Dynamics of Machines

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Module - 13 Lecture - 4

Approximate Method of Vibration Analysis:

Rayleigh's Method

We have started discussion on certain approximate method, using which we can get reasonably accurate values of the natural frequency of an elastic body system, where analytical formulation will not be possible. So, there are two approaches: one is in such cases, where the geometry is not simple enough, [amending] the problem or making the problem suitable for analytical solution as in case of uniform beams, bars, uniform shafts or that means two methods will be either used - computers are used or some quick approximate methods to get some idea about the natural frequency. For design purpose, sometimes this approximate method may be considered enough rather than going for a complete computation. So we have already demonstrated that we use the Rayleigh's principle.

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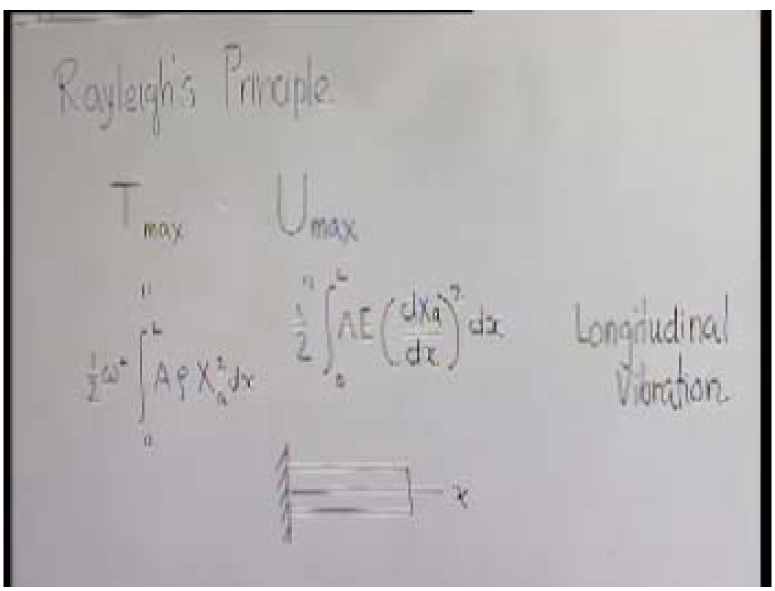

Rayleigh's principle says that for an unknown system, in the steady state, the maximum kinetic energy during free vibration must be equal to the maximum potential energy, which in case of elastic bodies is predominantly the strain energy involved. Using this principle, what we found in case of bars that, this is equal to… (Refer Slide Time: 02:32). That means, the total kinetic energy of the whole elastic body, in its peak value, in the normal mode oscillation can be expressed like this, where X_a is the assumed deformation shape or mode shape; whereas, this will be in case of longitudinal vibration and we have seen by applying this method, longitudinal vibration of a cantilever bar or one end fixed bar, that our result was reasonably accurate, even though we assumed the mode shape which was not satisfying all the conditions. But one thing we should keep in mind to get meaningful values, that whatever mode shape we assume, they must satisfy the geometric boundary conditions.

Geometric boundary conditions are those which depend on only these reflections or the slope of a particular cross-section; whereas, the other kind of boundary conditions, for example, (Refer Slide Time: 04:44) this end is free from a force, whereas, this end is free from any displacement. So, this is a geometric boundary condition. This end is free to move, but its force is 0 that is a non-geometric or dynamic boundary condition. So, at least the geometric boundary condition should be satisfied. So we solved something like this, very simple form and we reasonably got the answer. So, now I think let us go to the other type of vibration problem, which is more common, that is transverse vibration.

