Dynamics of Machines

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Module-2 Lecture-2

Space Motion of Rigid Bodies
Inertia Tensor and Angular Momentum

In the last lecture we have started discussing a motion of a rigid body in space. It has been found that the motion of a rigid body in space can be split into two parts: a pure translation and a rotation about a point. We have also seen that the part which contributes to translation is very similar to the motion of a particle and the whole body can be assumed to concentrate at a point. On the other hand, it is now necessary for us to analyze in details, that part of the motion of the rigid body which is about one point. It has been further shown that when a rigid body is moving with one point fixed, the motion at any instantaneous position or at any instant can be considered to be a rotation about an axis passing through that point.

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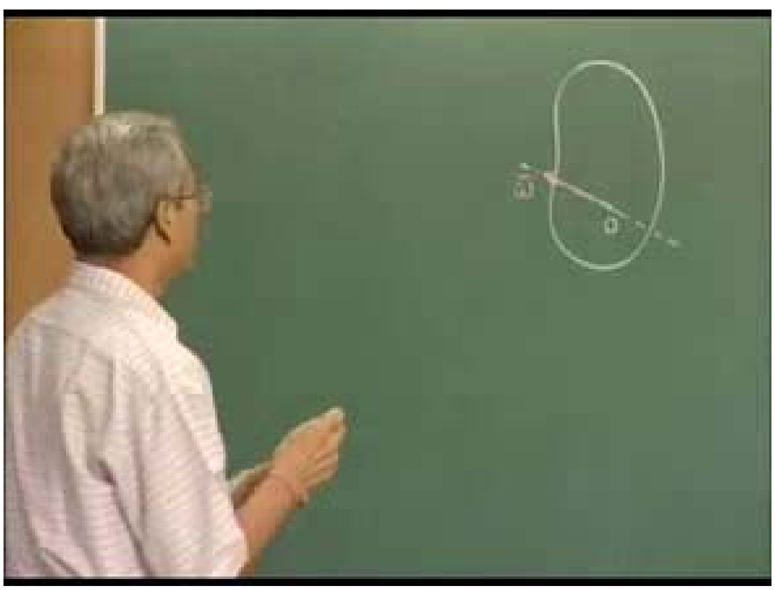

Thus, if this may be rigid body and this is the point about which it is executing rotation, then at any instant of time the body can be assumed to have a rotation about an axis and instantaneously the angular velocity of the rigid body can be expressed or represented by a vector omega along the axis, about which it can be considered rotating at this instant. It should be also remembered that this angular velocity vector is neither fixed in space nor fixed with respect to the material body. This executes a motion with this point O fixed in space generating a cone which you call the space cone.

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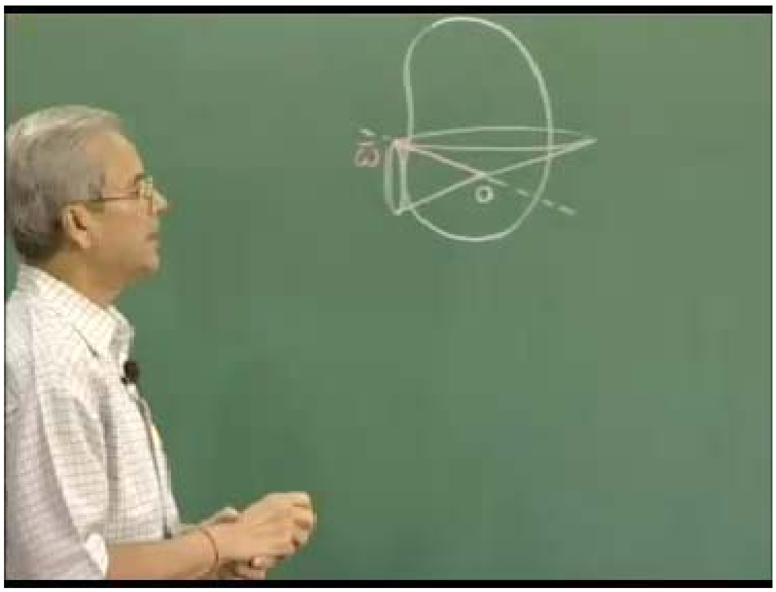

Say, for example this and so far as an observer a task to the body is concerned with respect to the body. This vector also executes another cone, which you call the body cone. The motion of the rigid body is equivalent to the motion generated by a pure rolling of the body cone over the space cone. At every instant the line along which the two cones are in contact is the instantaneous axis of rotation and the instantaneous angular velocity is along that. Today what we will do first is discuss about the angular acceleration.

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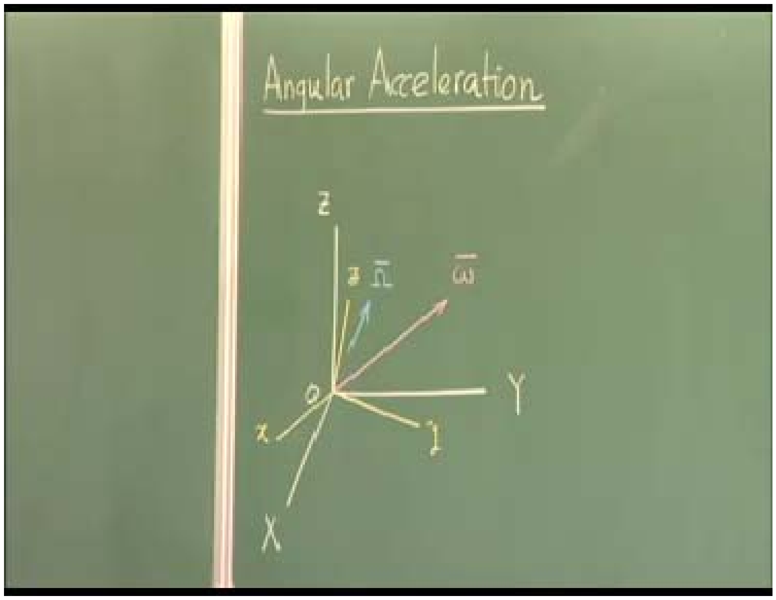

