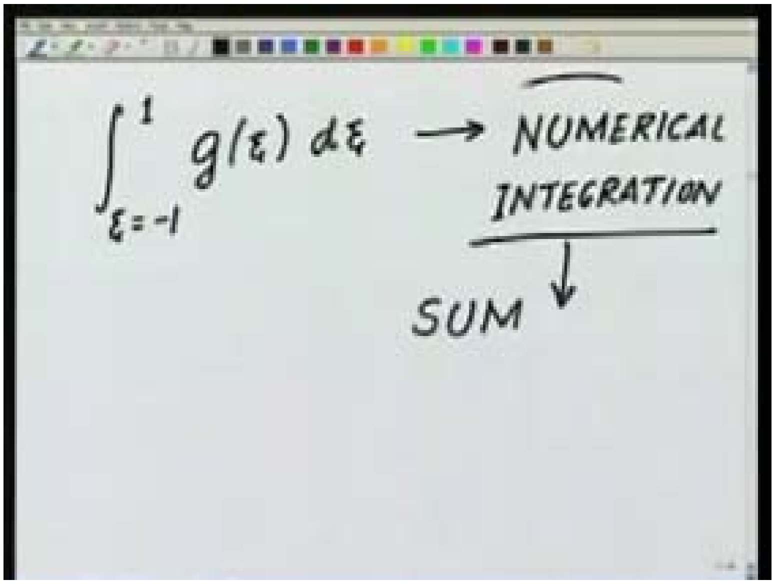

This integral that we have defined where g psi is generic integrant, this will be now given in terms of a numerical integration. Numerical integration means we will replace this integral by a sum. Sum of the value of the integrant evaluated at certain points. This is the topic of our next lecture where we will talk of numerical integration and how to do it for the given integrants that we face in finite element calculations.

Thank you.