**Dynamics of Machines**

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**Module No. #02**

**Lecture No. #03**

**Euler's Equation of Motion**

In the last class, we have discussed the motion quantities of a rigid body and how it moves about a point. We have found that the inertial properties of a rigid body are a far more complex quantity which can be represented by six independent quantities. We term it as a tensor and give the name as inertia tensor.

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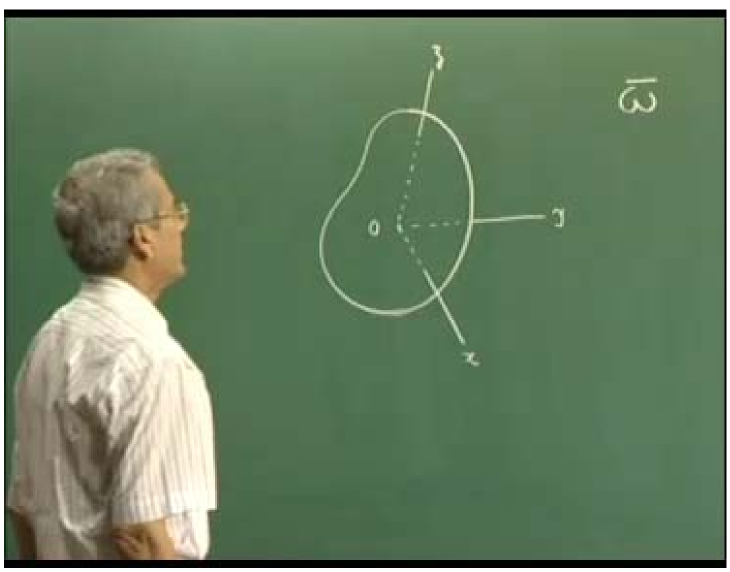

If this be the rigid body under consideration, its translational motion can be considered to be a particle type of motion, where the whole mass of the rigid body may be considered to be concentrated at this center of mass. The angular motion, about the center of mass or any other fixed point o can be represented by an instantaneous angular rotation about an instantaneous axis which changes its position both relative to space and time. If the angular velocity at a particular instant be omega vector, then we can resolve it into various components along three mutually perpendicular directions x, y, z as omega$_x$, omega$_y$ and omega$_z$. So, these are the components of angular velocity of the rigid body along these three directions small x, small y, small z.

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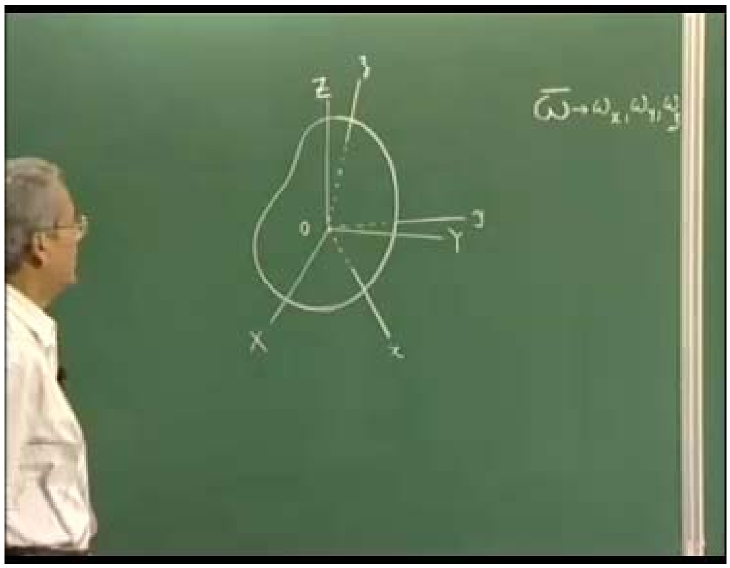


We should also remember that there is another set of coordinates - capital X, capital Y, capital Z. Without losing much generality in the results we can make the two origins coincide at o. This smaller x y z system has an instantaneous or an angular velocity omega with respect to the fixed frame of reference.

Now, we have to remember that the angular velocity of this frame x y z omega is different from the angular velocity of the rigid body which is omega. So, this is the situation, we have a fixed frame of reference capital X Y Z and moving frame of reference small x y z moving with an angular velocity capital omega and the rigid body which is rotating with an instantaneous angular velocity omega. So, we have the angular moment of vector l; it can be expressed in the form of three components $l_x$, $l_y$ and $l_z$.

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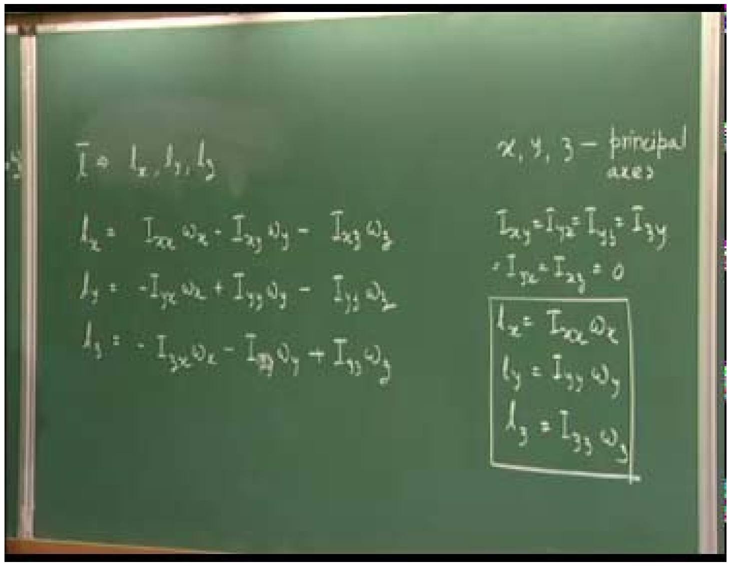


We have seen that the expression for each $l_x$, $l_y$, $l_z$ can be expressed as $I_{xx}$ omega$_x$ minus $I_{xy}$ omega$_y$ minus $I_{xz}$ omega$_z$, the component $I_y$ expressed as minus $I_{yx}$ omega$_x$ plus $I_{yy}$ omega$_y$ minus $I_{yz}$ omega$_z$ and this component $I_z$ expressed as minus $I_{zx}$ omega$_x$ minus $I_{zy}$ omega$_y$ plus $I_{zz}$ omega$_z$ where, $I_{xx}$ is the second mass moment of inertia or simply the moment of inertia of the rigid body about the x axis.