What is an angular acceleration? Angular acceleration is the rate of change of angular velocity. Now, we should define a bit more clearly that if we have a frame of reference X, Y & Z which is fixed in space. If this be the angular velocity vector at an instant then angular acceleration is the rate of change of angular velocity omega with time as observed by an observer fixed in this fixed frame of reference. Now, we have seen that this angular velocity vector since it is a vector it can change either by its change in magnitude without changing the direction it means, it can remain same in its orientation. Only its length will increase or it can remain constant in its magnitude but the direction of this vector will change or in general both happening simultaneously.

To find out the rate of change of this angular velocity vector therefore, what we do is we first consider a frame of reference small x, y & z with the origins of the two coordinate systems coinciding all the time, this angular velocity or this frame of reference in general, we may assume that it has an angular velocity capital omega with which this small x, y & z is rotating with respect to the fixed frame. Then there are two observers one fixed to the capital X, Y & Z frame one attached to the yellow x, y & z frame which is not fixed in space which is having a rotation with an angular velocity omega with respect to the fixed frame. So now if there is any vector A, its rate of change as observed by an observer in the fixed frame of reference dA bar dt X, Y and Z can be represented as the rate of change of the same vector as observed by an observer x, y & z plus the effect due to the rotation of x, y & z frame. So, see that if this vector A we are talking about is rigidly attached to the x, y & z then to an observer in x, y & z this vector will have no change.

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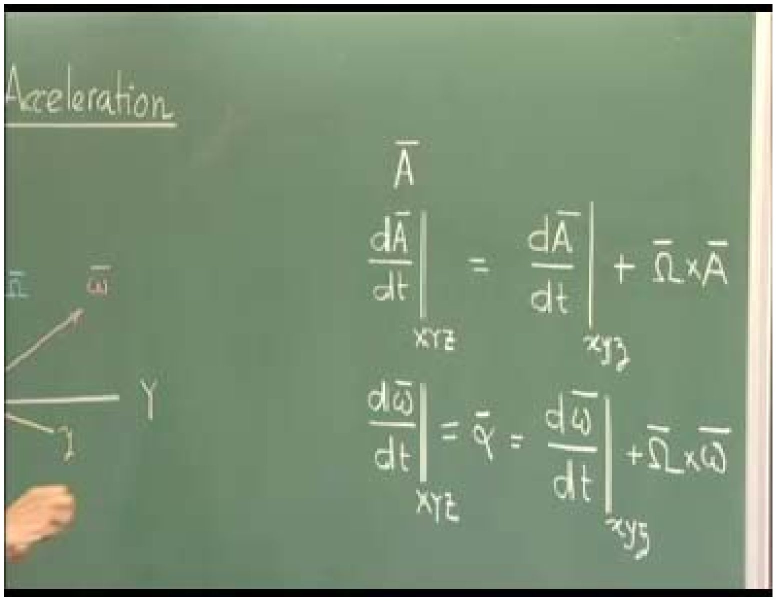

The observer sitting outside will observe the change because this vector is rotating along with this moving frame of reference with this and from vector calculus. We know that this rate of change can be written as this. So, the angular velocity vector omega will also have the same equation fulfilled that rate of change of angular velocity as seen by a fixed frame of reference is the angular acceleration by definition and it will be equal to…(Refer Slide Time: 08:50). So, this is the general expression for angular acceleration of a rigid body for a motion about a point O. Now, I think to explain this better, let us consider an example. Let us take this cylinder that is mounted on a shaft and the shaft is again mounted horizontally in this case on a vertical shaft which is rotating with an angular velocity omega$_2$. So, the whole drum or the cylinder is being carried by this horizontal arm, which is rotating with this angular velocity. At the same, we provide a rotation about this horizontal axis with an angular velocity omega$_1$. So, as you can see that this point which is on both the axis is fixed in space it cannot have any motion. Now, what is the angular acceleration of the cylinder, let us find out.

What we will do? We will do the same way, we will have one frame of reference which is fixed. This is the fixed frame of reference with this point as O. This is the moving frame of reference momentarily coinciding with the fixed frame of reference X, Y and Z. The angular velocity components that print the velocity of the body about the shaft is omega$_1$bar. The velocity of the shaft itself of the axis of rotation itself is omega$_2$bar. Now what we do, we consider this x, y & z frame to be rigidly attached to this horizontal axis carrying the drum. Therefore, this omega$_2$ is nothing but the angular velocity of this x, y & z frame, which is rigidly attached to the shaft carrying the drum. So now, from this relation that angular acceleration of this can be written as (Refer slide time: 12:18).

