Dynamics of Machines

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Module 13 Lecture 3

Vibration of Beams: Approximate Methods

We have solved the problem of transverse oscillation of uniform beam. We derived the equation of motion valid for a general element, which is this, where u represent the lateral displacement, which is a function of both its location, that means x and also the instant of time under consideration.

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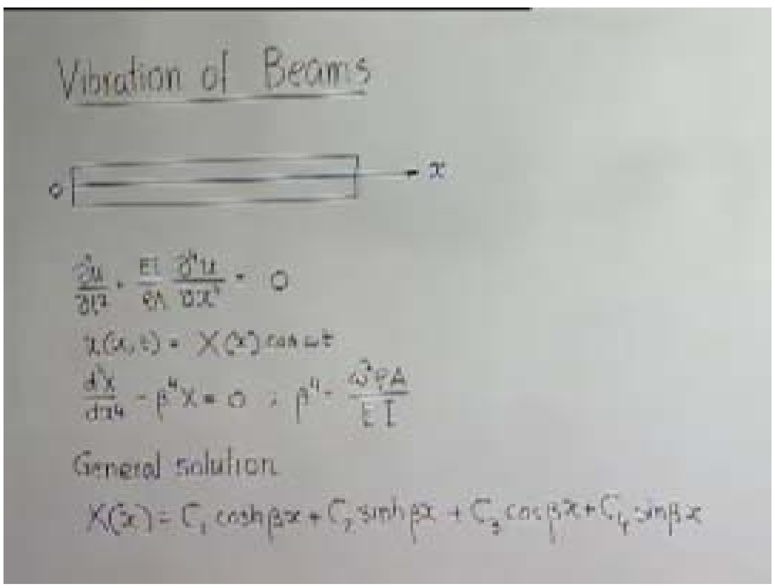

However, for a normal mode oscillation, we know that the solution will have to be in the form of a product of the function of x representing the mode shape and a harmonic function of time, sin omega t. When we substitute this in this equation, we get an ordinary differential equation in the x of this form where beta to the power 4 is this. We know that the general solution of x will be of this form; that means, four constants consecutively C_1, C_2, C_3, C_4. First two terms are hyperbolic solutions and the second two terms are harmonic solutions, we have to look at for this. Now, to make further progress, we have to take up individual cases and as we have found in the case of bars and shaft, the value of beta we can find out from the boundary conditions.

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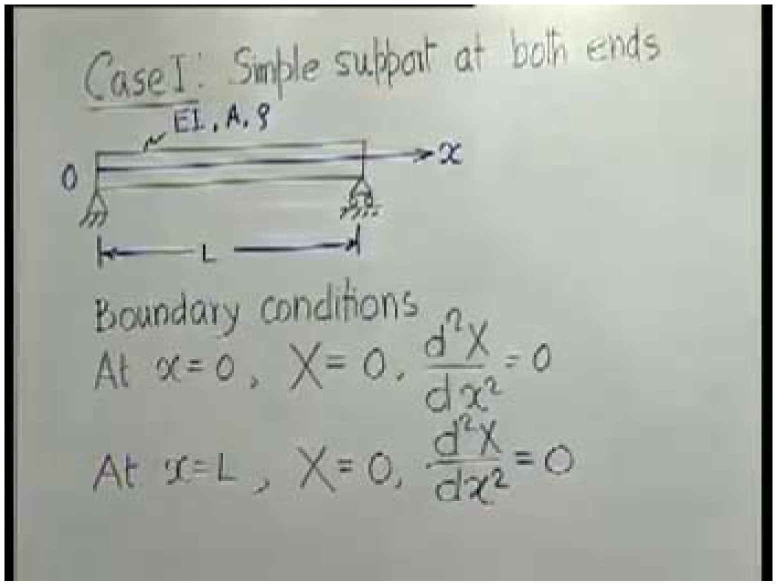

So, therefore, our next step will be to take up individual cases. First, we will take the case of simply supported beams at both ends. So, a beam with simple support at both ends, length of the beam is L and the flexural rigidity which is the product of the two terms E and I that appears here is called flexural rigidity EI, the cross sectional area is A and density of the material is rho. Now, we have to find out the natural frequencies and mode shape if possible. So, the boundary conditions here, we notice that four constants A, B, C, D or C_1, C_2, C_3, C_4 whatever, requires four conditions. So, on each end, we will have two conditions. So here, we find that if we take a coordinate system like this origin, when the conditions at x is equal to 0, that means this end now it is simple supported at both ends. What is the meaning of that? Meaning is that, at the simple support your displacement is 0 all the time; that means x will have to be 0 all the time.

Therefore, displacement is 0 and bending moment will be 0, because simple support cannot provide any bending moment. Now, if we remember during the derivation of this or bending moment was proportional to del 2 u by del x square, V was proportional del M by del x and rho proportional to del 3 u by del x cube. Therefore, now del 2 u by del x square, when you were doing, so, we can write this as x double dash cos omega t and this will be x triple dash cos omega t. If M has to be 0 at all instances, this term will have to be 0. This means the d square x by dx square is the double derivative of x with respect to x and similarly, at x is equal to L the same two conditions must be at (Refer Slide Time: 6:16 min). So, these are the four conditions expected to be. Now, x is equal to C_1 cosine hyperbolic beta x plus C_2 sin hyperbolic beta x plus C_3 cosine beta x plus C_4 sine beta x.