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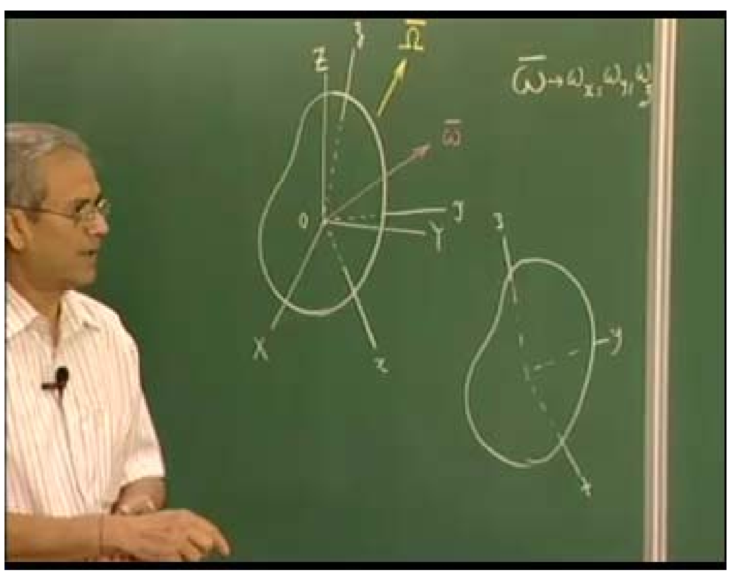

This rigid body is now, along this direction x, along this direction y and along this direction z. If I measure or define its moment of inertia about the x axis is $I_{xx}$, the moment of inertia about the rotation axis y is $I_{yy}$ and moment of inertia for the rotation about axis z is $I_{zz}$. These were familiar terms. We have seen the concept of moment of inertia even in planar motion of rigid bodies in our dynamical courses.

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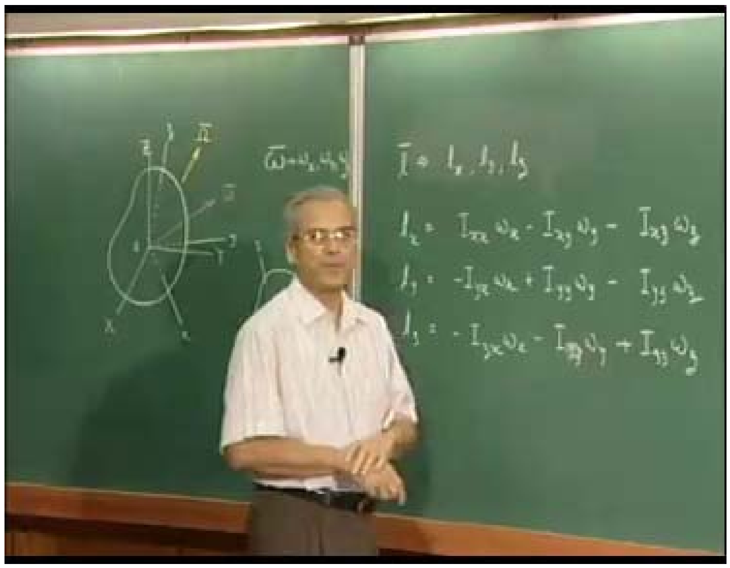

However, these quantities $I_{xy}$, $I_{xz}$, $I_{yz}$ they are something new, which we have not encountered before and they are called the product of inertia.

We have also seen that this complex situation can be made much simpler, if we select this small x small y and small z directions in a way, so that the product of inertia vanishes. When these directions x y z are along the principal axis of the rigid body, then $I_{xy}$ is equal to $I_{yx}$ equal to $I_{yz}$ equal to $I_{zy}$ equal to $I_{zx}$ equal to $I_{xz}$ equal to 0. So, all the products of inertia vanish. Therefore, the components of angular momentum of the rigid body $l_x$, $l_y$ and $l_z$ can be represented very simply in this form (Refer Slide Time: 07:35).

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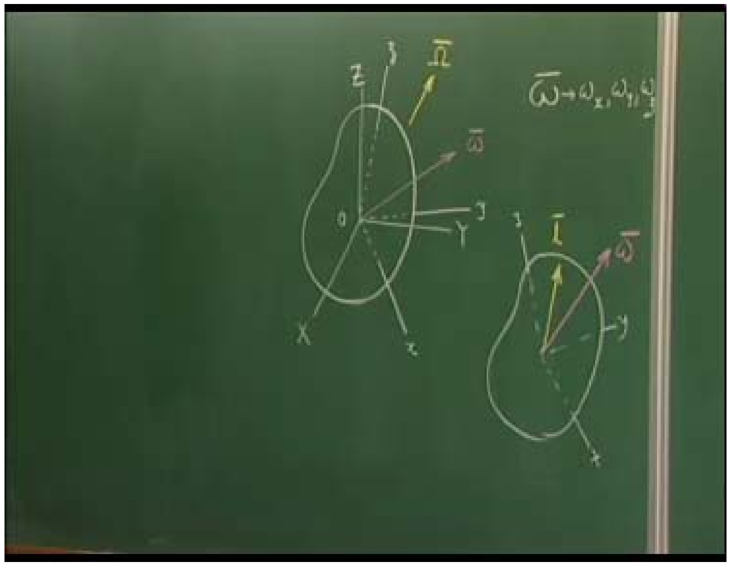


If this rigid body is having an instantaneous angular velocity of omega, its angular momentum l will be something like this. In planar motion it does not happen. There we have seen, the angular velocity vector and the angular momentum vector generally coincide, but in this case they are not coinciding except for the situation, where all the moments of inertia or principle moments of inertia are same. Then only $l_x$ $l_y$ $l_z$ components will be proportional to $\text{omega}_x$ $\text{omega}_y$ $\text{omega}_z$ which means that both the vectors will be along the same direction, but in general it does not happen.

