Dynamics of Machines

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Module-13 Lecture-5

Approximate Method of Vibration Analysis: Rayleigh - Ritz Method

It will be better explained with the help of an example.

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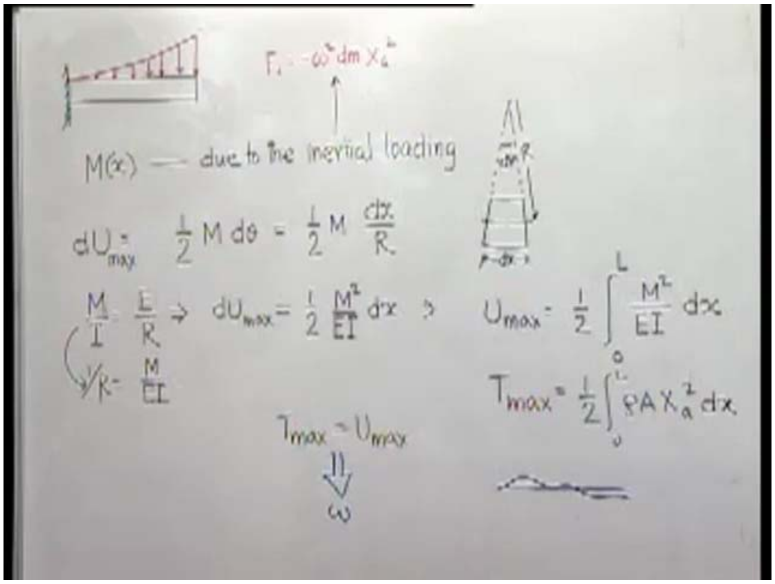

So, to compare the values what we get by this approach, that is Rayleigh's method by Grammel's modification, let us take up again a simple problem of a cantilever beam uniform for which we know the answer. Our objective is to compare the Rayleigh's method and Rayleigh's method using the Grammel's modification. Just to demonstrate that, we will take a simple [example] and we can also compare the exact value of the answer.

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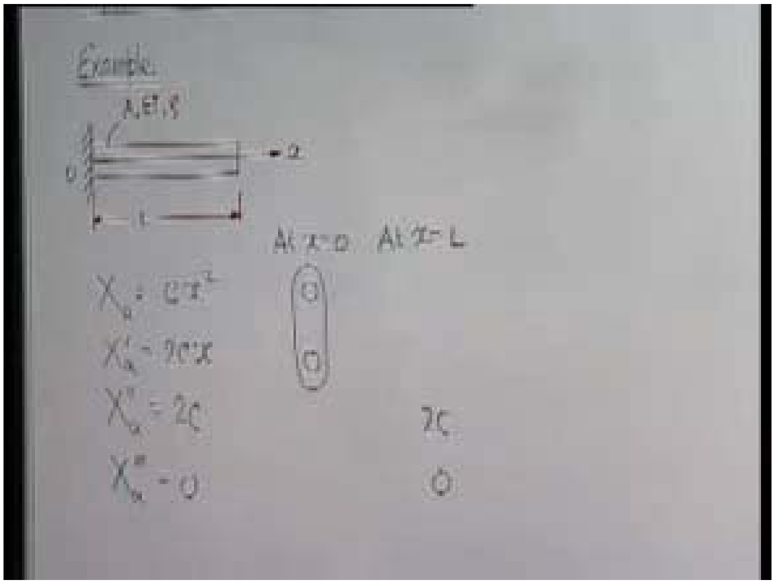

Let us take a cantilever beam of uniform cross-section and of length L and they are all uniform. Now, we assume the deflection satisfying the condition, at least the geometric condition will have to be satisfied to know that, so we just take it like this: X_a is equal to cx square. So, the slope is this and x we do not need that anymore. So, let us see… of course, we can do that. At x is equal to 0, X_a is equal to 0 that satisfies the condition. The slope is also 0, that is another geometric condition that is also being satisfied. At the free end, we are supposed to have this as 0, this is 2C, this is 0; but, it should have been 0, because, there is no momentum acting here, but it is not satisfying that condition, but anyhow it is satisfying the geometric boundary condition, which are very important and they must be always satisfied.