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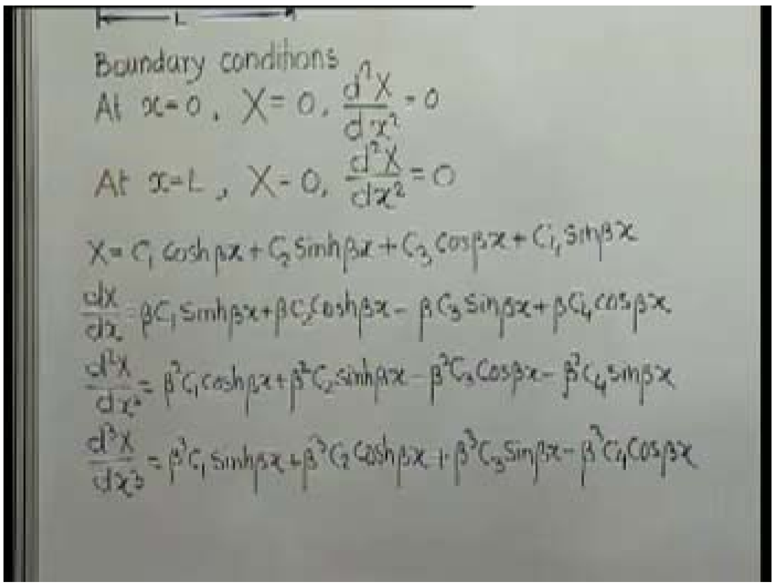

If you differentiate once with respect to x this will be C_1 beta sin hyperbolic beta x plus C_2 beta cosine hyperbolic beta x minus C_3 beta sin hyperbolic beta x plus C_4 beta cosine hyperbolic beta x. differentiating once more (Refer slide Time: 07:37 to 09:27 min) we may require for other conditions with third derivative also so let us wait here, so we will not require this for the present problem. Now, if we apply the first two conditions: first condition at x is equal to 0; capital X is 0. Now at x is equal to 0 this term vanishes; this term vanishes; this becomes 1; this becomes 1. So, C_1 plus C_3 equal to 0. Similarly, we find that at x is equal to 0 d 2 square x dx square that is this term is also 0. If you put 0 then obviously, this term goes, this term also goes, this becomes 1 and this becomes 1.

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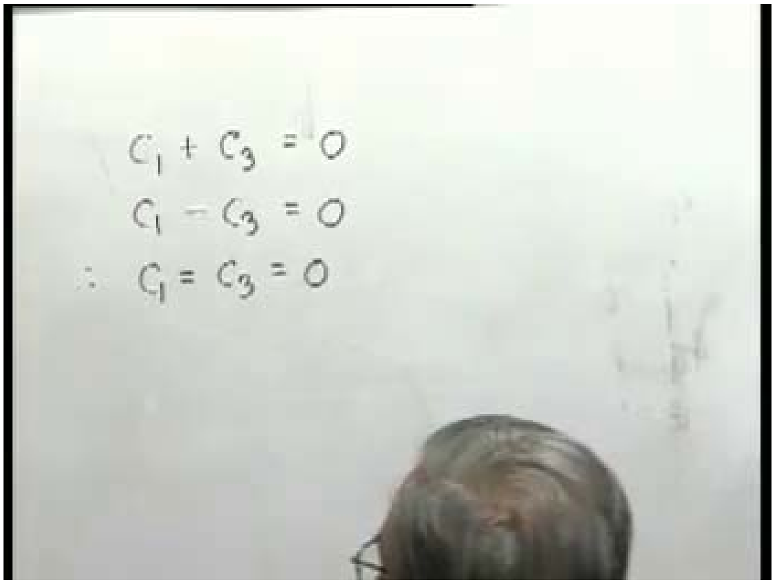

So, the third condition, that means, this moment condition is C_1 minus C_3 equal to 0 as beta is not equal to 0. So, when C_1 plus C_3 is 0, C_1 minus C_3 is 0. This obviously is that, C_1 equal to C_3.

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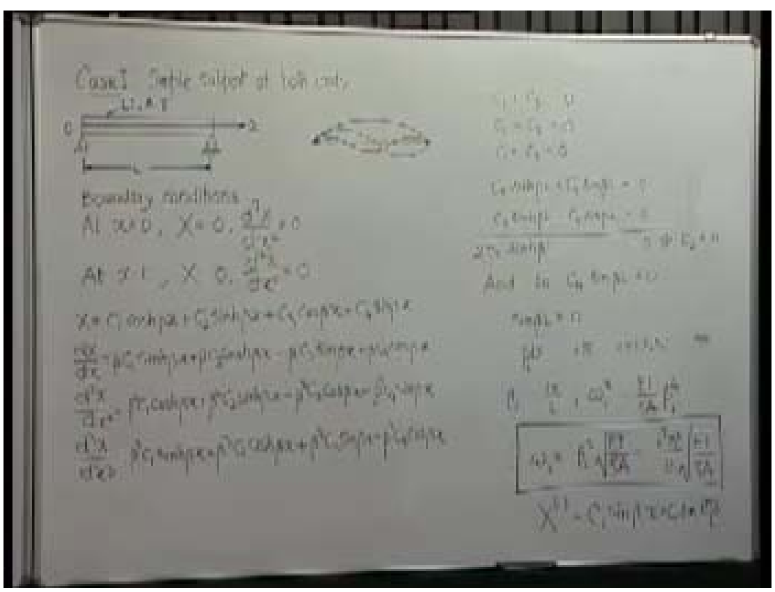

So, now, let us put the condition at x is equal to L. At x is equal to L also x is equal to 0 capital X; so, remember that C_1 and C_3 terms are gone. So only these two terms remain.