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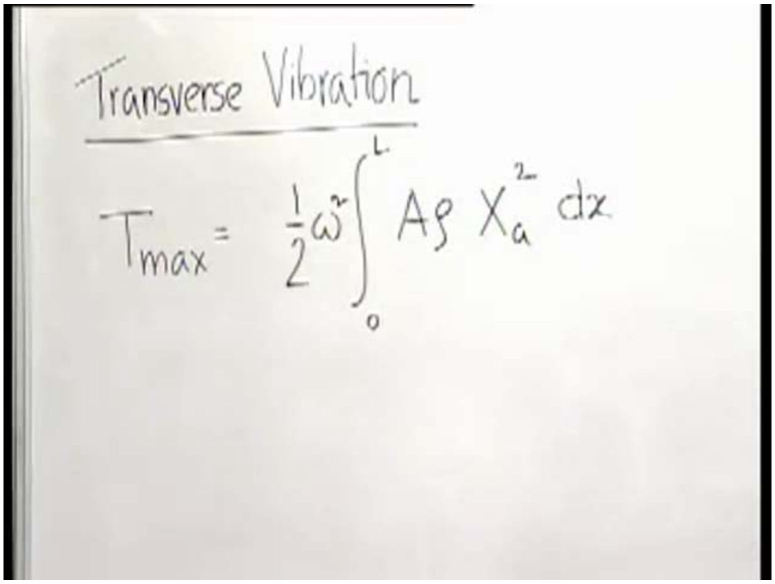

Now, in case of transverse vibration, the kinetic energy expression at the peak value will be same, that means, an element with x and the element mass is given by its volume, that is, A into dx that is the length of the element.

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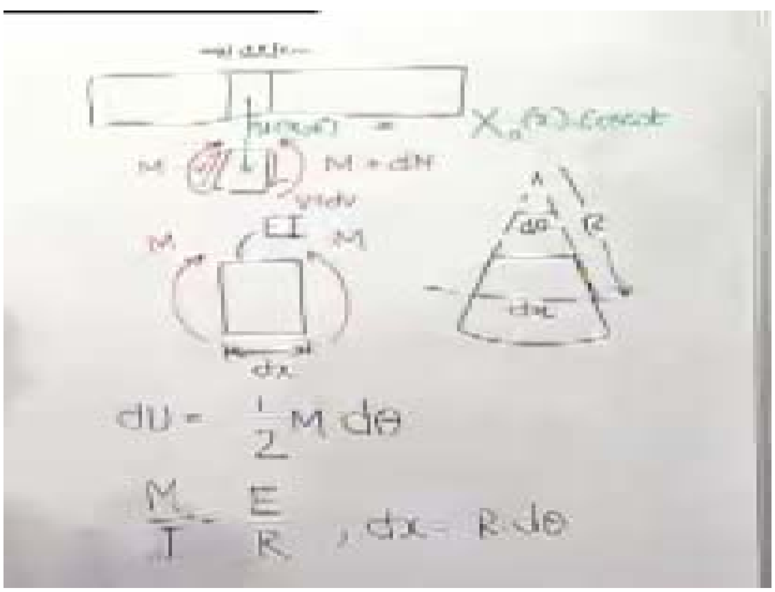

If we take one element of the length dx, in bend condition from here, it gets subjected to bending moment, which is M and here it is M plus is small [06:49 min] dM. The force is here, say for example V and here is again it is V plus, the resultant force is also very small and the displacement of this from here is given by U, which is a function of x and t, but which you write as this as a function of x and cosine omega t, we assume this normal mode oscillation to derive at the Rayleigh's quotient. Now, in this case the maximum velocity will be same as omega X_a. So the square of that will be, mass will be A dx into rho then half into velocity square for the whole minimum. So that will remain same as we have in this, but however, how much is the maximum strain energy that you have to derive and let us find out this element.

Now, we all know that if element is like this (Refer Slide Time: 08:14) length is dx, its flexural rigidity at this particular location, it will vary along this path, but flexural rigidity is say EI, that means E is the modulus of elasticity of the material and I is the second moment of [area]. Now, this is subjected to a bending moment. We know that in such case the energy in this member will be half M into… [09:10 min] Now. the whole thing we know is going to be in the form of bending. So, if we fix one end, then the other end is going to rotate by an amount d theta, if we try to match it with this and try to rotate the other end, the total amount of angular rotation of the end where we are applying the moment is d theta. Therefore, it will be half M d theta, we know that, because it is a gradually applied moment and so the total work done by this moment will be half into M into d theta, because moment also slowly increases from 0 to maximum value, when the deformation or the rotation of the end is already there.