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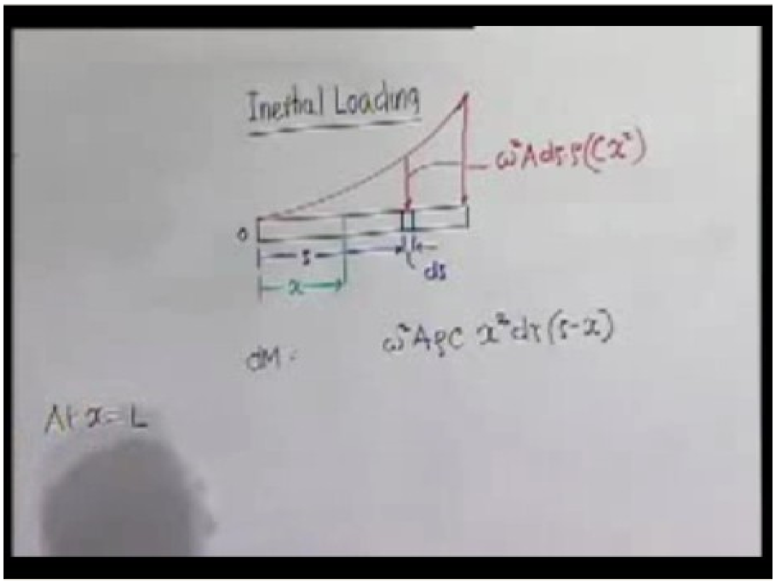

Now, with this what will be the inertial loading? Inertial loading is x square. It will be like parabola and we consider an element of length d zeta, at a distance zeta. Now, let us find out the moment of this inertia force because of this, produces how much bending moment at distance x from the origin? So, what is the inertia force of this element? That is equal to the mass of the element that is A into d zeta into rho. Then, omega square into X_a square; now X_a is this part, so it will be C x square. So that is the inertia force. Moment of that at x is equal to X, due to this element will be omega square A rho C x square d zeta into zeta minus x.

So, this is a force, inertial load and this is the distance. So, this will be the moment and since this moment or the positive direction of moment was like this (Refer Slide Time: 06:53) and for that the slope of the neutral axis, as x increases, gradually decreases. Here it will be gradually increasing. So, they will be in negative. Therefore, total moment will be the moment of all the inertia loading of all the elements from zeta is equal to x zeta is equal to L. So, therefore, moment as a function of x at distance x will be: minus omega square A rho C integral x to l x square zeta minus x d zeta. So, we have taken coordinate [08:11 min] zeta is the variable not x, x is a parameter; at zeta is equal to x what is the [08:31 min] if you want to find out [08:35 min]. So, this if we estimate, we will find this is: minus omega square rho AC by 12 into x to the power 4 plus 3L to the power 4 minus

4xL cube. Now, if we use this: omega square is equal to M square by EI dx by x 0 to L, again now x is the variable 0 to L rho AX square dx. If we now use this M here and x, of course, is we have assumed like this then this will lead…. Now of course, if you remember that omega will also come in this side, because, of this twice omega to the power four. So, when you solve the whole thing, we will get omega square is equal to 12.46 1 by L to the power of 4 EI by rho A or omega is approximately square root of this 3.53 1 by L square root of EI by rho A. So, this is the result we get using the Grammel's modification that means we find out the moment at x due to the inertial loading which is caused by an assumed deflection X_a and substitute it here in the original equation of Rayleigh. Then solve the equation for omega square because, omega square will be on both sides and we get this (Refer Slide Time: 10:54).

Now, if we try to find out with the same function or same displacement function X_a just by following Rayleigh's method and if we take, then what we get?