Next therefore, you have to find out what is the angular velocity of this drum? We have seen in the previous lecture that at any instant the angular velocity of the drum is nothing but the vector sum of the two components (Refer slide time: 12:40). Substituting this here we get (Refer slide time: 12:53) because omega$_2$ cross omega$_2$ is 0. So, what remains there is only omega$_2$ cross omega$_1$. If we consider a situation, where the magnitudes of the angular velocity components omega$_1$ and omega$_2$ are constant, then we will find that both this term and this term will be 0. Because, there is no change in omega$_1$ as observed by an observer sitting on the moving frame of reference though with respect to fixed frame, it is changing direction, but an observer sitting with the moving horizontal shaft will not see any change in the angular velocity omega$_1$. So this will be 0.

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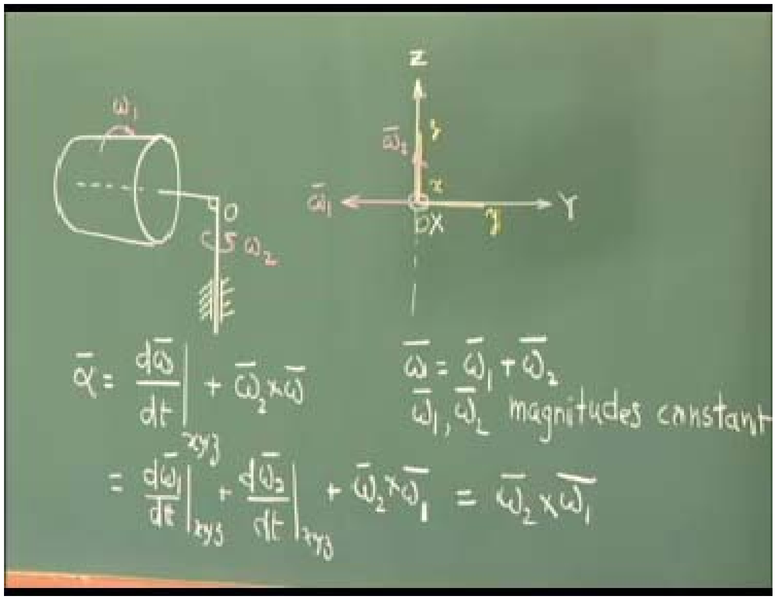

Similarly, this observer will also not see any change in the omega_2 angular velocity component so that will be also 0. So, this is a particular case where omega_1 omega_2 are constant (Refer slide time: 14:23). Under that condition, it will be simply, (Refer slide time: 14:23) we can simplify further in this way we can represent omega_1 and omega_2 in terms of the unit vectors along this so unit vector along x will be I, unit vector along y direction is j and unit vector along the z direction will be k.

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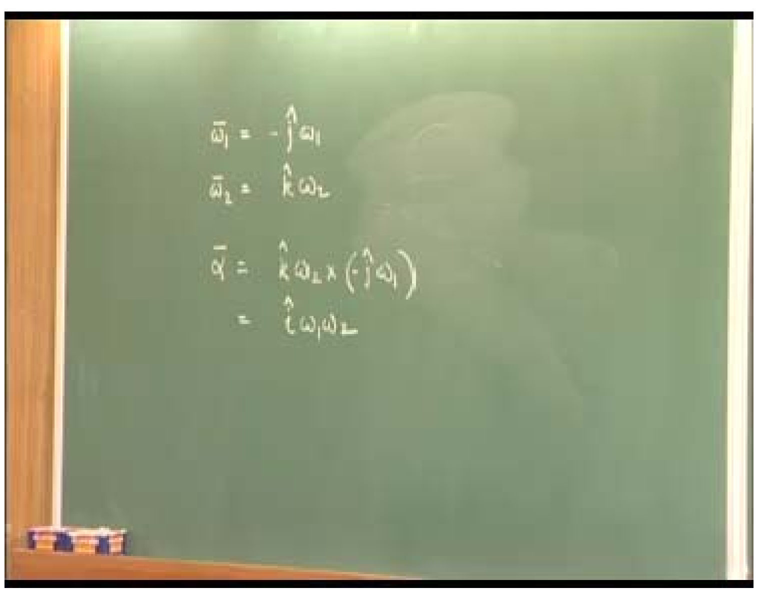

So $omega_1$ is equal to as per this is minus j $omega_1$ both z and x components being 0 and $omega_2$ is equal to k $omega_2$. Other two components that is x and y components being 0. And the angular acceleration will be (Refer slide time: 15:54) and this is equal to k cross j which is minus i and this minus and that minus becomes positive 1. So angular acceleration of the drum is given by this and we can see it is directed towards x axis it is also a vector. This completes our discussion on the motion of a rigid body. Particularly we are interested in the motion of rigid body with one point fixed as we see that any general motion can be split into two parts. One is the translational motion of any point on the body and a rotation about that point and the translational part is trivial in the sense that it represents a particle like motion, on the other hand the motion due to rotation needs further investigation. Therefore, let us now take up the question of the quantity of motion in a rigid body when it is executing a space motion with one point fixed or in general a space motion.

Before we discuss the dynamics, we know that we have certain quantities which represent the magnitude of motion that is momentum definition. There are two kinds of momentum we know one is linear momentum, other is the angular momentum. Now again, this is a quick recapitulation of our knowledge of dynamics, we are not going to have detail discussion. We all need to derive the basic relations which we will be using for an analyzing or studying the dynamics of machines or machine components. Now, here you know that rigid body during its discussion on kinematics we have mentioned that a rigid body is defined by a collection of infinite number of infinitesimally small point masses whose relative distances remain unchanged.