Now if this is dx and if the distance of neutral axis is R, R is nothing but the radius of curvature, we know from bending theory M by I is equal to E by R. From geometry we also know that if this is dx and this is R then the relationship between these quantities are given like this (Refer Slide Time: 10:46). Therefore, dU_{max} that means maximum strain energy stored in this element is given by half into… [11:12 min]. Now here d theta we can write this now using dx by R. Now, R is nothing but EI by M; so, it becomes half M square by EI into dx. So this has to be now integrated for the whole member starting x is equal to 0 to L.

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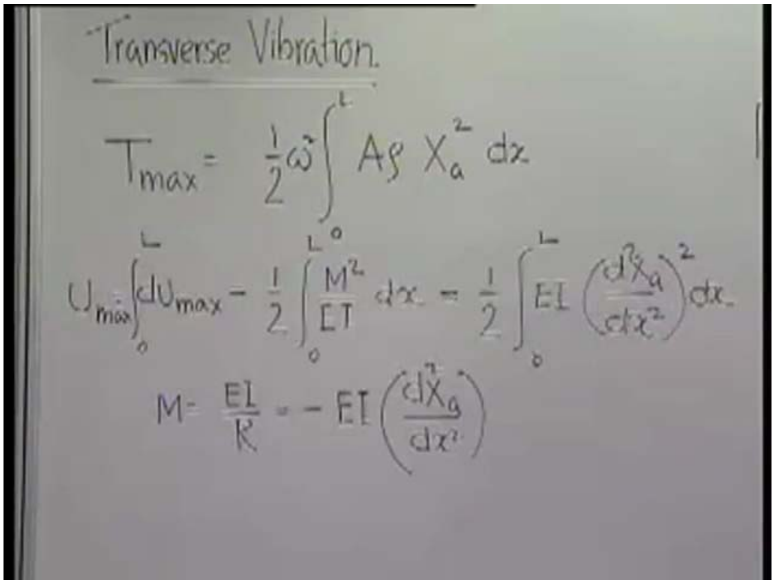

Now, how much is M? M we already have found out, let us first write this. Now, we have already shown earlier that M is nothing but EI by R and with the sign convention which we used for positive moment there and when the neutral axis bends in such a way that with x (Refer Slide Time: 12:34) its slope gradually, decreases, so then minus sign comes, and 1 by R we have already derived.

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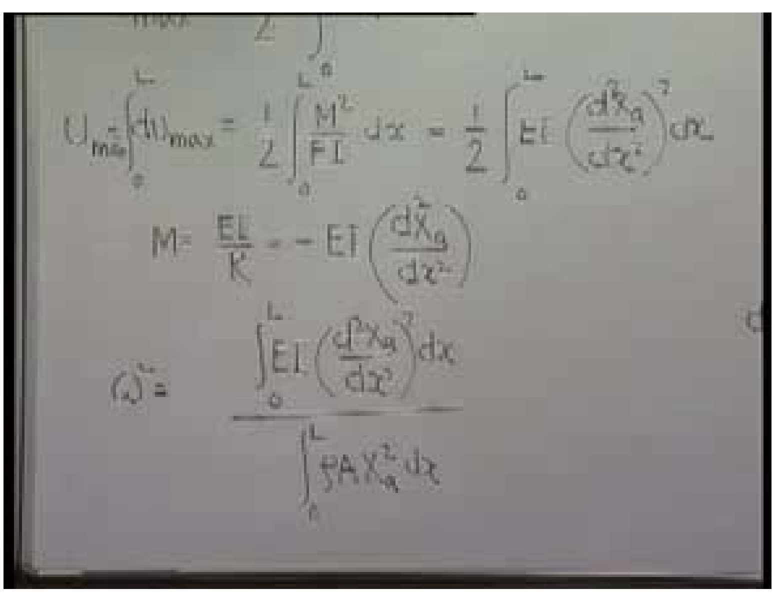

So once we have integrated, then this becomes (Refer Slide Time: 13:08). Now putting the same here we get half EI into this [Refer Slide Time: 13:30]. Thus the Rayleigh's when we equate the two equations the natural frequency square we get as this. So it is slightly different in this case we are having half EI into d 2 X_a dx whole square into dx. We get the expression for the natural frequency. So, this is for transverse vibration. Let us see what kind of result we can get from this.