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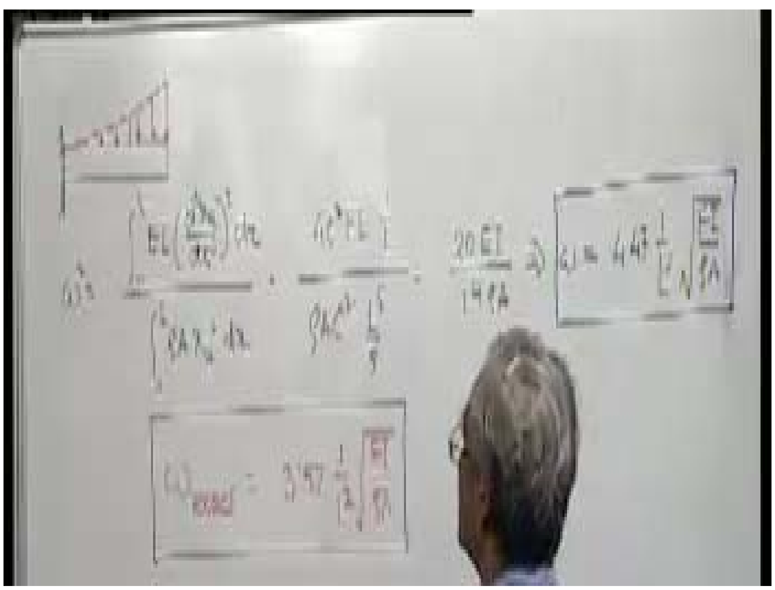

Omega square is simply 0 to L EI d square X_a dx square, whole square dx by 0 to L rho A X_a square dx. Now, we know dx square dX_a is 2C; so this will be simply 4C square into EI dx that is 1 by L - that is the numerator. What will the denominator be? The denominator will be: rho A x square that is C square x to the power 4, if you integrate it will be simply L to the power 5 by 5. So, that is 20 EI by L 4 rho A or omega is approximately square root of 20 is 4.47. So, this is the value we get by using simple or straight forward Rayleigh's method. For a cantilever beam, we have already solved by exact method. The exact value we already calculated earlier it was equal to this (Refer Slide Time: 13:18). Now, if you compare, this is the exact. I am using this simple straight forward assumption about the displacement function dx square, the Rayleigh's principle gives this. Even in the same assumption about this, if we use the modification proposed by Grammel, we get this. Now, you can see that how close it goes to the exact value. Therefore, the accuracy can be substantially improved in the Rayleigh's method if we use the moment expression directly, which is determined by finding out the inertial loading based on the assumed mode shape and that results in substantial improvement of the value of the [14:30 min].

So, I think this will be enough for the time being, but we have to now keep in mind that whether it is possible to find out the higher mode or higher frequency. All these cases, this Rayleigh's principle is valid in case of fundamental frequency as we know. Therefore, all the time we are assuming something for the first mode deflection, if we can find out the second mode shape and put it there, obviously, we will get the second mode, but most of the time you see that it is convenient to solve the problems satisfying the boundary conditions, it will give us the first mode frequency. So, in the next presentation, we will try to see if it is possible to get a methodology by which we can have some approximate value of the higher frequency.

So, following the approach proposed by Rayleigh and even considering the modification suggested by Grammel, it is always found that we get the fundamental natural frequency or the first mode natural frequency. However, when it is required to find out higher frequencies or higher mode frequencies, it is possible to devise a method of this approximate nature following the same principle and employing one of the important properties of Rayleigh's quotient.