So, at x is equal to L, C_2 sin hyperbolic beta L plus C_4 sin beta L equal to and again at x is equal to L, moment is 0; but anyways C_1 and C_3 are already gone. So, what remains is beta square if you take common, C_2 minus $C_{4.}$ Again we find that for a nontrivial solution. Now, C_2 and C_4 they cannot be all zeroes. Since, there is no vibration at all. So, for a non-trivial solution, we find that if we pick up this, what we will get? We add these two. We get two C_2 sine hyperbolic beta L equal to 0.

Now, we know that sine hyperbolic beta L cannot be 0, so it is an explanatory function; so therefore, the only possible thing is that C_2 equal to 0, if C_2 equal to 0 and this has to be 0 and so C_4 sine beta L equal to 0. Therefore, we find the situation is like this C_1 equal to 0, C_3 equal to 0 C_2 equal to 0, but C_4 cannot be equal to 0.

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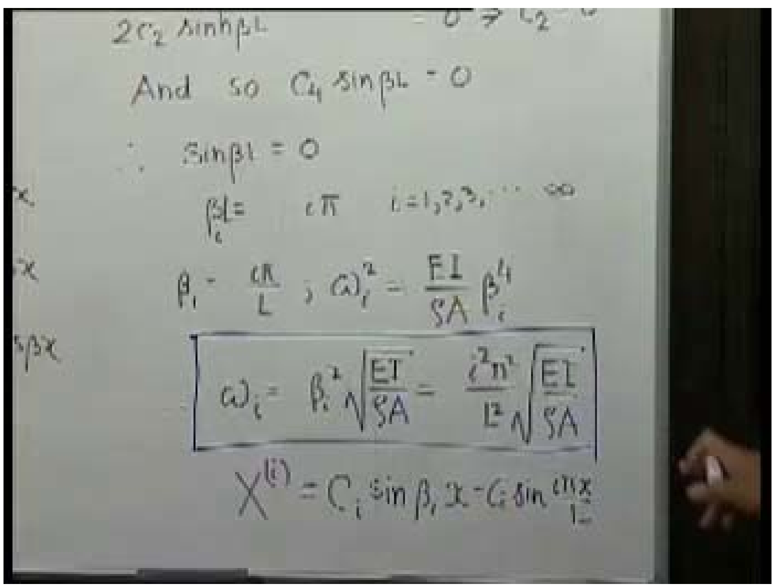

Therefore, the only possible solution will be sine beta L or beta L will be equal to some integer multiple of phi, i phi. So, a $beta_{i;}$ so, $beta_i$ is equal to i phi by L and this we know from here, is the fourth root of this. So, from this, if you know $beta_i$ here we get now we know that $omega_i$ square is equal to (Refer Slide Time: 13:32 to 14:08 min). From that equation will be EI by rho A into beta to the power 4. So, $beta_i$ to the power 4.So, $omega_i$ will be $beta_i$ square into square root of $beta_i$ square is nothing but i square L square. So, this is the expression which gives us the natural frequency of a simply supported beam on each lateral. So, when i is equal to 1 that is the fundamental frequency that is equal to phi square, which will be something 0.9. Something second frequency will be obviously 2 into phi square, 2 squares into phi square that means four times that and so on.

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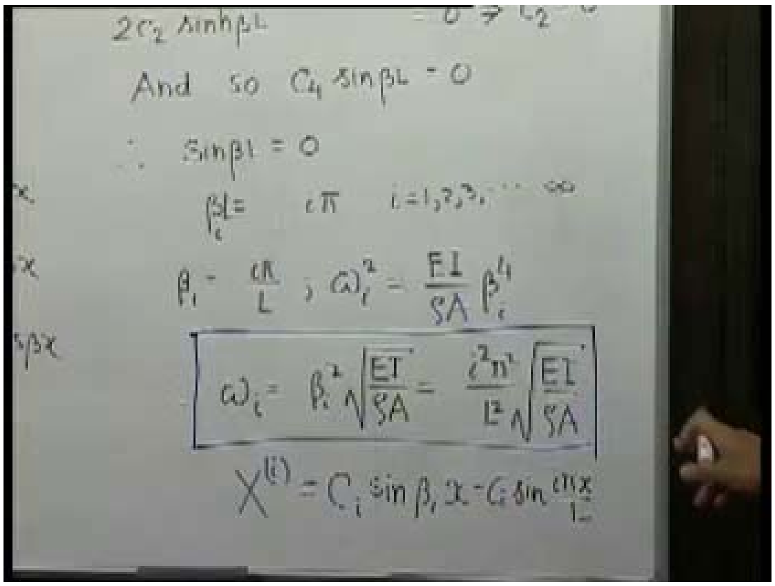

Also, we can find out here, in this case, it is possible that x is equal to nothing but since C_1equal to C_2 equal to C_3 equal to 0. So, X in ith mode is nothing but C_i fourth A sin $beta_i$ into x or since $beta_i$ equal to this, C sin i phi x by L.