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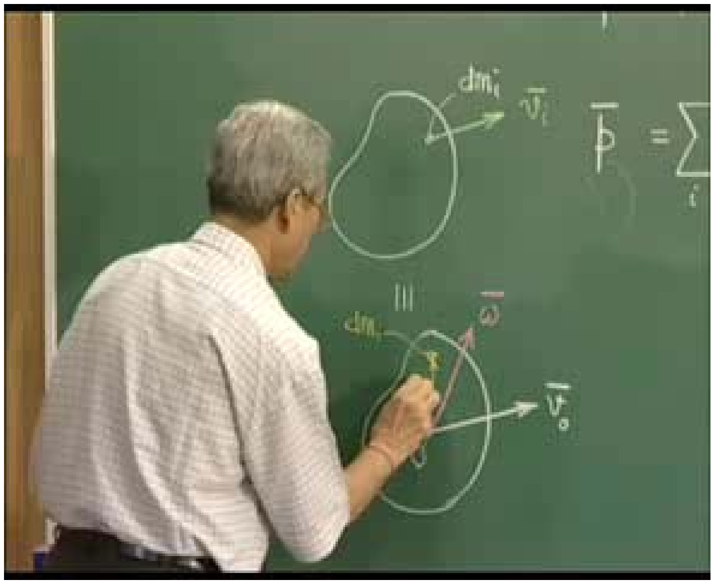

So what we can do, we can consider just one such infinitesimal mass say dm_i and if its velocity be v_i then the linear momentum of this particle is simply (Refer slide time: 19:02) and if we sum up this quantity for all the particles constituting the body we will get the linear momentum of the rigid body. So, if we use symbol p to represent linear momentum of course it is vector so it is p vector, we have to just sum it up for all the particles. Now, we have seen that we can treat the motion of this rigid body as a translational motion of a point and a rotation of the rigid body about that point, so this rigid body's motion will represent that a point O it is having a velocity v_o and a rotation of this rigid body instantaneously about an axis and its angular velocity at this instant being omega, if this be the case then a particle of mass dm_i at a location from this fixed point say row_i vector.

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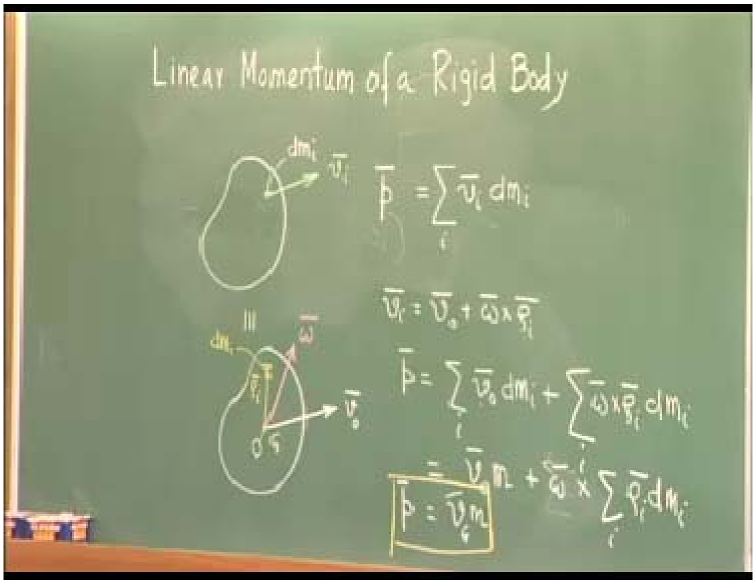

Then we know the velocity of this with respect to this point is given by omega cross row. Therefore, the total velocity of this point will be velocity of the point v_o plus omega cross row_i. So the velocity of this point v_o plus the velocity of this particle with respect to this point, which is given by omega cross row_i and we know that any velocity between two points on a rigid body has to be due to rotation because the distance between the two points cannot change in a rigid body. So with this we can write now p is given by (Refer slide time: 21:40) summed over i all the particles and summed over all the particles. Now in this case we find that this summation is over all the particles where this v_o rather is common to all particles. So, we can take this v_o outside and sum over all the particles their mass which is nothing but the total mass of the object plus omega. Just a moment here also we can see omega is constant for all the particles.

We can take omega term outside the summation and we get this. Now if this particular point O we have chosen happened to be the center of mass then by definition of center of mass this term is 0. Therefore, the linear momentum of a rigid body is the velocity of this point which is now the center of mass G, rather than being any point into m and this part is 0. So, in effect the linear momentum of the rigid body is represented or is determined by the consideration that the whole body is concentrated into a particle of point mass at the center of mass of the body or this is nothing but the linear momentum of a particle located at the center of mass having the same mass as the whole body. This is an interesting result we have seen. Now, find out the angular momentum of a rigid body. When do we talk about angular momentum you know that we must have a point about which we consider that angular momentum, unlike linear momentum which can be defined without any reference to a specific point. But in case of angular momentum we need a reference point about which we are defining the angular momentum, therefore first let us find out the angular momentum of a rigid body about a fixed point.

This is the rigid body and we have seen that the center of mass G is a very convenient point for defining certain quantities as you will see. So, we first define its distance from a fixed point O by the vector r_G and the particle under consideration dm_i which is at a distance row_i from the center of mass and at a distance r_i from the fixed point O.