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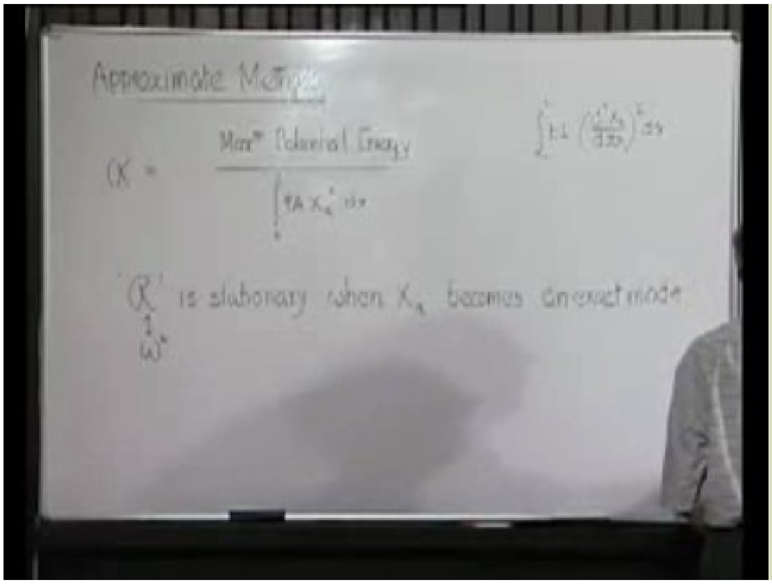

We have seen that Rayleigh's coefficient is equal to this (Refer Slide Time: 16:40). So, this is the maximum potential energy during normal mode oscillation and this is the maximum kinetic energy divided by the natural frequency square. So, this is the Rayleigh's coefficient and with this Rayleigh's coefficient as the natural frequency, omega square, square of that. The maximum potential energy expression will depend whether it is a transverse oscillation in that case it becomes this (Refer Slide Time: 17:44). If it is longitudinal oscillation, then again its expression is different, we have seen that. So, in general you can keep it as the maximum potential energy during normal mode oscillation divided by this quantity.

Now, there is one property of this Rayleigh's coefficient which says: R is stationary when X_a becomes an exact mode. So, whatever we may assume for this displacement function or the deformation pattern or mode shape whatever you may say, R of this quantity will be always higher than a quantity which will be minimum or stationary as X_a becomes a real mode, that is exactly it coincides with the natural mode. So, therefore the omega square what we will get which we associate with this, this omega square will become lower and lower as X_a becomes a better and better approximation and when X_a becomes in-distinguishable from the exact mode, then the real natural frequency will be the lower. So, this property of Rayleigh's coefficient is used and this proposal was first suggested by

Ritz. The whole process will be also called Rayleigh Ritz process of Rayleigh Ritz method.

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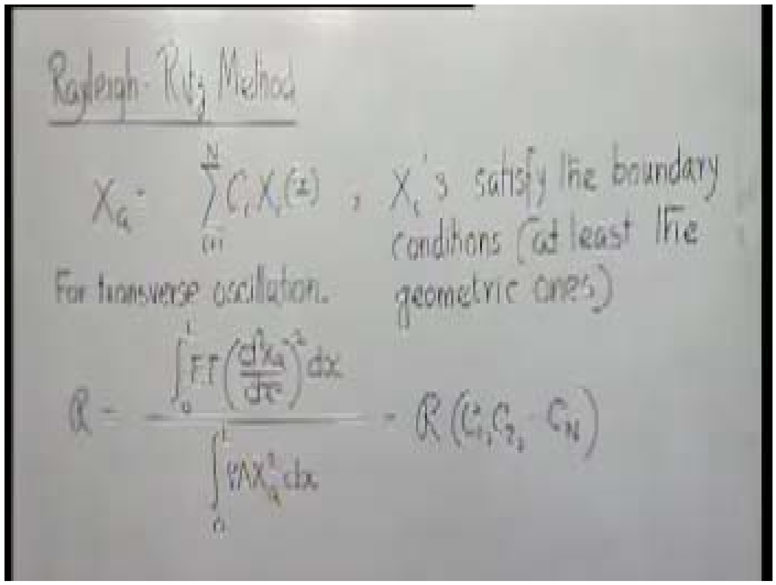

So, in Rayleigh Ritz method, this assumed deformation shape is expressed in the form of a series or in terms of a number of functions. So, where, each X_i satisfies the boundary conditions as much as possible. Here, better approximation means satisfying more boundary conditions, better results will be obtained and at least the geometric boundary conditions. Now, if we do that, then what will happen if we substitute, this is also a function dependent on quantity depend on X_a and derivative; here also it is dependent on X_a; so as a whole when this quantity is found out, so, the Rayleigh's coefficient will be equal to this (Refer Slide Time: 22:43), for transverse oscillation in beam mode.