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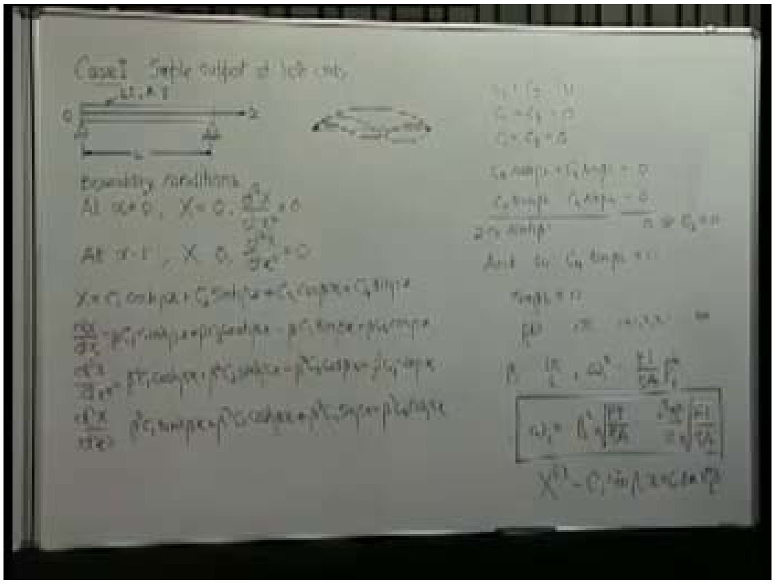

Therefore, it is possible in this particular case, to find out the mode shape in this closed form and we can see it is nothing but sine functions and the various modes we can plot. First mode is obviously this second mode will be third mode will be and so on. Therefore, we find that for a beam simply supported at both ends, the natural frequencies given by this are interesting to note that mode shape, same as the kind of mode shape we had in case of longitudinal vibration of bars, you remember that both ends.

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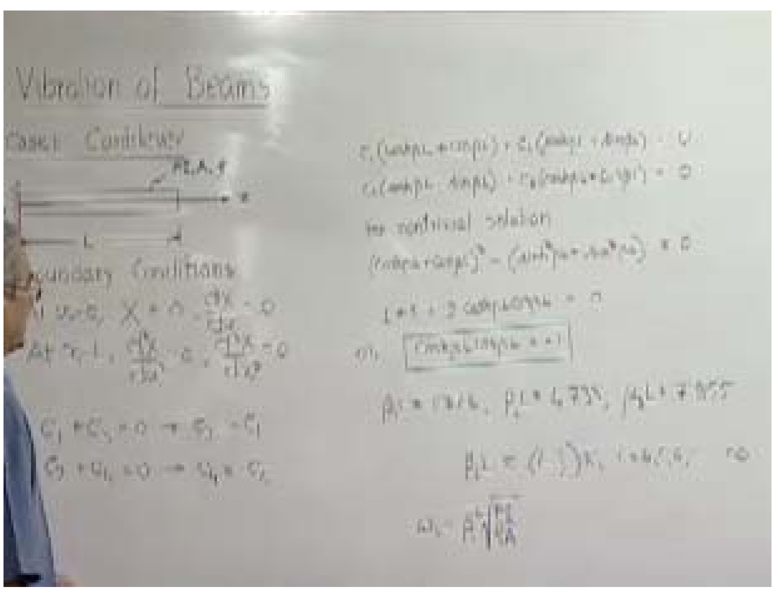

So, if you want to consider a few other cases without going through this detailed solution, Let us consider the second case, which is another very commonly found equation that is called Cantilever; that means one end fixed and the other end free.

The length of the L it is flexural rigidity is EI, then the cross sectional area is A and density of the material of the beam is rho. So, now, in this particular case, the boundary condition at x equal to 0 displacement is 0 at all times. So, x is equal to 0. Another thing, what happens here that whatever way it may go, its slope will have to be 0. (Refer Slide Time: 18:39 to 21:46 min) Since, it is rigidly connected here, so dX. dx; slope of the beam will have to be 0 at all times; that means at dX, dx is equal to 0. At x is equal to L at the free end, that free end means for there is neither any force acting on it nor any moment acting on it. So, since there is no moment acting on it, d square X dx square is equal to 0, as you have seen in previous case. Since there is no shear force acting, shear force is proportional to third derivative if you remember. So, these are the four boundary conditions in the case of the Cantilever. When we use that so like say these two to be applied. Obviously, at x is equal to 0; this term will vanish; these will be 1. So, C_1 plus C_3 is equal to 0.

Next, we go to the second condition that is here, so now this term and this term goes. So, it is C_2 beta C_2 plus beta C_4. So, C_2 plus C_4 equal to 0. So, this will give C_3 is equal to minus C_1 and this will tell us C_4 is equal to minus C_2, then we can use these in the two other equations and put x is equal to L and equate them to 0. So, we will get these two equations.