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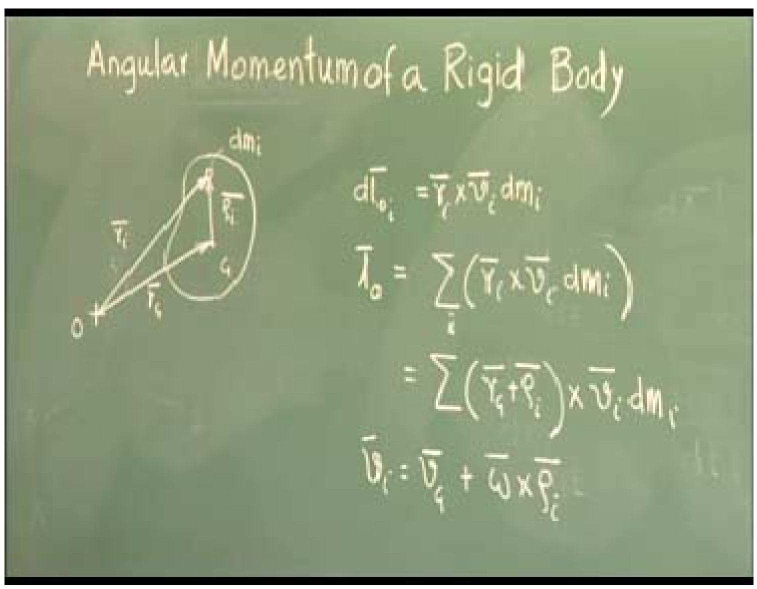

Then the definition of angular momentum tells that an momentum of this particle about point O, dl_{oi} if we define that as the angular momentum of this elementary mass about this point will be equal to the linear momentum of this which is and the moment of that linear momentum about this point, which is given by (Refer Slide Time: 26:59). This is the moment of linear momentum of this particle mass and to get the angular momentum of the whole body about this we have to only sum it up for all the particles constituting the body. So, l_o will be summed over all the particles. Now, we have seen that the point of this mass can be also represented by the sum of two vectors. One is the position of the center of mass r_G plus its location with respect to the center of mass row_i. If we do that this represents this (Refer Slide Time: 27:59) cross. Now again, the angular velocity of this point, we know can be represented as velocity of the center of mass v_G plus the velocity of this point mass with respect to this which is given by omega and row_i is the distance or location of the point mass with respect to the center of mass.

If you substitute this, then the angular momentum about point O will be given by (Refer Slide Time: 28:57) it is a substitution of the item there. Now, if you expand you will get r_g cross v_G dm_i plus r_g cross omega cross row_i dm_i summed over $_i$ plus row_i cross v_G dm_i and plus (Refer slide time: 30: 14). After expansion, we get this four summed up terms. Let us examine these four terms. So, here you find that r_g v_G they are all constant and they do not change or depend on the particle we are considering. So, summation can be taken and this represents nothing but r_g cross v_G into m that is the linear momentum of the object.

As you have seen linear momentum object is nothing but v_j into m. The second term you see that this is something which does not depend on the particle, omega also is constant. So the summation will be taken over here and by the definition of center of mass we know that sum row_i dm_i is 0 so this term will vanish. Same will be the situation with this because v_g is constant. So what we can do we can take dm_i here and do the sum and then take the cross product with v_g and again by definition the sum dm_i row_i is 0. So, the other term which remains will be thus angular momentum of a rigid body about a point O is given by this.

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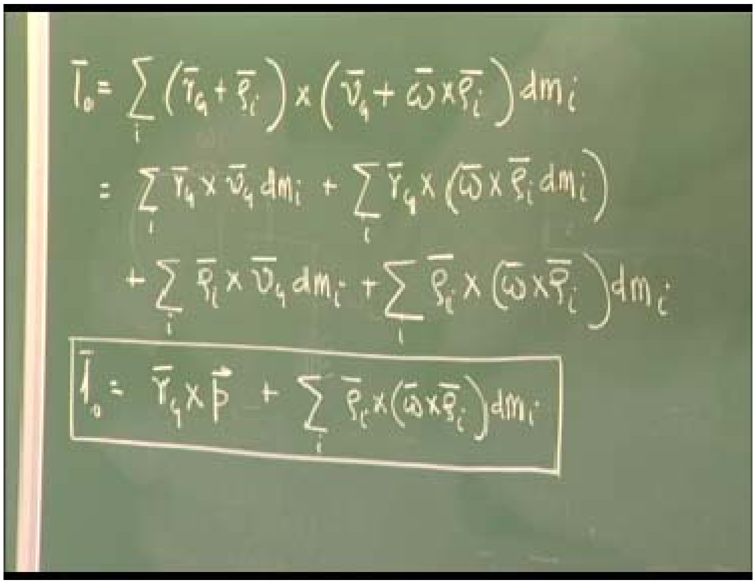

We consider two more situations, this is a perfectly general case two more cases where the motion of the rigid body about point O as a fixed point. Let us take the case. Now as I said let us consider this case where a point O of the body is fixed in space and body is rotating about this point. This is a simpler situation and the particle mass dm_i is considered to be at a location r_i with respect to this point O then its velocity will be simply omega cross r_i and the angular momentum about O will be (Refer slide time: 33:26). Therefore, straight forward we come to this expression from the definition, that it is nothing but the moment of all the linear momentum of all the particles summed over the whole rigid body.