We will take up an example, first, let us for the sake of comparison, we will take up that type of example for which we know the exact solution.

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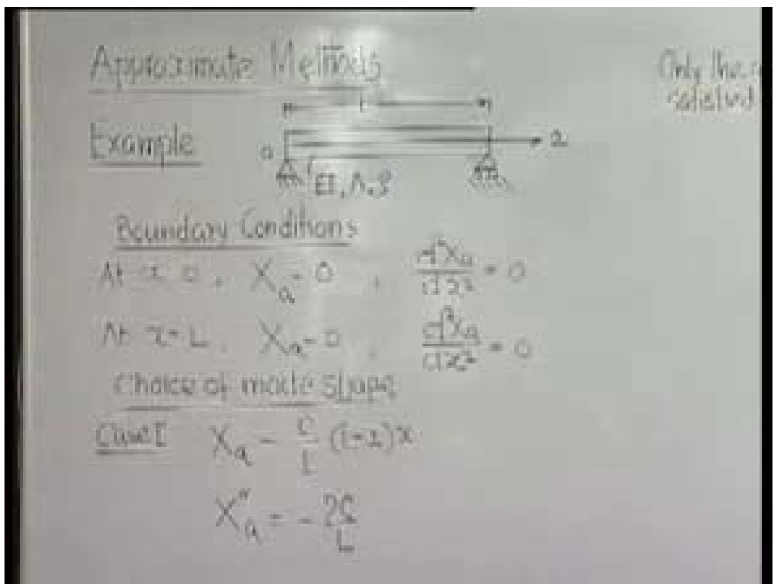

Let us take this example - a simply supported coordinate system is like this, the total length - L and modulus of elasticity is EI, cross-sectional area is A and density is rho. It happens to be so, that in this particular case of simply supported uniform beam EI, A and rho - they are all constant.

So, now what are the boundary conditions? Because the assumed mode shape X_a or the deformation pattern has to satisfy as many boundary conditions as possible and at least the geometric boundary condition. So boundary conditions are like at x is equal to 0. We know that at this end (Refer Slide Time: 17:07), what we know is deformation or displacement is 0; similarly, at x equal to L, again at the other end displacement is 0;

these are the boundary conditions of geometric [17:35 min]. Now the other boundary conditions which comes that here again there is no moment acting; that means, if moment is not here that means, the second derivative d square X_a which is proportional to the moment should be also 0 and since it is simply supported at both ends, so these are the four boundary conditions.

Now, we have to assume mode shape. Let us take a case I, X_a is simply this (Refer Slide Time: 18:49). If you assume like this, the first boundary condition that X is equal to 0, capital X_a is 0…. Similarly, at x is equal to L, X_a is 0, which is again true. So the geometric conditions are satisfied. Let us see how much is d 2 X_a dx square? That is equal to minus 2C by L. If we differentiate this twice with respect to X_a we get this, so which is not equal to 0; neither at x is equal to 0 nor at x is equal to L. So these two boundary conditions are not being satisfied. Now once we have assumed X_a and we have found out the expression for x two prime a only thing we have to do is to substitute X_a in this expression and get the value of omega square.

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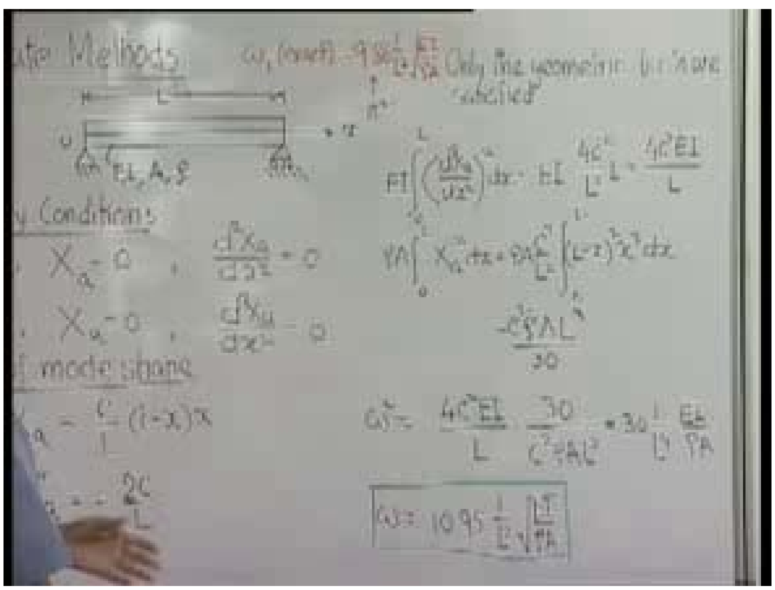