So, when you substitute this, this becomes a function of C_1, C_2 up to C_n. So, this X_i's are definite functions E, I, rho, A these are all known quantities, L is known; so, only thing what remains as a parameter on which R will depend will be the N number of coefficients. So, now if R be a function of all these coefficients we have chosen, so what will be the best way to select this coefficient? Here comes the utilization of this property.

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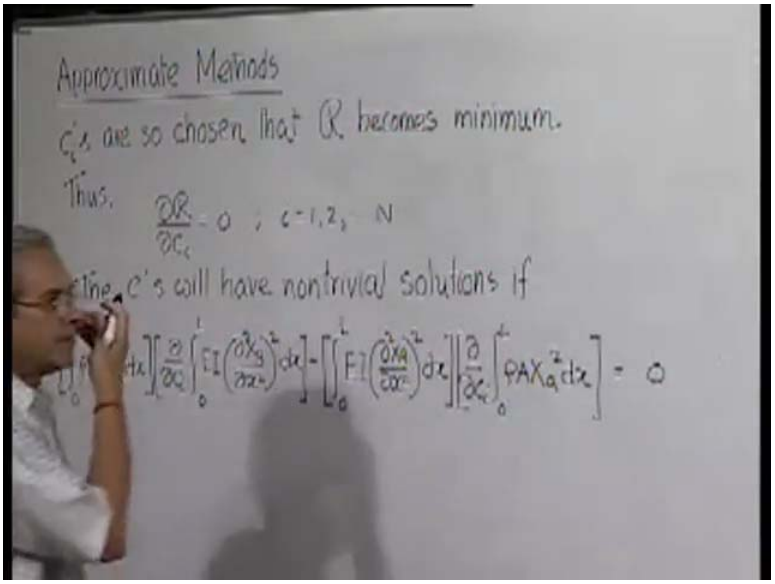

We should select C_1 C_2 and so on up to C_n in such a way that we get a minimum value of R. That means it will be nearest to the real value of the natural frequency square. Thus a condition which satisfies that, they are so chosen that the R will be always higher than the natural frequency square. So, the best possible choice of R, for the given values or the given functions X_i's, can be those for which C_1, C_2, C_n are so that R is minimum. That condition says: thus del R del C_i will be equal to 0, if R takes a stationary value at the chosen position then, all these [25:55 min] N number of that will be 0. Therefore, this represents N number of homogeneous equations in C and therefore a non-trivial solution of the C will be possible, only if the determinant is 0. Therefore, when this is done, this represents this N equation. Now the determinant of these equations if we differentiate this, this is the numerator and this is the denominator it will be this (Refer Slide Time: 27:03). How do we get it? Let us consider (Refer Slide Time: 28:25) this as the numerator and this is the denominator. So del R del C_r will be equal to this.

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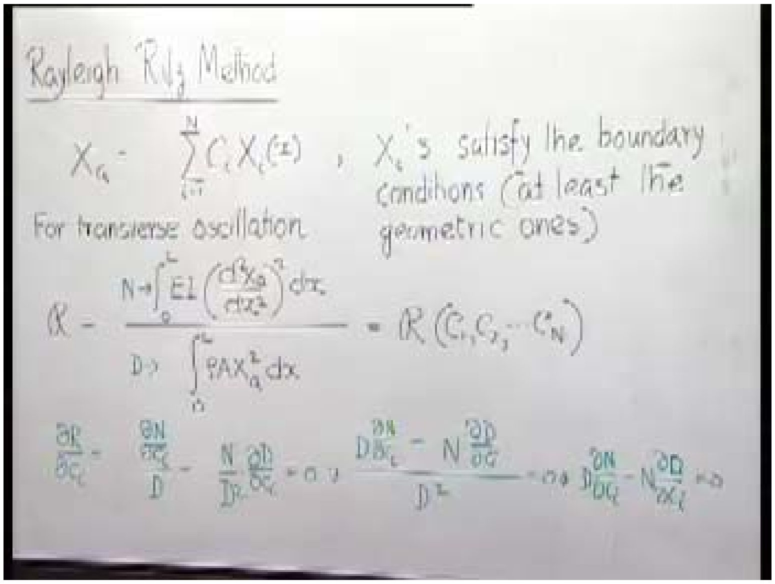