Therefore, now onwards we will always try to express our quantities and equations through the use of this principal axis or through the use of this condition, when the products of inertia vanish. Determination of this principal axis is a very important exercise in dynamics. There is a general procedure with the help of which we know for a particular x y z system all these quantities $I_{xx}$ $I_{yy}$ $I_{zz}$ and the products of inertia; there is a method of determining the principal axis, for which the products of inertia will vanish.

In most of the cases, in engineering applications, you will find somewhat happier situation because we do not generally deal with bodies of very general shape. In most engineering applications objects will have certain symmetry, like say for example, a roller or a disk, a gear clutch plate or simple rod like objects.

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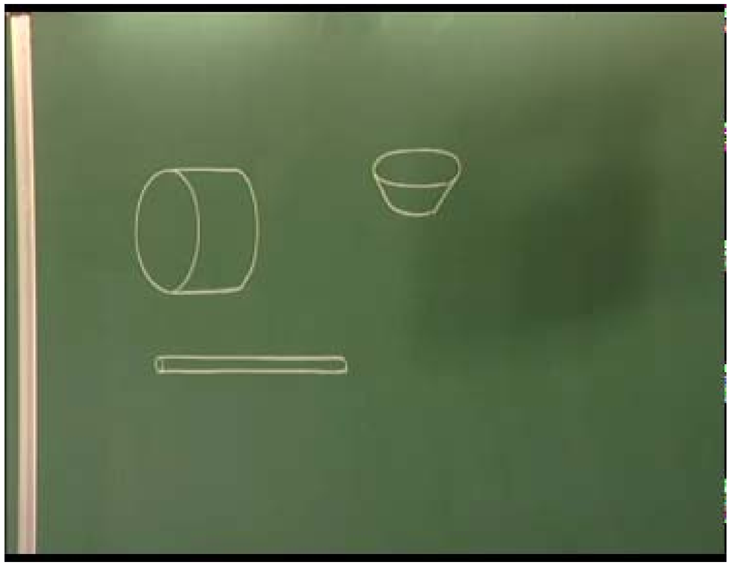

In most cases, these are the kind of component shapes we will be dealing with and for such cases it is easy to determine the principal axis by observation. If the body is symmetric, then all axis of symmetry will be principal axis. So, in the case of that roller, for example, Y axis of symmetry is very obviously this one. So, this will be one principal axis (Refer Slide Time: 10:33).

Then any two mutually perpendicular lines, any other two mutually perpendicular lines will be principal axis. Similarly, if the body is not an axis-symmetric body like this, but say a rectangular object like this with no circular or rotational symmetry (Refer Slide Time: 11:24). Then principal axis will be obviously - this is one axis of symmetry, this is one axis of symmetry and this is the other axis of symmetry. They all meet at the center of the body. So, here again we can use x y z as principal axis.

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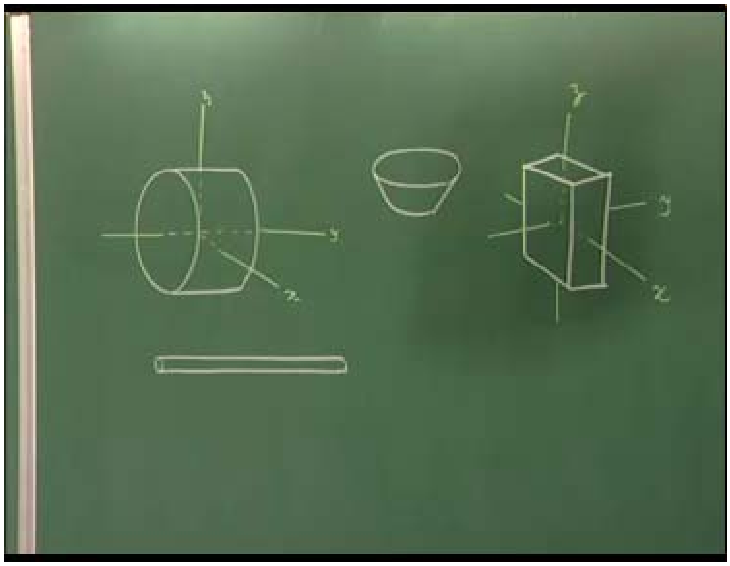


The difference between this and this is that here only y direction is fixed, x and z can be any two mutually perpendicular directions, where both x and z are at right angles to y. In this case that is not so, they are the direction of x y z are all uniquely determined. Similarly, we will find that in this case again it is an axis-symmetry body. So, this will be one principal axis and then any two mutually perpendicular lines because of this circular symmetry can be principal axis.

In case of a rod like objects again, this is one axis of symmetry and this if it is a line, then any two mutually perpendicular direction or lines can be chosen as principal axis. So, except in rare situations where the bodies do not have degree of symmetry, we can find out the principal axis by observation.