If O happens to be the center of mass, then the angular momentum of a rigid body about the center of mass (Refer Slide Time: 34:13) summed over all the particles. These are the two cases, where a point on the rigid body is fixed and in a special situation when that particular point is the center of mass we will require these expressions at later time. Now, let us come to the question of the inertial properties. We know that in case of simple fixed axis rotation of a rigid body, the inertial property of the body is defined by a quantity we call moment of inertia, which is nothing but the second moment of the masses of all the particles about that axis. Angular velocity into the moment of inertia gives us the angular momentum in case of planer rotation as you have seen.

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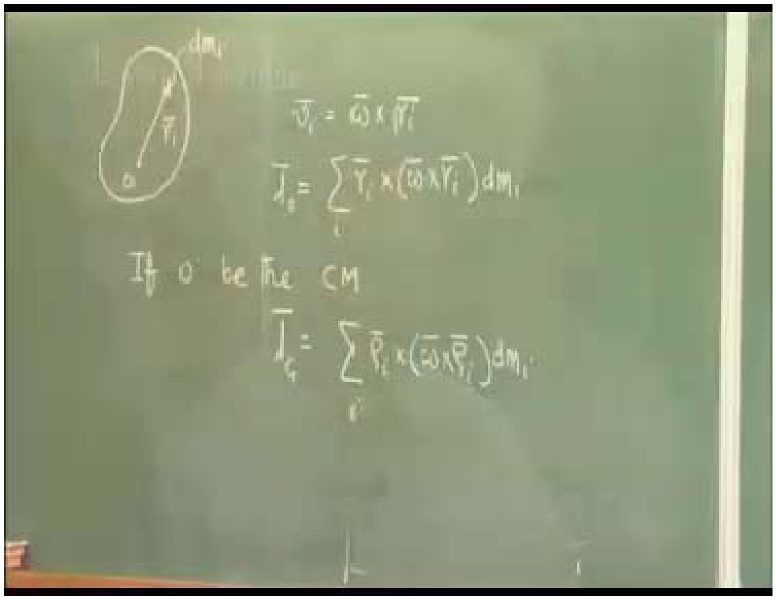

Let us now investigate little bit about the inertial property of a rigid body in space motion. If we take the angular momentum of a rigid body, if the particles be infinitesimally small and the number become infinitesimally large, then we know that the summation can be represented or replaced by integration over the whole body (Refer slide time: 35:58). Now we would like to have a more clear definition or rather in more concrete terms this quantity.

To do that, let us consider this positioned vectors r in terms of coordinate of the position of a point which is simply x, y & z be the coordinates. We have to fix up a coordinate system on the body so any point its location will be x, y, z and the vector r will be given by this where i hat, j hat, k hat are nothing but the unit vectors along the axis x, y and z respectively. Similarly, vector omega will be also expressed in terms of the three components $omega_x$, $omega_y$ and $omega_z$. Now with this term omega cross r becomes when you take the cross product when i cross i is zero, i cross j is k $omega_{xy}$ plus i cross k is minus j $omega_{xz}$ then next j cross i is minus k, $omega_{yx}$ j cross j is zero, j cross k is plus i $omega_{yz}$ plus k cross i is j $omega_{z\ x}$ plus k cross j is minus i $omega_{zy}$. So this is equal to i $omega_y$ into z minus $omega_z$ into y plus j $omega_z$ into x minus $omega_x$ into z plus k into $omega_x$ into y minus $omega_y$ into x this is omega cross r. Now if you take this term r cross omega cross r then we have to again multiply this by r. So it will be i cross i zero, i cross j will be k and this x into this.

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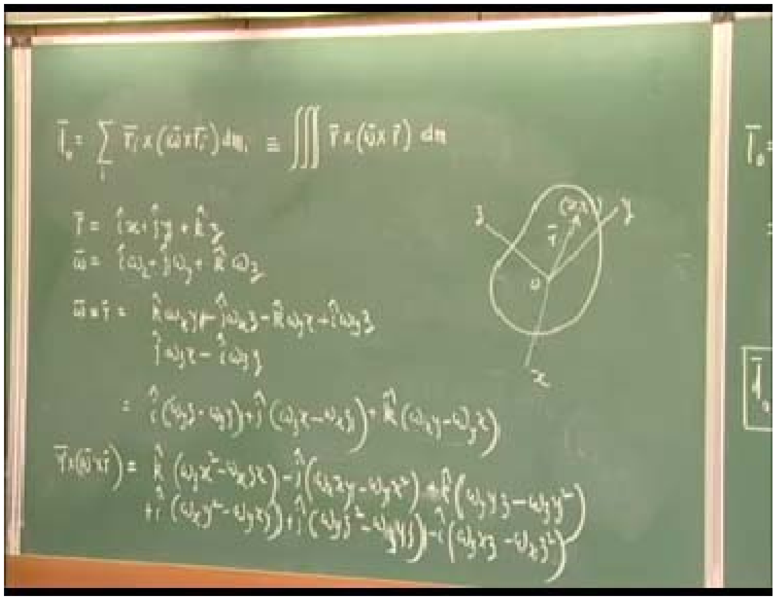

Next, will be i cross k will be minus j omega$_x$ xy minus omega$_y$ x square. Next, let us take j cross this j y cross this so this will be plus j cross i is minus k into this into y, that is, omega$_y$ yz minus omega$_z$ y square plus j cross k that is plus i into omega$_x$ y square minus omega$_y$ xy the last term will be this cross this, so k cross i is plus j omega$_y$ z square minus omega$_z$ yz plus k cross j is minus i to omega$_z$ xz minus omega$_x$ z square and k cross k is 0. So, this is the term let us put it in a neater form so we can say that about the whole body can be written as i omega$_x$ y square minus omega$_y$ xy minus omega$_z$ xz minus omega$_x$ z square so it will be omega $_x$ into y square plus z square dm that is one term it is minus omega$_y$ xy dm minus omega$_z$ xz dm plus j omega$_y$ z square plus omega$_y$ x square. Next omega $_z$ yz minus omega$_x$ xy so this is the jth component of the angular momentum remember that plus k that is the z component k into omega$_z$ x square plus omega$_z$ y square this minus makes plus minus omega$_x$ xz minus omega$_y$ yz dm. ok now let us see what we have got? So, here if we further examine we will find this is nothing but i.