So, to do that what we find that EI is constant, but in reality if it is not constant and it is a function of x we can keep it inside that. This is equal to EI into… now d 2 X_a dx square is minus 2C by L; if we make it square, we will get this (Refer Slide Time: 21:15). So what remains inside is simply 0 to L dx which is nothing but simply L; that is, 4C square EI by L - that is the numerator; the denominator is - rho A is again constant, because it is a uniform beam what remains inside is this. If we do this integration we will get this (Refer Slide Time: 21:29). Equating the two or rather finding out the omega square like this, omega square will be 4C square EI by L or omega is approximately equal to this. So this gives the value we obtained from this assumed mode shape X_a which satisfies only the geometric boundary conditions.

We should remember our exact value of omega. What you found out from our mathematical analysis, it was phi to the power two phi square; so it is 9.86. This is actually phi square; that is the exact. You see that it is reasonably accurate - 9.86 and 8.95 - approximately 0.1 difference in 10, that means, if it is 0.1, it is just 1 percent error, in spite of the fact we made a very approximate assumption. If we know, for example, if we try to improve upon that suppose let us take up an expression for X_a, which satisfy all the four conditions. So let us see what happens. [26:10 min]

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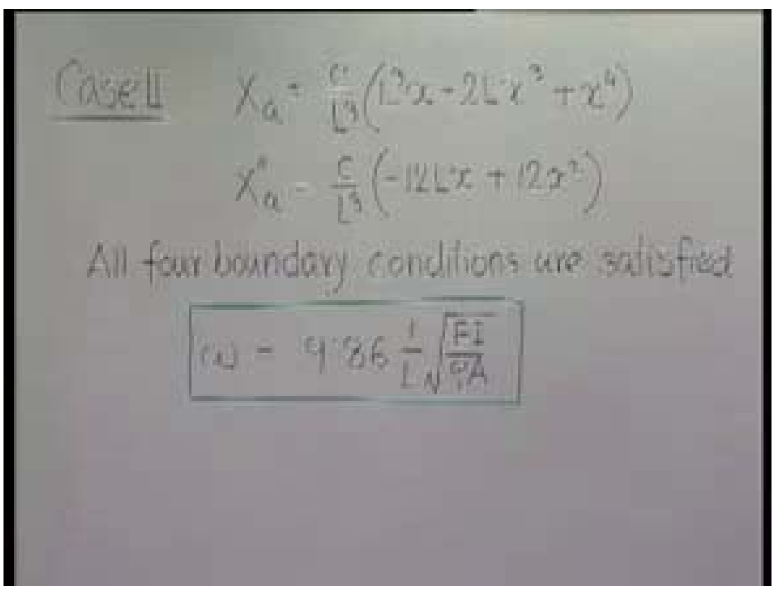

If we take another choice X_a, if we make this choice, then when X_a is that and d 2 X_a dx square is this. Now you see that if we put x equal to 0, X_a is 0 and if you put x is equal to 0, d 2 X_a dx square is also 0. So it satisfies both the conditions at x equal to 0. Next, if you put x is equal to L, this is L4, so 2L4 minus 2L into L cube. So minus 2L4; so X is 0 at x is equal to L; this is satisfied.