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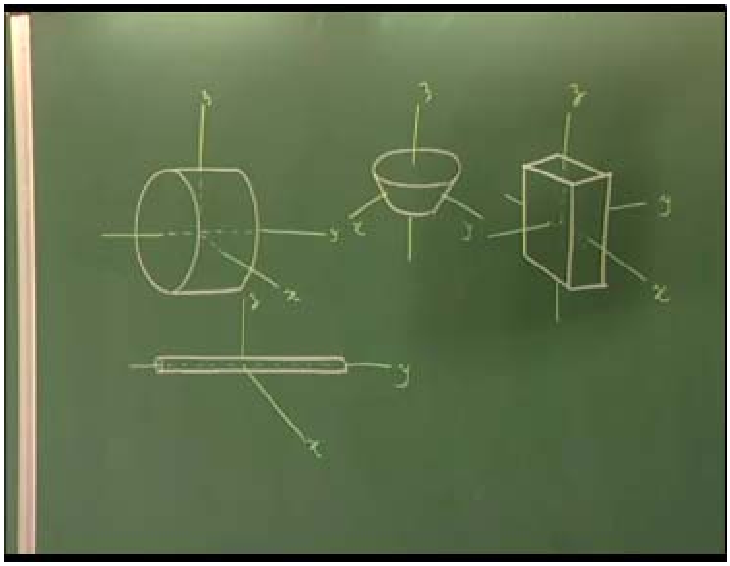

In these cases again, there are two types of objects. In one case, the bodies are axis symmetric where only one principal axis is fixed. The other two can be chosen and the moments of inertia do not change, when you vary this x and z along this direction. Same is the case here, but here the body; it has three unique directions which can be taken as principal axis. When we consider a body under external forces and moments, we can find out what will be the subsequent motion of that rigid body; that is what we want to do next.

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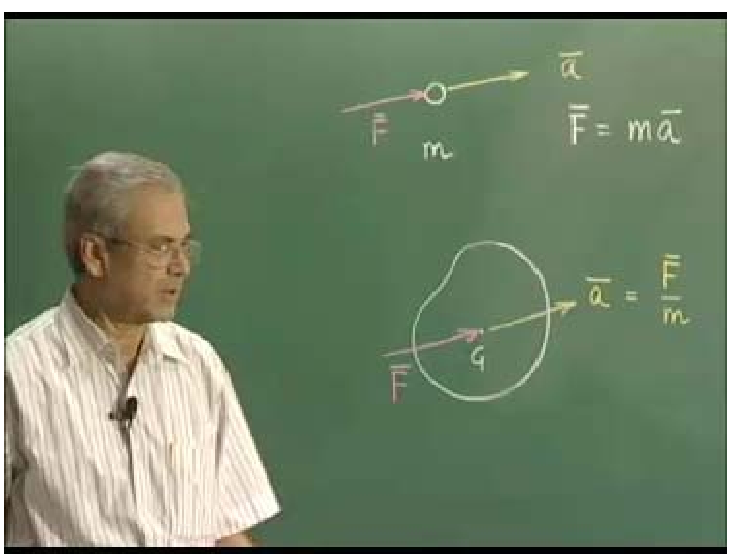

We know very simple case is a particle of mass m, when subjected to a force F, will produce acceleration in this direction; that is simple Newton's law. The relationship between these three quantities is very well known to all of us.

Instead of a particle if we take a rigid body, you have already seen in our first module lectures that if the resultant force, which means the resultant of all the external forces acting on the body, can be represented by a force F, then the linear acceleration of this center of mass will be again mass of the body. So, till this case is very similar, in transitional motion the rigid body behaves like a particle as if the whole mass has been concentrated at the center of mass where the resultant of all the external forces are acting and the acceleration of the center of mass will be this.

Real difficult task or complicated task will be to find out the angular acceleration of a body, when it is subjected to external moment and the body as its own angular velocity at a particular instant. So, that is what now we will do.

If the three components of the externally applied moments, acting on a rigid body about its center of mass, then $M_x$ $M_y$ $M_z$ are the components of this. Then we can show that this will be nothing but rate of change of the angular momentum x component, $M_y$ is nothing but rate of change of angular momentum y component and $M_z$ is rate of change of angular momentum z component.

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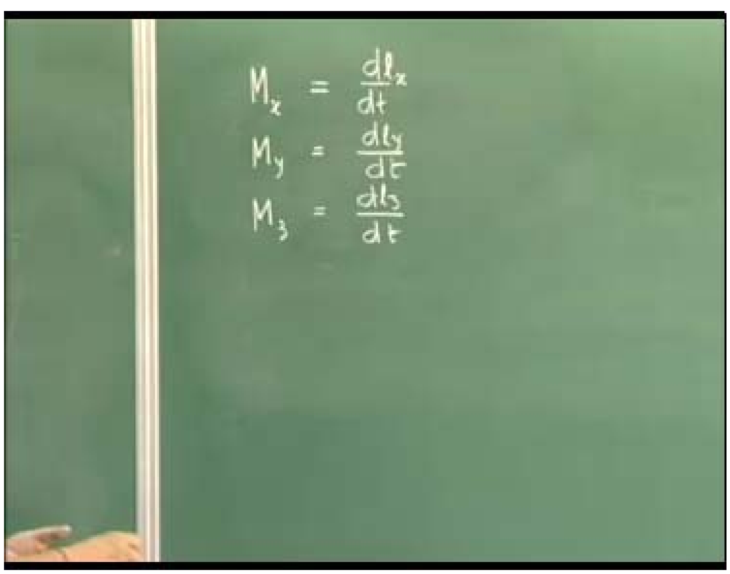

Because you have already seen, we can easily prove that M is equal to l vector; proving this is not very difficult. If we treat this rigid body as a system of particles, where the distance between any two particles remains constant; as the definition of rigid body says, then we can always derive this relation.