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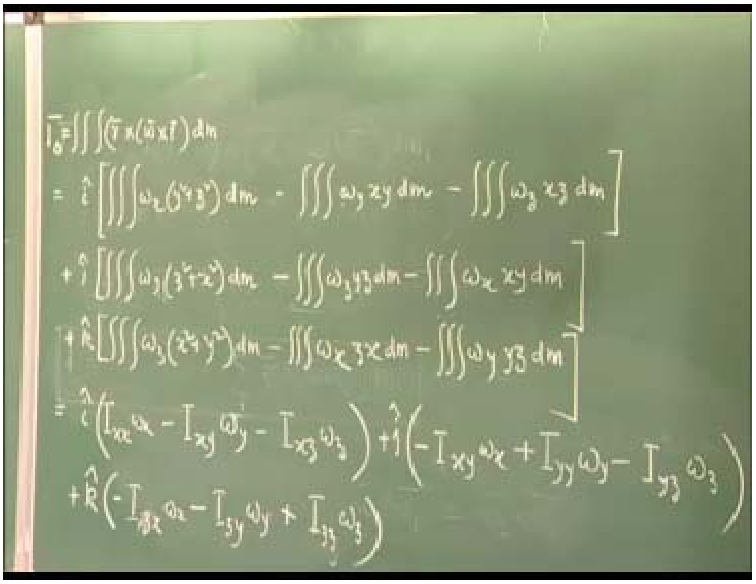

Now, omega x is the component of the rotation performance angular velocity of the rigid body, it does not depend on particle to particle so it can be taken out. If I take out this then what remains inside is dm into y square plus z square. You all know that if the particle be here then y square plus z square this is y square. This is y and this is z. Now y square plus z square is nothing but the perpendicular distance of this point square of that about x. So mass of this particle into distance square from x if you sum it over then it is nothing but the moment of inertia of the rigid body about the x axis and we denote it by I_{xx}. So, this is nothing but $omega_x$ into i_{xx} minus this $omega_y$ can be also again taken out because it does not depend on the particle position but this quantity x y dm that means, the product of the x and y coordinate of this point multiplied by mass summed over the whole thing is something which we not encountered before and we call it a product of inertia I_{xy}.

And similarly this will be another product of inertia I_{xz}. so, the Ith component looks like this, jth component look like this, minus I $omega_i$ xy $omega_x$ plus this is again nothing but the moment of inertia of the rotating body about the y axis and we (not audible) define it by I_{yy} and the third remaining term this one is I_{yz} $omega_z$ and the k component is minus this, this integration is nothing but I_{xz} or $_{zx}$ $omega_x$ this term is nothing but I_{zy} $omega_y$ and this term is nothing but the moment of inertia about z axis I_{zz} $omega_z$. So we can also see from here that this product of inertia terms that I_{xy} and I_{yx} will be same I_{xz} and I_{zx} are same. So it is a symmetrical situation therefore now you find that in a slightly more compact form the components of the angular momentum vectors can be written as this.

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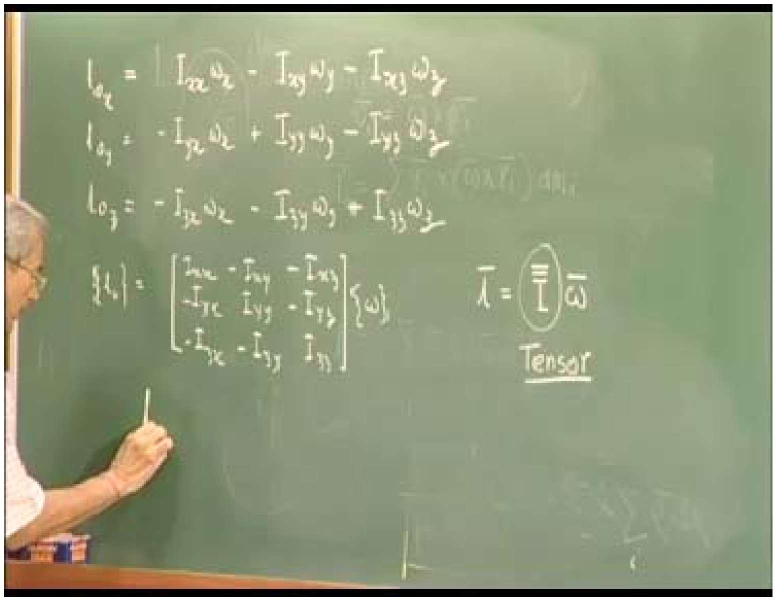

Now, we should also see that angular momentum of the object is a vector quantity. Angular velocity is also a vector quantity. So, here you could represent in matrix form so if you write in matrix form we find that this column metrics which is representing the angular momentum vector is a product of this square matrix, which represents the inertia of the system multiplied by the column matrix representing the angular velocity vector. So we come across a new quantity that angular velocity vector is a product of a quantity I, with angular velocity vector omega. Now this cannot be a vector we can see because the product of two vectors are either a scalar in case of product or a vector in case of cross product but it is always at right angles to the two vectors whose product we are taking but in this particular case there is no such compulsion.