Now of course, we know that for a non-trivial solution for C_1 and C_2, the determinant of the coefficients must be 0, or determinant of the coefficients of $C_1$and C_2 will be 0, which is this into this minus this into this; (Refer Slide Time: 22:18 to 23:12 min) that becoming must be equal to 0. So, if you expand cosine hyperbolic square beta L minus sine hyperbolic square beta L is equal to 1, then cosine square beta L plus sine square beta L will be another 1. What will be left with 2 or so this coincidental equation its solution will be values of beta L or so this has to be solved numerically all looking into standard table. I will give you the first three values: $beta_1$ L is equal to 1.876; $beta_2$ L equal to 4.733;

beta$_3$ L equal to 7. 855 and when you go above beta$_3$, then there is relationship beta$_i$ L is approximately equal to i minus 1 by 2 phi, for i equal to 4, 5, 6. So, once we know the beta omega is always like this. So, here, if you find, the first route we are going to get will be approximately 3. Something divided by L and L square and this. So, that is the natural frequency of a Cantilever beam of length L in second mode and so on.

Only thing here, you find that it is not possible to express the natural mode of oscillation in a close form as it was possible in that case, because we do not have this kind of only numerical solutions. Therefore, we do not have any expression for beta M. That is why, it is not possible to express the natural mode in close form, as it was possible in the case of simply support.

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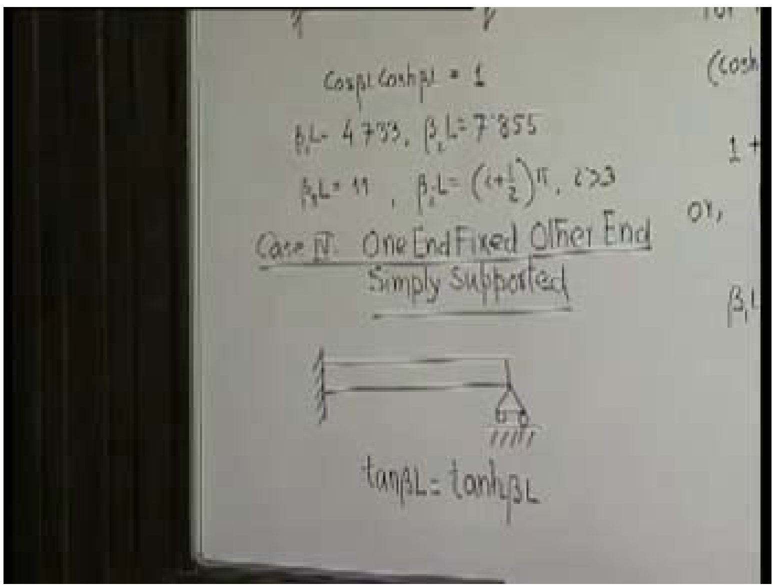

Same way, if we take a case where we have a case three; for both ends fixed, our equation will be, here it was minus 1, now we have plus 1 and beta$_1$ L is equal to 4.733, beta$_2$ L is 7.855; beta$_3$ L was 11; and beta$_i$ L equal to i plus 1 by 2 into phi greater than 3. So, from this, we can find out the sequences. We can take another case; one end fixed other end simply supported; this is the situation. Here, our equation is tan beta L equal to tan hyperbolic beta L.

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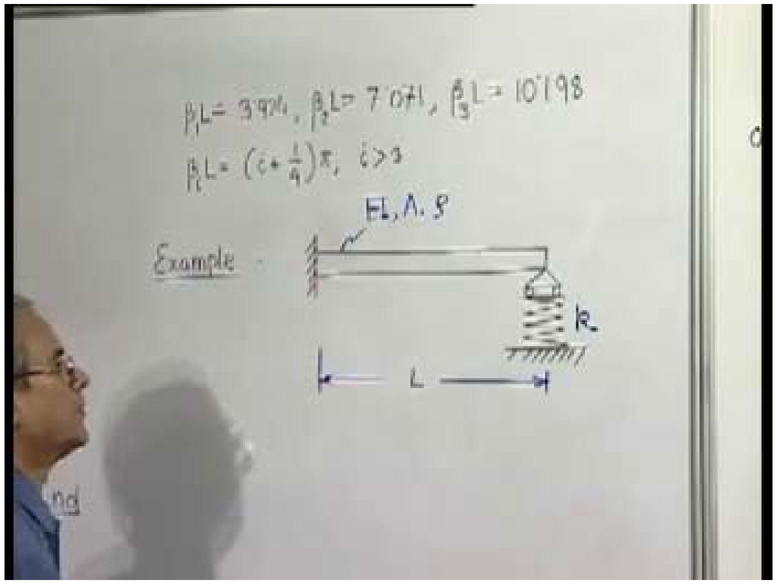

For this, the solutions are beta$_1$ L equal to (Refer Slide Time: 29:12). So we can find outlet to solve a particular case example. So, the example we take is one end is fixed and the other end is elastically supported. So, this thickness is k of the spring; modulus degree is the rigidity modulus EI; on this beam cross sectional area is A, and density of the material is rho and length of the beam is L. So, what will be the natural frequency? So here, the boundary condition will be what? Because we have to always start from the boundary condition may be general solution within that form. Now here, we find that at x is equal to 0.

Obviously, we have taken the same kind of coordinate system as we have at x is equal to 0, the deflection is 0 and slope also is 0. So, at x equal to 0; X is 0 and dX by dx at x is equal to L. This is interesting and what we find? That bending moment is 0 and bending moment is proportional to the second derivative we have seen. The force is not 0, but suppose if it is deflected by an amount x, then, what will be the force? It will be k into x.