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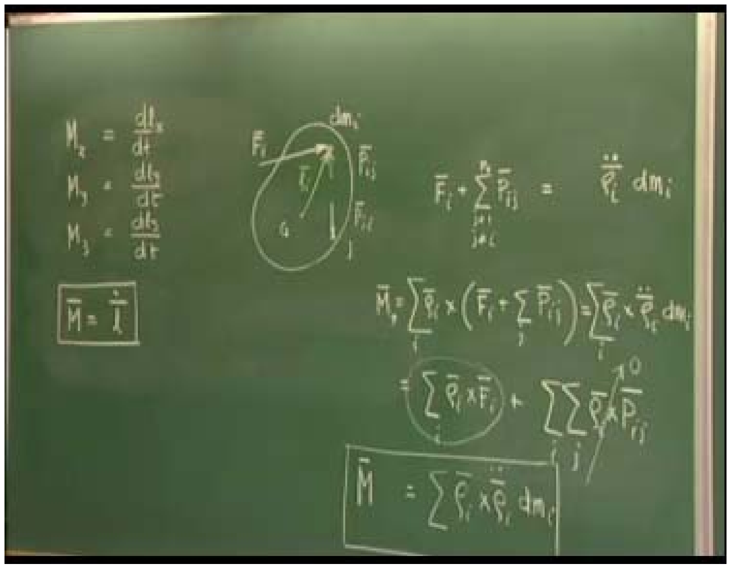

This is the center of mass G; say the ith particle with a mass $dm_i$. The external force which is acting on this ith particle is $F_i$ and this particle is subjected to interactive force with other particles apart from this external force.

If we take a jth particle, then this will be subjected to a force which you call say $P_{ij}$. It means that it is the force on this ith particle due to the jth particle. So, here to get the total force on this ith particle, we have to sum it up for all the other particles of the rigid body. That is j goes from one to all except i, because one particle cannot force, exert force on itself.

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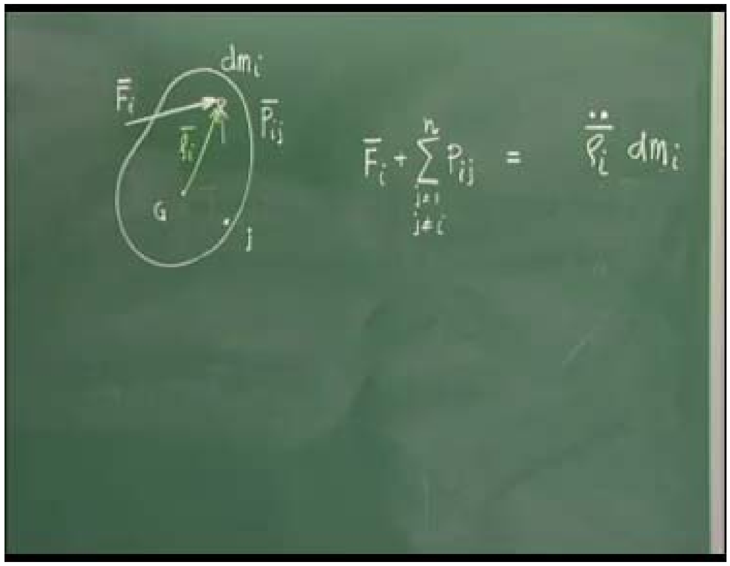


Therefore, you will find that if the position vector of this be rho i, then the moment or rather the total force, first of all, is $F_i$ plus $P_{ij}$ j from 1 to n except j is not equal to i. So, that is the total force acting on the particle. If this is a fixed point, we can show that this will be nothing but $dm_i$ is the mass and $P_{ij}$ is the acceleration of the particle.

Treating the center of mass as a fixed point; if it is not a fixed point, then also we can show that the same kind of formulation will be valid; that can be shown later. Therefore, if you want to now find out, what is the moment of all this forces about point G?

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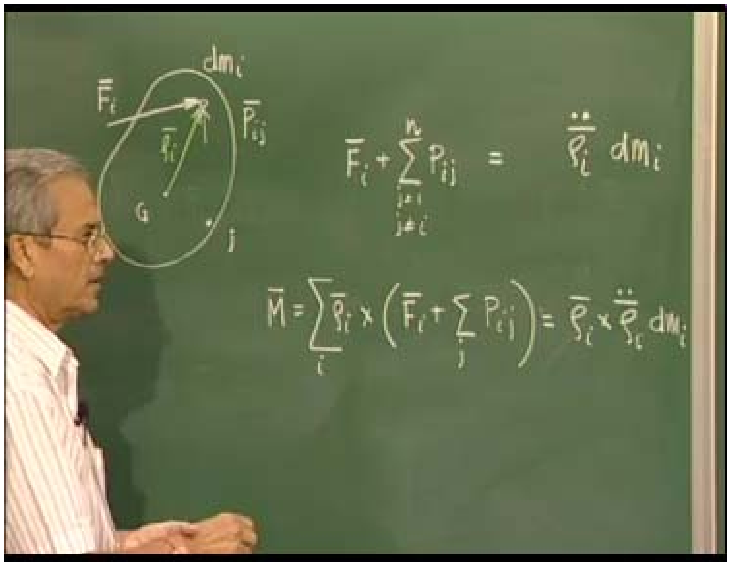


This will be rho i cross $F_i$ plus $P_{ij}$. Remember that since, we are finding the total we have to sum it up for all particles. Here, you will find that this is going to be rho i cross $F_i$ summed over i plus; when we do this, we find that it is going to be a double sum i and j of rho i cross $P_{ij.}$

This is an interesting thing because $P_{ij}$ is a force and because of Newton's third law the equal and opposite force $P_{ji}$ will be just equal and opposite. So, when you vary this, the index for all the particles you will find for each $P_{ij}$ there will be one negative $P_{ji}$ which will oppose the effect of this. Therefore, the resultant of this will be equal to 0 and this is equal to rho i cross rho i double dot $dm_i$ and this is nothing but the moment of the external forces. Here it was the total, but actually we have seen that the internal forces cannot produce any general moment. So, this is the equivalent of Newton's law for a rigid body in rotary motion (Refer slide Time: 23:03). This quantity can be shown as l dot; to do that I think you have to only express rho into its components x y z.