This will be: del N del C_i by D minus which will be N by D square del D and this equal to 0. So what we can do now? This gives us below there will be D square, so there will be DN DC_i minus N. Since D square is not 0 [29:35 min] D and this is nothing but that. This is the numerator and this is the denominator (Refer Slide Time: 29:55).

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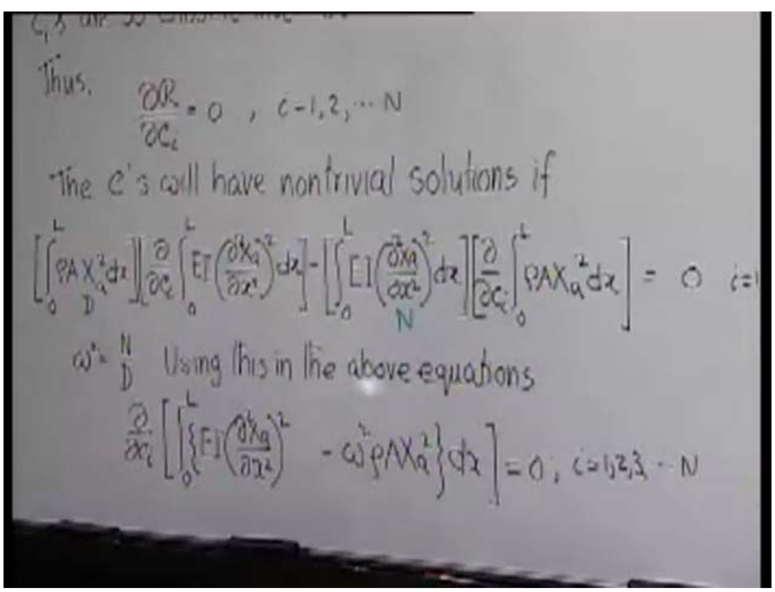

Now, we further note that omega square is equal to… this is the denominator, this is the numerator. So we know that omega square is equal to numerator by denominator. So, when you use this quantity here then, using this… now of course, you should remember that. Using this in the above equation what we get finally is this (Refer Slide Time: 31:28). So, there will be N such equations, it is very easy to see if we use this. Therefore we can always take this common and pick out this. Once we pick out this, it goes below here and this becomes omega square. So, it becomes omega square and then we take this (Refer Slide Time: 32:40). So, therefore it is straight forward. So, then what is done is that these N equations in C_i - C_1, C_2, C_3 up to C_n - and these homogeneous equations can have a solution only when determinants becomes 0. We will try to explain the whole process with the help of a specific example.

Let us take the same case of a paper beam, we have found out the fundamental frequency by Rayleigh's method, by Rayleigh's method we did Gravel modifications; we also know the exact value. So, it will be easier for us to assess or answer if we get from this at least the first mode. So, what we will try to do, because, we have to keep our analysis, board work and also objective is to demonstrate the use of the principle to find out higher mode; we will take only a case where we will find out up to second mode. So, the example is the same.