Next if you put x is equal to L here (Refer Slide Time: 27:57) this is minus 12L square, this is plus 12L square; so d 2 X_a dx square is also 0 at x is equal to L; thus all four boundary conditions are satisfied. Now, still it is not a very complicated expression, but obviously integration etc., will be a little bit involved and let us see what we get. If we substitute X_a and [X the] whole prime a in the expression and find out this numerator and denominator we get this (Refer Slide Time: 28:55).

Now let us compare the result with the exact value what we received from analytical method done in the previous module. If you compare this with the exact which is this (Refer Slide Time: 29:27) you find after second decimal place, there is no decimal. The difference will be found only in the third decimal place and that to again having the… So when you satisfy all the boundary conditions, the result what we normally receive is very accurate except for some special situation. One thing of course you should remember we have not calculated up to this third decimal place. If we do it, we will find still this will be slightly higher than the first natural mode we find by exact method. Therefore, what we get from this is that whatever value we get by the method of Rayleigh's approach - we have shown it earlier also - it is always slightly higher than the exact value. So, as a result what is happening, this Rayleigh's method will give us a limit, whatever value we get the actual value of the first natural frequency will have to be below this.

By Dunkerley's method if we get the value of $omega_1$ like this, then we know the actual value of $omega_1$ will have to be above this. Thus we get a band and the exact $omega_1$ must lie in this. Therefore, if we can make by some method this band as narrow as possible, then $omega_1$ can be estimated in a very accurate way. Now, the question is that in all these cases we could solve the problem exactly, so what we have gained by this? We only got an approximate answer; that will be apparent if we take another example.

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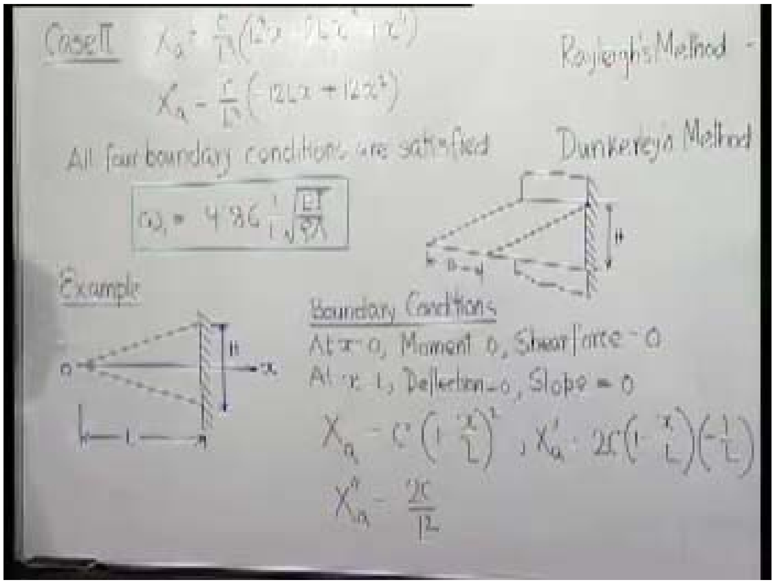

This example is that let us take a tapered cantilever beam, the length of the beam is x, length of beam is L as before and the depth, that means, in reality this beam is a cantilever beam with constant width. So the width is constant and the thickness is uniformly varying and at the other end it becomes H and at this end it is 0. So, that means the depth is increasing and it becomes H; the width remains constant. So the material property remains constant, which is rho, that is the density and E that is the modulus of elasticity.

So, with this, one thing we have to show - what are the boundary conditions? Now this is an example to indicate that in reality boundary conditions are not on X_a or these things; boundary conditions are deflection, slope, shear force and moment. These are the real conditions, but they get translated into conditions on X_a. So here we will find sometimes it may be important to note the condition on the actual condition; that means, at x equal to 0 moment is 0, bending moment is 0, because it is a free end, shear force is 0. At x is equal to L that is the fixed end, deflection is 0 because it is a fixed end and slope is 0. So, we now we assume a mode shape X_a in such a way that all the boundary conditions are satisfied.