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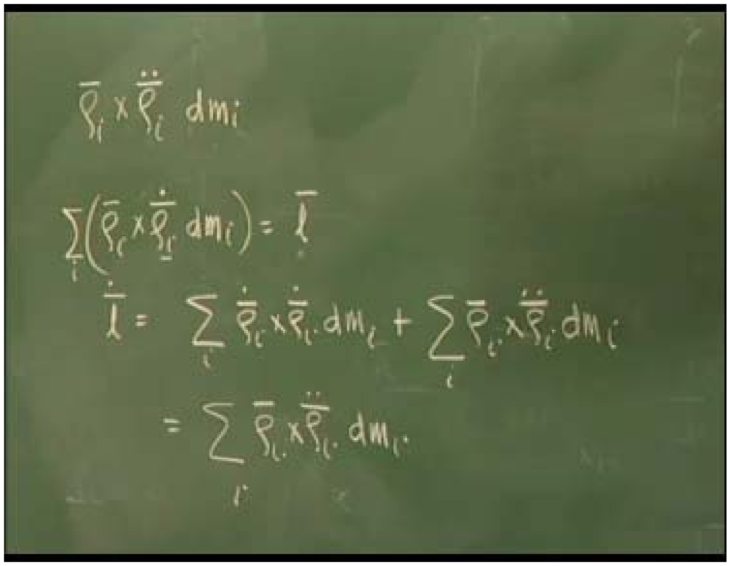

Let me see; to show this, we will start with rho i dot is the velocity of the particle, when multiplied by $dm_i$ we get the linear momentum of the particle. So, rho i cross the linear momentum is by definition the angular momentum. If you sum it up for all the particles, we get the angular momentum vector for the rigid body. This is the expression for angular momentum of the rigid body about point g from where rho i is being measured.

If I differentiate $l_i$ with respect to time what we will get. We will get rho i dot cross rho i dot $dm_i$ plus rho i cross rho i two dot $dm_i$ and summed over all particles, but we all know that the cross product of the same identical vectors will be 0. So, this term becomes 0 now, only this term remains (Refer Slide Time: 25:36).

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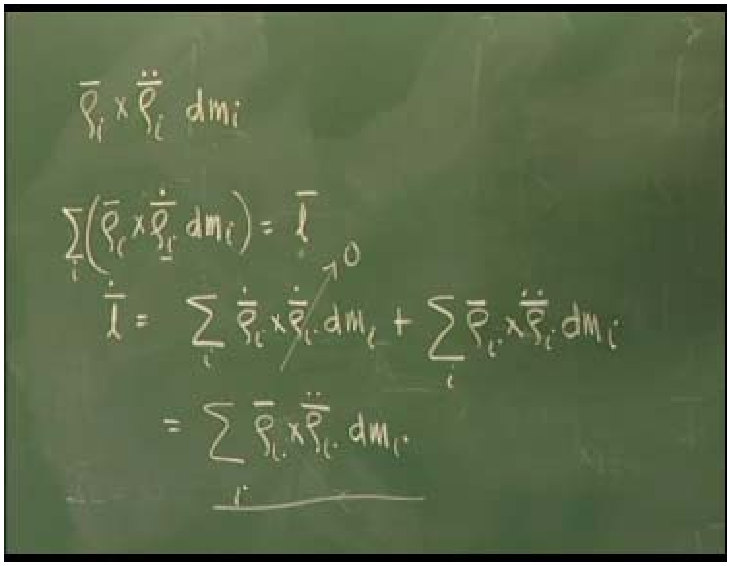

We find that this is nothing but rate of change of angular momentum which is equal to the resultant moment of the external forces or resultant externally applied moment (Refer slide Time: 26:02). Therefore, this is equivalent to Newton's law, that moment applied is equal to rate of change of angular momentum. Now, we have to express this in the kind of quantities which you have been using. Here, it is just a very general form. So, what we do here? We represent the angular momentum in terms of the angular velocity components and the inertial property of the rigid body.

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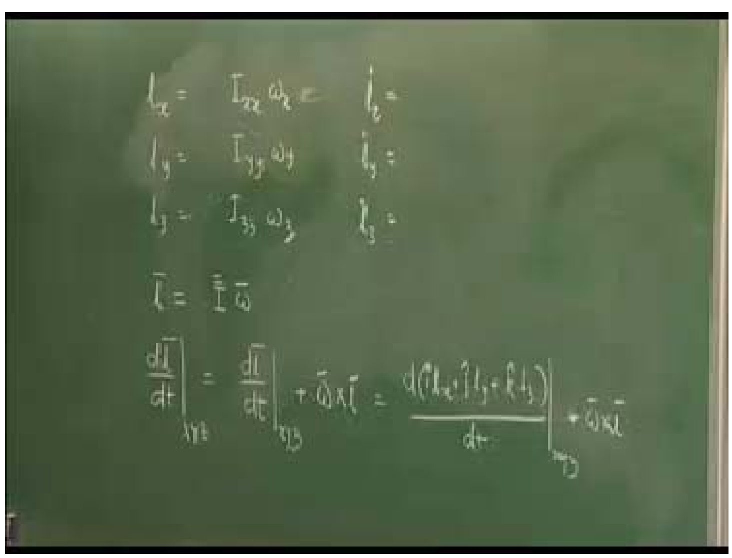

Therefore, $l_x$ $l_y$ $l_z$ we have already seen, $l_x$ is equal to $I_{xx}$ omega$_x$ minus, sorry, $l_y$ is equal to $I_{yy}$ omega$_y$ and $l_z$ equal to $I_{zz}$ omega z. All the time, we have to keep in mind that small x y z system is rigidly attached to the rigid body under consideration. So far, as the small x y z axis that observer is concerned it has no change in $I_{xx}$ $I_{yy}$ $I_{zz}$ can be seen or in other words $I_{xx}$ $I_{yy}$ $I_{zz}$ do not change with time as seen from small x y z system. So, l is equal to I omega vector.