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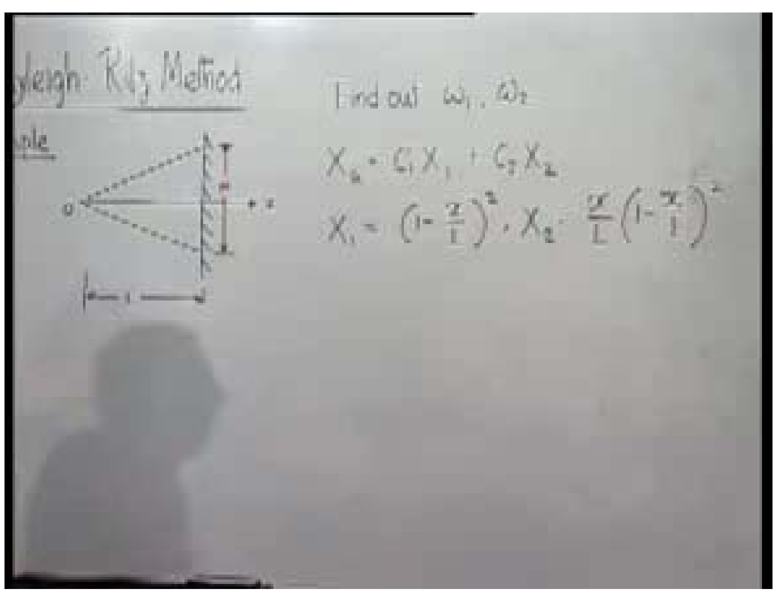

So, the same paper beam, the size of thickness is varying linearly and reaching the value eight after a length L and it is symmetric. So, if you fix the coordinate here the origin and the x-axis. Now, the question comes that if we were interested only up to two modes - $omega_1$ and $omega_2$ only. Therefore we would need only two terms in the expression X_a is equal to C_1X_1 plus C_2X_2 and X_1 and X_2 should satisfy the boundary conditions, at least a geometric one. So what do we do? We select X_1 as this (Refer Slide Time: 36:02). So, these are the two functions of X; at X is equal to L, X_1 is equal to 0.

If we differentiate once with respect to X, again this term 1 minus x by L remains and at x is equal to L that will be 0. So, slope is also 0. So, these are the two boundary conditions, geometric conditions, which are satisfied. This one also say at x is equal to 0 also it is becoming 0. So, it is not really satisfying the condition, but other end if you see that x is equal to L it is satisfying and both the slope and the value of x deflection is 0. So therefore, X_1 and X_2, X_1 is satisfying the geometric conditions, X_2 also is satisfying the geometric boundary conditions, but neither X_1 nor X_2 will satisfy all conditions. Any how our objective was that whether it satisfies the two conditions here onwards. So, we have to keep that in mind only two conditions are being partly satisfied.

Now, with these two things our equations become this (Refer Slide Time: 37:35). So, we substitute this quantity which becomes this after integration of this.

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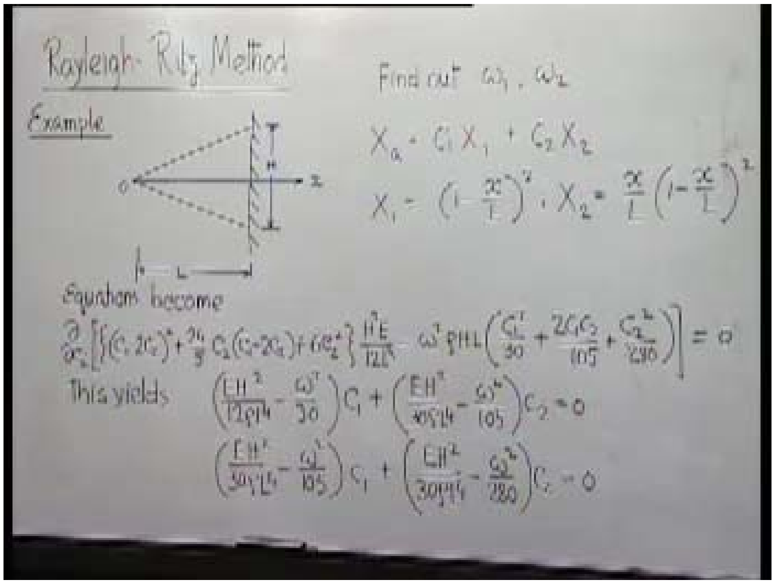

This after integrating becomes this (Refer Slide Time: 39:19). So, now this yields two equations. These two equations will be now algebraic equations. So, here it is C_i; therefore, if you differentiate with respect to C_1, we get this equation. Now, if you differentiate with respect to C_2 partially, we get the other equation (Refer Slide Time: 40:22).