So this quantity is something different from a vector, it cannot be a scalar also then the two vectors would have been in the same direction and we call such quantities, which are represented by the matrix of three into three metrics we call this is a quantity which is of a higher level and we call it a tensor. We need not bother much about the tensor except for the fact knowing that this inertial property of a rigid body needs to be defined by a tensor it cannot be defined by a scalar as in case of a particle or planer rotation of rigid bodies nor can it be represented by a vector something higher than that and that is why we called it the tensor.

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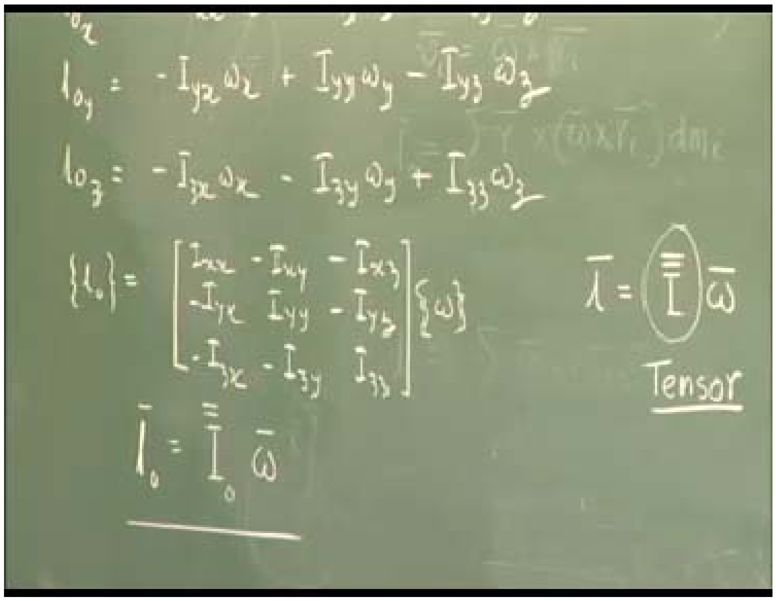

Angular momentum of a rigid body is inertia tensor with that point O as the reference point multiplied by the angular velocity. It is a very complicated situation as you can see and handling this kind of expressions is not very convenient, but we are lucky in one sense. In case of all rigid bodies it can be proved we did not go into the proof. It can be proved that we can always select the x, y and z axis in such a manner that the product of inertia vanishes and these set of axis are called principle axis.

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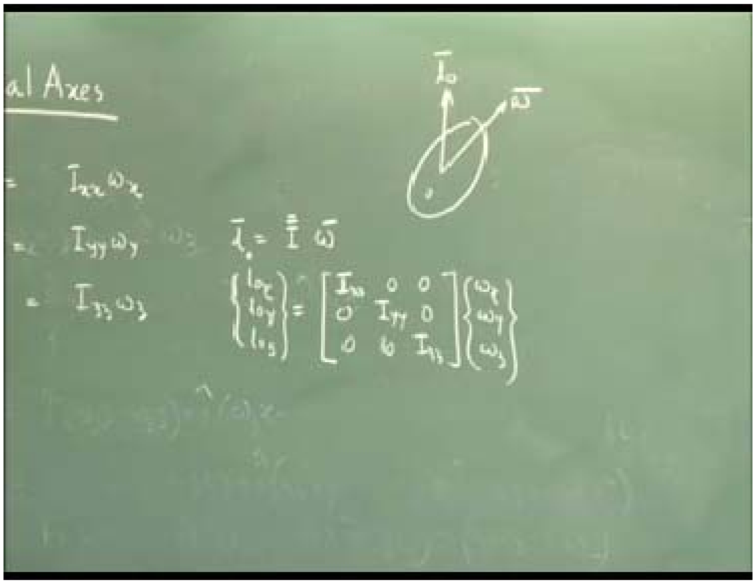

If we allow the mutually perpendicular three axis x, y & z along the principle axis which always exist then the products of inertia will vanish and the angular momentum vector can be then represented into three components. Three components will be I_{xx}, omega$_x$ and I_{yy} omega$_y$. It is a very happy situation the expressions are much simpler but still we should remember that it is a tensor and or in matrix form, still we need to have a matrix and even in such a situation we have to keep in mind that the angular momentum vector and angular velocity vectors do not coincide except for the special situation where I_{xx}, I_{yy} and I_{zz} are all equal in magnitude. So, even for the simplest possible situation we represent the whole thing with reference to the principle axis.

The angular momentum vector and the angular velocity vector they will not be same. I think this gives us some idea about the inertia property of a rigid body we find that it is a very complex situation than a particle or single particle or even planer motion of a rigid body. It needs in general since six independent elements I_{xx}, I_{yy} and I_{zz} and three products of inertia, because this is a symmetric case. Now I think when you go to the discussion on the dynamics of a rigid body then obviously we have to find out the rate of change of this momentum which you have found out the expressions which we intend to take in the subsequent lecture.