We have seen that rate of change of angular momentum; we have to remember all the time that this rate of change is with respect to the absolute frame or capital X Y Z. So, here it will be plus; since, x y z system is rigidly attached that frames angular velocity is also omega.

Here, if you want to now find out what will be dl dt. Now, l we have seen is the components are like this (Refer Slide Time: 28:24). So, we can write it at least one more $il_x$ $jl_y$ plus $kl_z$ dt xyz plus omega cross l. Split up this into these three components, i is the unit vector along the small x direction. Therefore, with respect to small x y z system that vector i do not have any change. So di hat dt will be 0.

It will be i hat $dl_x$ dt in x y z system plus similarly, j hat that unit vector also in x y z system is fixed. So, it will not have any change. When you differentiate this you will get, as shown in slide below.

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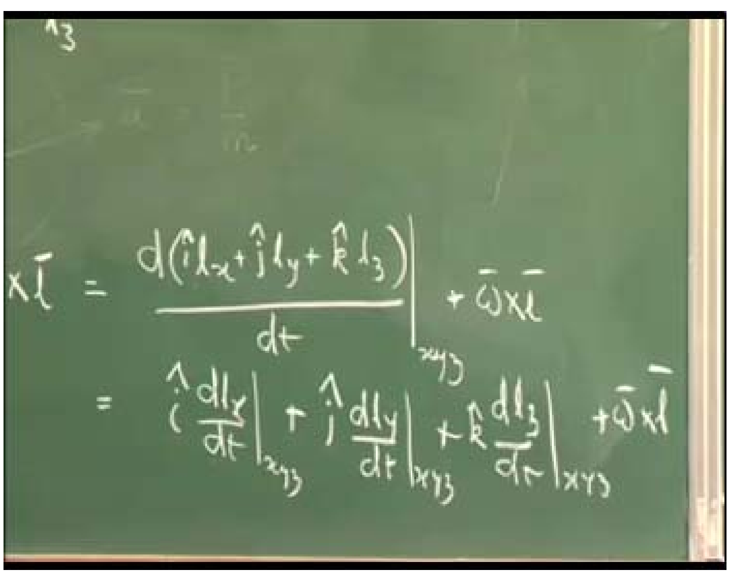

Now, d lx dt $l_x$ magnitude, we are talking about the magnitude of $l_x$ which is given by this (Refer Slide Time: 30:06). If we differentiate this with respect to time in the small x y z frame, then this is a constant quantity in the small x y z system only this can change.

So what we will get? This l dot rate of change of the angular momentum with respect to an absolute frame of reference is equal to i hat $I_{xx}$. When I differentiate this, only this can change. So, it will be omega x dot that the magnitude of the angular velocity component plus j $I_{yy}$ omega y dot plus k $I_{zz}$ omega z dot plus omega cross l.

Omega cross l will be what? We can write it like this; i hat omega x plus j hat omega y plus k hat $\text{omega}_z$ cross i hat $l_x$ plus j hat $l_y$ plus k hat $l_z$. When we do this cross multiplication here this cross product then, i cross i is 0, i cross j is k $\text{omega}_x$ $l_y$, i cross k is minus j $\text{omega}_x$ $l_z$, j cross i is minus k $\text{omega}_y$ $l_x$ . So, this is the complete result. We can next combine terms under i, under j and under k. So, rather than writing a long expression, we can also find out the components of this.

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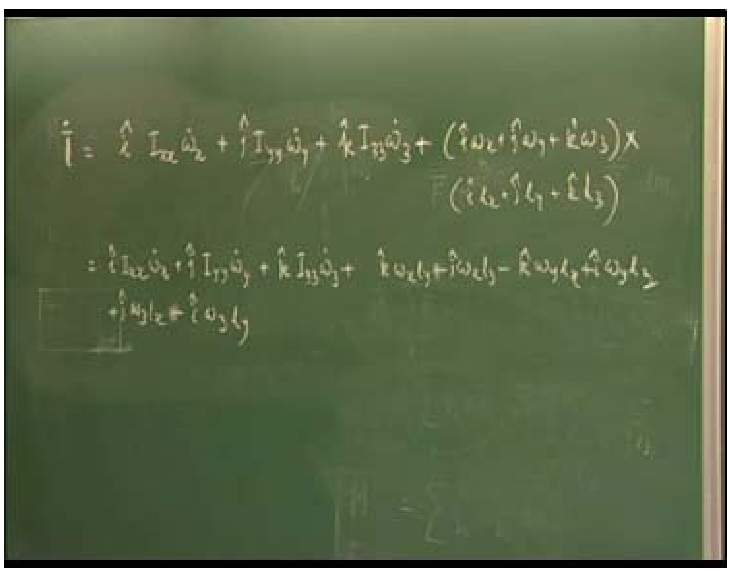

Now, remember that this l, rate of change of angular momentum is nothing but the externally applied moment. So, $M_x$ will be the x component of this. That means the terms if we take i hat as common, the term with which i is associated and that will be $I_{xx}$ omega x dot plus i $\text{omega}_y$ $l_z$. Now, $l_z$ is nothing but $I_{zz}$ $\text{omega}_z$ and the other term is this minus $\text{omega}_z$ I $l_y$ is $I_{yy}$ $\text{omega}_y$.