So, now these are our final equations - homogeneous equations - in C_1 and C_2 and as I have been mentioning that non-trivial solutions will be possible only when the determinants of the coefficients equal to 0. We can see the determinants of these coefficients contain this term omega square. So this will ultimately give us an equation in omega square. So, the determinants of this - we will find out.

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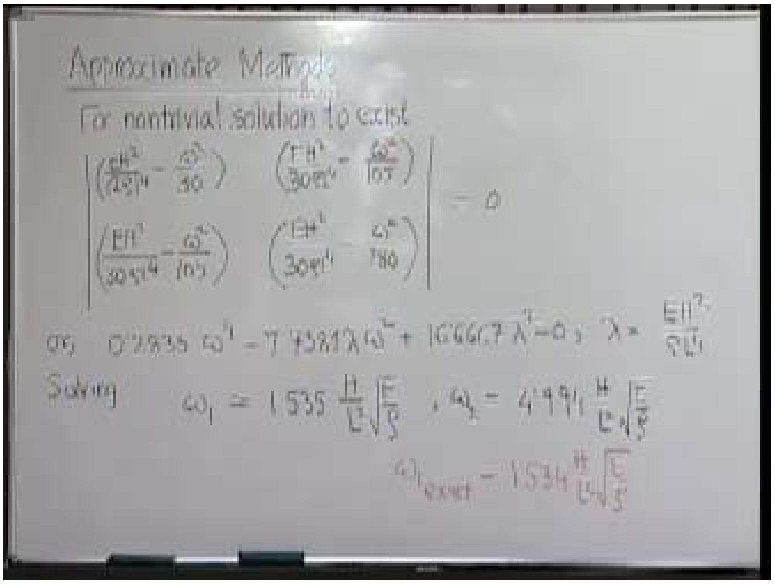

So, for non-trivial solutions, this becomes like our characteristic equation or this becomes… (Refer Slide Time: 44:11). So, this quadratic equation in omega square, we ultimately obtained [45:00 min] while lambda is equal to this. Solving this equation, we get two values of omega square and one value is omega$_1$ which is equal to this (Refer Slide Time: 45:19). So, the first mode frequency and this is omega$_2$. We get the second mode frequency also, which is approximately 5. If we try to compare this, the exact value if we remember is this. Thus, we find that, here the difference is only in the third decimal place and that so again you can see it is less than 1 percent maybe 0.1 percent. Therefore, the first mode we get quite accurately; the accuracy of the second mode has to be again found out by analysis, but it can be shown it is also reasonable accurate. We could find out even higher frequencies, only then it will be necessary to take or add one more term here like [47:11 min]. Obviously, computation will be more; there will be three equations C_1 C_2 C_3 and the determinants will give rise to a third degree equation omega square which has to be solved for the [47:26 min].

So, this brings us to the end of our discussion on approximate methods for solving continuous systems, which are otherwise difficult or otherwise not so easy for solution using analytical exact method as we have discussed. I think normally what is done in a real life system will be approximated by a model systems and in the model system, one tries to keep as many paths idealistic as possible. Then, of course in some cases, it may not be possible to have a theoretical solution or analytical proof or solution. Approximate methods like this can be used to make a quick estimate. However, now-a-days with the advent of computer and computers being so cheap, generally a complicated structure can be always modeled and using a finite element technique we have complete [vibration] analysis can be solved; standard software are available. In spite of that, the designers sometimes may find this quite handy and convenient for getting approximate answers.