So, our equation for shear force was EI into d cube x by dx cube that was the shear force. Therefore, d cube x by dx cube will be shear force by EI. Now, shear force or the force here will be whatever as the deflection stiffness into that deflection will be the force and that divided by EI will be the third derivative. So, these are the boundary conditions.

When you solve, first of all these two, if we substitute one thing at x is equal to 0 these two terms vanish. So, C_1 plus C_3 equal to 0.

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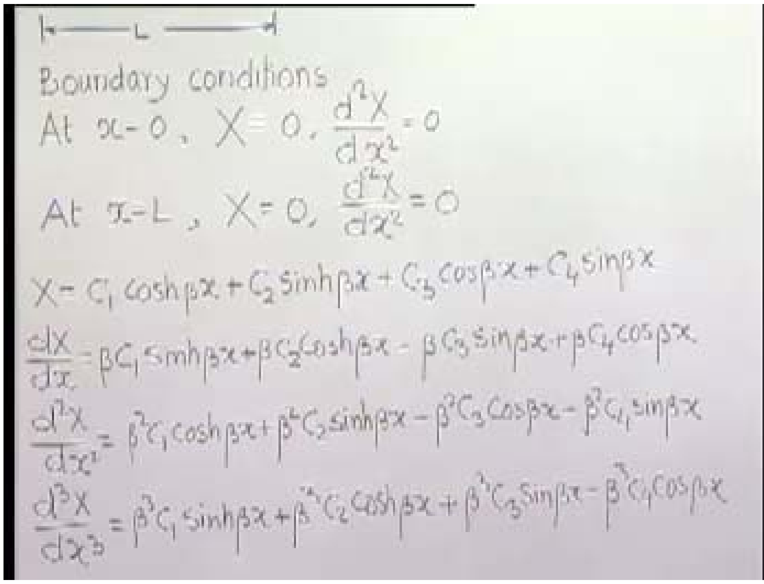

Then again, here at x is equal to 0, these two terms vanish. Beta into C_2 plus beta into C_4 is 0; beta goes, so therefore C_2 plus C_4 is also equal to 0. Therefore, just like previous case, we can express C_3 as minus C_1, C_4 as minus C_2. Then, d square x, dx square and d cube x by dx cube can be then expressed in terms of only C_1 and C. If you do that, we will get somewhat complicated equation.

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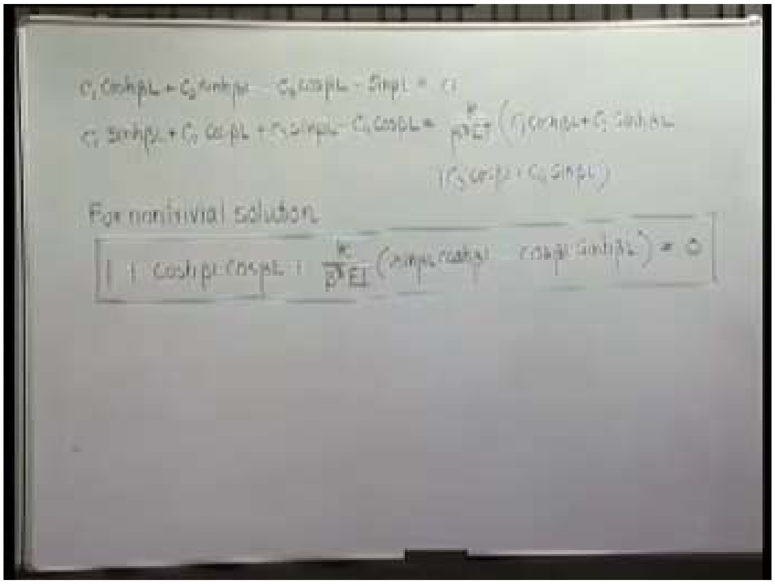

So, if you now use the third condition that means here, at x is equal to L; this is equal to 0. Therefore, that gives us C_1 cosine hyperbolic beta L plus beta L minus beta square term gets. So, till this third condition, there is no difficulty. We can always use this C_3 and C_4. The last condition is that, when we use this part at x is equal to L, what we get? (Refer Slide Time: 35:41 to 37:52 min)We get $C_{1;}$ this is equal to the value of x at small x is equal to L into k divide by EI and that beta cube which we take from common. So, it will be k by beta cube EI and the value of X at x is equal to L. So, it will be C_1. So, we can replace C_2 by minus C_1 or C_3 by minus C_1 and C_4 by minus C_2. So, we will get two equations; two unknowns C_1 and C_2 and obviously, we will finally get for non-trivial solution. For non-trivial solution, it exists if condition has to be satisfied. So, for this problem, this is our equation or frequency equation. Whatever we may say, this has to be solved numerically; obviously, when all the values are given and values of betas can be found out infinite number of betas like beta$_1$ beta $_2$ and so on. So, in this case, we can see, it is no simple matter. It is only possible to solve numerically.

We can check some limiting cases, say for example, if we find that this spring stiffness, for example is 0. If the spring stiffness is 0, it means that this end is free; that means, it is nothing but a Cantilever beam. So, this must go to Cantilever beam. You can see, when k is 0, this whole term goes and cosine beta L into cosine hyperbolic beta L must be equal to minus 1. That is the condition for Cantilever beam; we have already seen.