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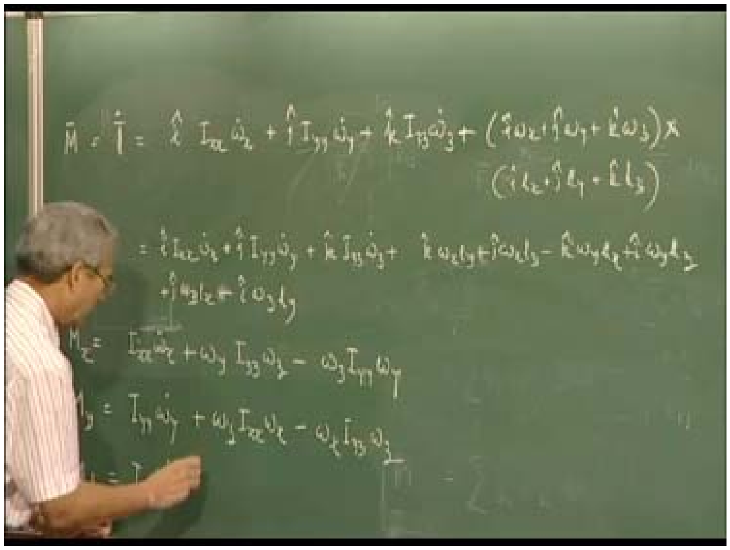

So, we have to keep in mind from the very beginning that the small x y z coordinate systems are being chosen in such a way that they are principal axis. Similarly, My will be the j part that is $I_{yy}$ omega$_y$ dot, then this term was i cross k is minus j, K cross i is j. Therefore, this plus omega$_z$ $L_x$ is $I_{xx}$ omega$_x$ and this is omega$_x$ $l_z$ which is $I_{zz}$ omega$_z$ and $M_z$ will be the k part that is, $I_{zz}$ omega$_z$ dot plus omega$_x$ $I_{yy}$ omega$_y$ this term minus this term. So, we can remember this when it is in a form. Generally it is written like this. $M_x$ is $I_{xx}$ omega$_x$ dot here, we can put this minus $I_{yy}$ omega$_y$ omega$_z$ plus $I_{zz}$ omega$_z$ omega$_y$ and $M_y$ is $I_{yy}$ omega$_y$ dot minus $I_{zz}$ omega$_z$ omega$_x$ plus $I_{xx}$ omega$_x$ omega$_z$. This form, it is easy to remember because x y z is in circular form x y z, y z x, z x y. So, that’s how one can easily remember.

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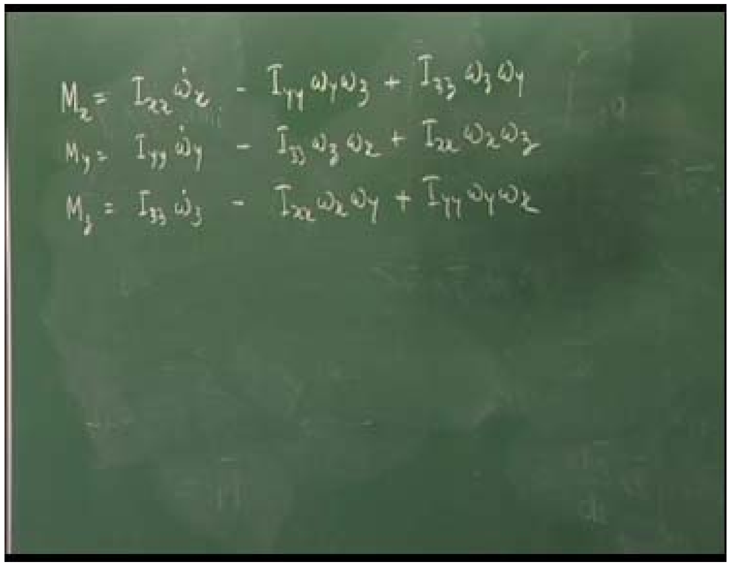


For the situation, where the small x y z system is rigidly attached to the body; that means its angular velocity and the bodies at angular velocities are same. Then, obviously this can be taken common minus $I_{yy}$ $I_{zz}$ omega$_y$ omega$_z$. So, this is the set of equations where the components of the externally applied resultant moment on the body and its kinematic quantities and the inertia properties are related. This set of equations is sometimes called Euler's equations.

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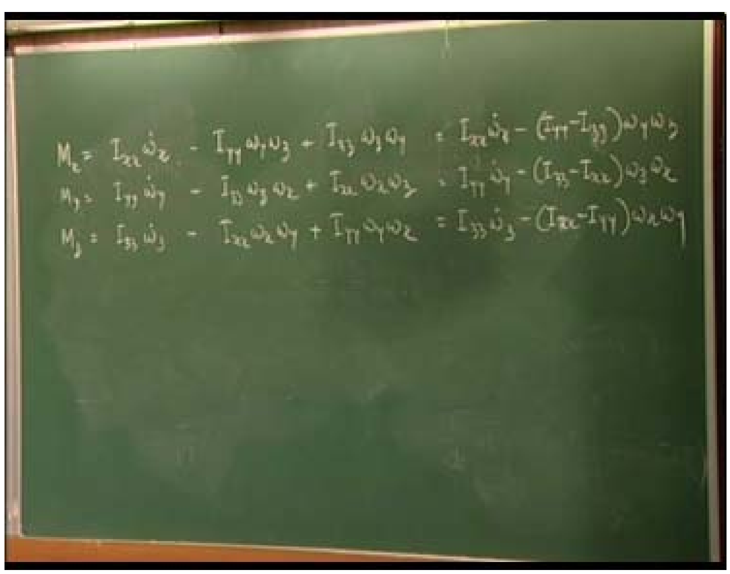

As it was first derived by Euler and this is the form of equation, though it is not perfectly general. The perfectly general form will be like that where the small x y z system is not rigidly attached to the body. We will do some such examples in the next lecture, if we get time, but this will be the governing equations for all the rigid body motions in three dimensions; we will be doing and we will be applying to machines.

In the next lecture, what we intend to do is to solve problems and see what motions are produced by certain externally applied moments. But more interestingly, what we will see is that, certain motions in machines and its parts components produce some moments or some reactions which otherwise we do not find. So, we will find some counter intuitive phenomena to take place when we consider the three dimensional motion of a rigid body.