We will do it here like this - if X_a is this, then X_a prime that is dX_a dx will be equal to this (Refer Slide Time: 36:03). d 2 X_a dx square is this. Now let us see the conditions which are being satisfied. At x is equal to 0, is the moment 0? Now, if you try to see whether this is 0 or not, that is not the condition here. Here the cross-sectional area is 0; since cross-sectional is 0 because it is coming through a point or thin edge, so whatever maybe the value of this, moment applied will be not thus this, but we have to the take care of EI. So let EI is 0 because I is 0 at this end. Therefore, bending moment at this end is 0, though the second derivative is not 0; this is a very important point to observe.

Secondly, shear force will be then whatever; you will find shear force will be given by this third derivative, which is 0 here; of course, even otherwise if it had been 0 because the area is 0. So, both the boundary conditions at this end (Refer Slide Time: 37:37) are being satisfied. Let us go to x is equal to L, x is equal to L, what happens? Deflection is 0; so we find from this expression (Refer Slide Time: 37:50) that deflection is 0. Similarly, the slope dX_a dx is also 0 at x is equal to L. Therefore, all four boundary conditions are being satisfied. So this is a very special case, so keep in mind that we need not all the time be carried away by the expression of this and trying to see…; it also depends on the cross-sectional area and this is a case where we find that without satisfying all the conditions on X_a because I being and A being 0 here….[38:29 min] So four boundary conditions are being satisfied. Now let us apply this to the expression for… [38:33 min].

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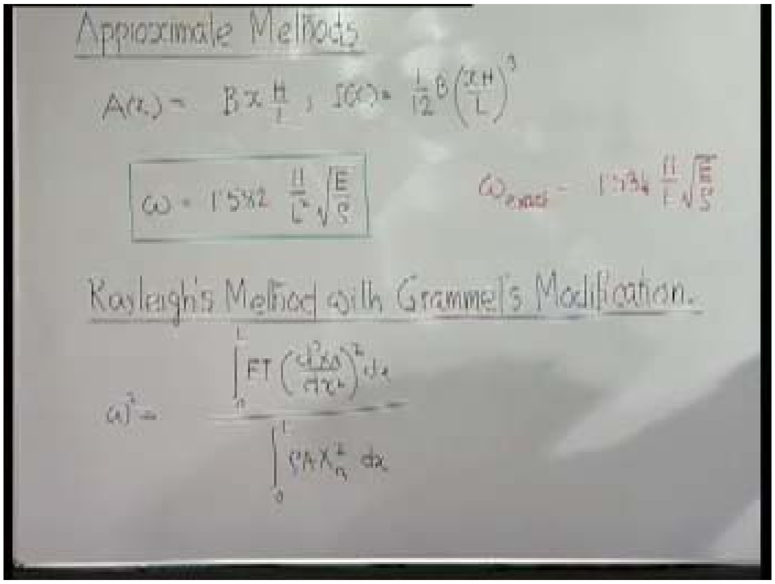

Before we do that, the two quantities A is a function of x, which is B that is the width into the depth; depth will be x into H by L. That means, if we go here, at x, this will be linearly at x is equal to L depth is H, so at x is equal to x this will be H by L into x and if the width is double B, then we just multiply by B; so this is the cross-sectional area.

Similarly, the second moment of area, that is I, will be xH by L cube; B is a constant. So if we substitute this in the expression for omega we will get omega equal to this (Refer Slide Time: 40:44). So this is the answer we obtained with that function as the assumed mode. Now this is a particular case, where it is possible though it is a non-uniform cross-section being linear, you can solve the problem exactly by analytical method and exact value of omega is found out to be this (Refer Slide Time: 41:36). Thus, we find that the result we get by a reasonably simple mode shape assumption is quite accurate and the error is very low, but all the time we notice that the frequency, which we get by assuming a mode shape which is not the exact one, will be always higher than the exact value.

I think more problems can be solved, but that is predominantly an algebraic manipulation most of the time and solving integrals. So we will not go further in that direction, rather what we will try to see that if there are methods by which you can improve the accuracy - that should be our concern. The one very important or rather I will say very convenient way of improving accuracy of the method following the Rayleigh's approach is if we use Grammel's modification.