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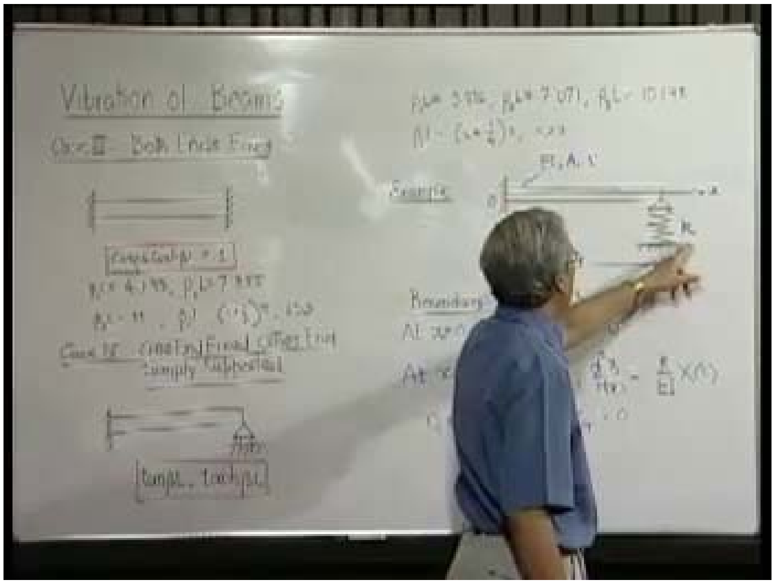

So, on the other hand, if k is infinite, it means that this end is simply supported. So, it should match the case, where one end is fixed the other end is simply supported. If k is extremely large, then obviously, this part is the larger part and this can be ignored. (Refer Slide Time: 39:38 to 39:48 min) So, when k is infinitely large, this part goes out. So, this part will have to be 0. Now, this can be 0 only if this condition is satisfied.

So, sine beta L cosine hyperbolic beta L must be equal to cosine beta L sine hyperbolic beta L or tan beta L equal to tan hyperbolic beta L. You can see is the case, where one end is fixed other end is simply supported. So, we can see the limiting cases have been satisfied, but otherwise solution will be possible only when k is given and all other numerical quantities here.

Therefore, this tells us, that the method of solving a problem which can reach our subject to transfer vibration. So, we have seen that the solution in closed form in the analytical way is possible only in case of very simple situation, but unfortunately, in our real life engineering, there will be many cases, where the members or the elements do not satisfy the implicit equation. They do not satisfy the conditions of and also therefore, in such cases, the solution we can get either by numerical methods and nowadays, with computers, you can employ finite element techniques. Otherwise, when the designer of the machine is interested only in the fundamental frequency and also only an approximate estimate within plus minus two percent of error, then the problem can be solved in an approximate method.

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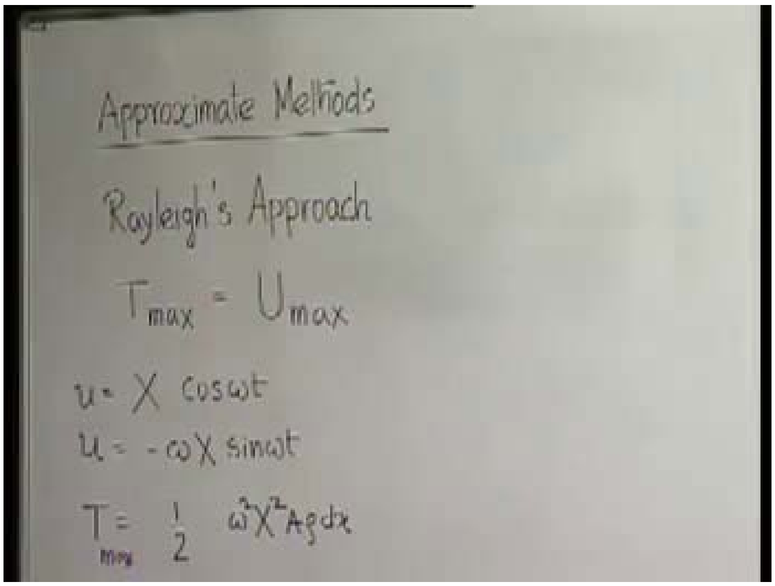

Therefore, now, we will take up those kinds of techniques which will give us approximate values of the natural frequency. In cases of more complicated geometric situation like the beam is not uniform. We have already got some introduction to the approximate methods, when we were crossing lump parameter systems. The similar techniques will be developed on some similar continuous bodies also. The approximate method what we have discussed in the case of lump parameter system can be extended continuous bodies also, because continuous bodies can be considered to be lump parameter systems with infinite degrees of freedom. So, one of the most prominent methods which we have discussed then, we will apply here. I am slightly modifying that method with suitable for continuous bodies and there we come to our Rayleigh.

We have seen that in Rayleigh's approach for solving free vibration problems without any damping, we took the condition that means we considered the condition that in normal mode oscillation, all the particles reach their maximum velocity situation or 0 velocity situations simultaneously; that means, the total energy is in the form of a potential energy when all the particles are 0 velocity. Similarly, when all the particles simultaneously attend the maximum speed then the whole energy of the system, mechanical energy of the system is in the form of kinetic energy. Since there is no dissipation or no change, so the maximum kinetic energy, which is equal to the total energy must be equal to the maximum potential energy, which is also equal to the total energy.