Now, we have seen that omega square we are expressing, which is the Rayleigh's quotient as 0 to L (Refer Slide Time: 44:13); here with little bit of thinking it can be found out that the error is getting introduced, because of this double differentiation - one is of course that X_a itself is not the exact mode shape, which is valid; it is slightly different from that, that is a source of error plus whatever may be that this double differentiation is a source of error in the whole process. So, what Grammel did, he tried to formulate the same thing that means equating the maximum kinetic energy or maximum potential energy during a normal mode oscillation, but formulate it in such a way that this double derivative is not necessary. So, what was done is that it considered that the system or the beam or flexural member, is subjected to a peak inertial loading, that means, during every cycle there is one period when it goes to the extreme, it is subjected to inertia forces.

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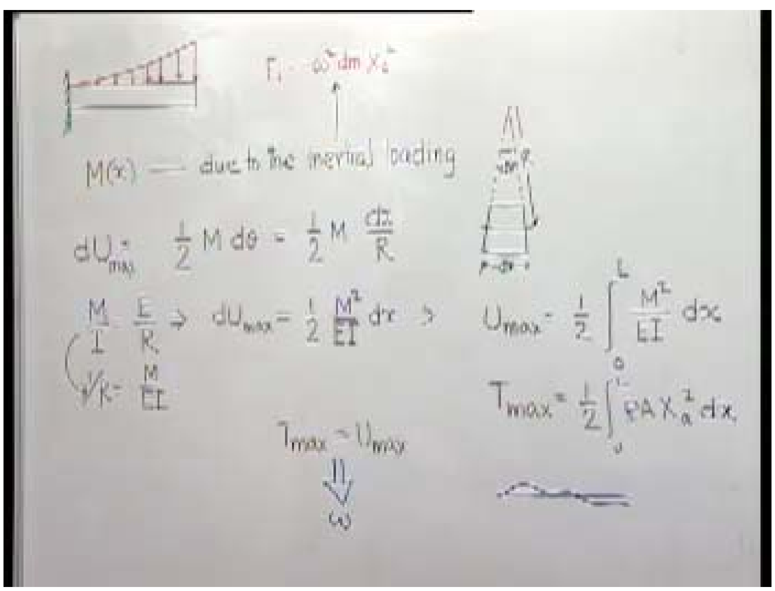

Say for example, if we take any kind of beam, but we will take an example like the cantilever. So what will the inertia force acting on any particular element be? It will be the mass of the element into acceleration of the element at that peak point, that means.

peak acceleration minus. So, it can be either in that direction or in this direction; so it will be something like this - inertial loading. Inertial loading can be written as omega square and the elements here will be the d mass dm into this (Refer Slide Time: 46:36). So it will be a distributed one; this is the inertial loading. What we will do, we will consider that this is the loading on the whole thing. Now, if this is the loading on the flexural member, we can find out the bending moment due to this as a function of x due to the inertial loading; it is a distributed load and let M be the bending moment due to that.

Now, we have already seen that dU of an element is half M into d theta, which was equal to half M into dx by R, where R is the radius of curvature, at that bend condition, at one extreme end. Now, from [beam] theory or rather simple expression that M by I is equal to E by R. If we keep this element as a beam, the length of the beam is dx, radius of curvature in bending is R; then we have seen just now, that R is or rather we can replace R as dx by d theta or simply we can write this as R d theta is dx equal to…. We will come to that later. So, using this now, we get for the elements maximum strain energy stored is half. Here R is equal to M by EI. So now let us use this here; it will be half into M square by EI dx. So now from this we will get maximum potential energy whole will be this (Refer Slide Time: 51:10) and T_{max} of course remains as it is.

So the procedure is now that if this will be the inertial loading, which is in terms of omega and X_a square, let us find out the bending moment in terms of X_a square omega, no differentiation is involved and then substitute it here and integrate it. Then when we equate U_{max} and T_{max}, this will yield the value of omega; an equation, which will be better in accuracy. The reason is this: that we are avoiding this double differentiation of X_a which has been assumed. So, in any kind of assumption of a car, the slope, the order of error in that will be more than the order of error in the actual function. That is why we are now getting a better result, as we are doing a whole calculation without differentiating X_a.

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