Therefore, our approach had been kinetic energy of the system maximum value is equal to potential energy of the system maximum and to find out this kinetic energy and potential energy, we assume the shape of deformation or deflection. It has been shown earlier that, even if this assumed form is not matching the exact one or even if it is as large error from large deviation the answer for the natural frequency will be reasonable. That means, the natural frequency which you find out from this is reasonably insensitive to the errors in the assumption of the deformation shape.

So, if the elastic body, if mode shape is say X then what happens? (Refer Slide Time: 45:26 to 46:17 min) Since, it is the deformation so the velocity of a particle is the kinetic energy is half. If the element is of length dx then, its mass will be A into rho into d x and maximum velocity will be omega x. Therefore, square of this has to be summed up or all the elements; that means we integrate or the whole bar or beam or shaft whatever it may be, of course since it is linear deformation we are taking, you should take either beam or bar.

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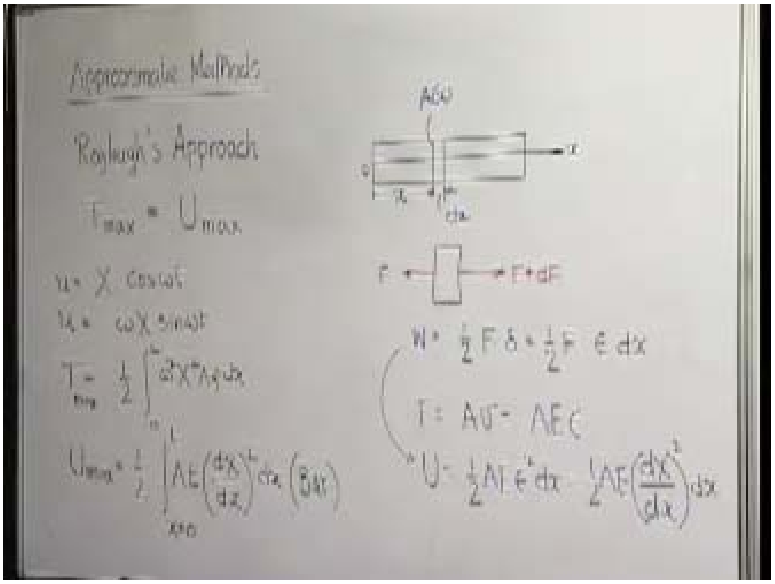

Similarly, now say T_{max} is the expression for U_{max}, say for case of a bar. In the case of a bar, which is having longitudinal oscillation then at a distance x. If you consider an element of length dx and if the cross sectional area A, which need not be constant which may be a function of x but at this distance. Therefore, at this location the elements is this is subjected to forces, which is in this direction and in this direction, (Refer Slide Time: 47:55 min) it is F plus dF.

So, you know that if we take an elastic body and stretch it by applying a force, so what will be the work done? It will be 1 by 2 into force into force into its elongation. How much is the elongation, it is nothing, but it is into its length of the element is dx. So, if the strain here and the length of the element be dx and this is the amount of stretch. So, that into F. so again we know that F is equal to A into sigma, where sigma is the stress into area. (Refer Slide Time: 49:12 to 50: 38)

So, this is A into stress is equal to we know that stress by strain is equal to E, so stress is equal to E into strain. So, work done or the energy whatever you may say, this work done is nothing, but the energy stored in the object in the form of strain energy. It is nothing but 1 by 2 into AE into square dx. Now, the strain we have already derived is nothing but maximum strain is this, so for the element, the energy stored is in that form, so for the whole beam or bar rather. So, this is for bar. longitudinal oscillation not for beam.

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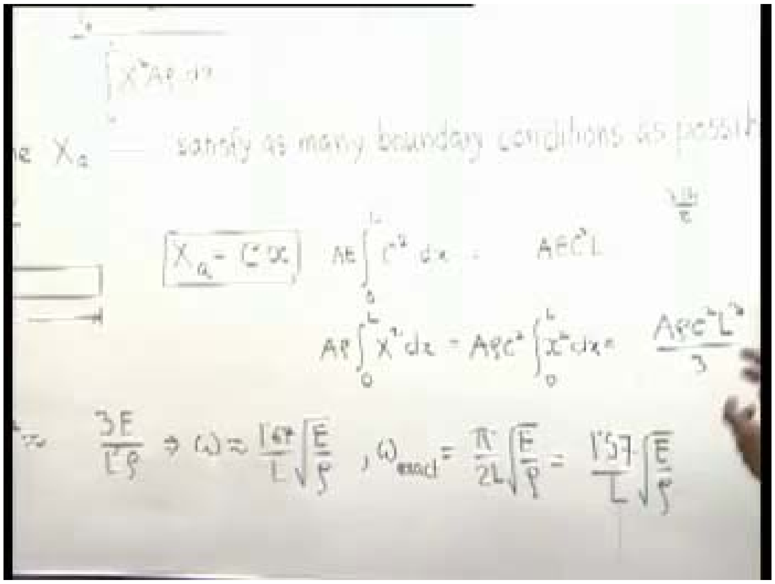

So, now, what we find that omega being same for all elements, omega square will be approximately written as, from that T_{max} is equal to U_{max} condition. (Refer Slide Time: 51:00) You can write that omega square equal to U_{max} by A, E, rho, everything they can be all functions of x. So, this is valid exactly, but in reality what happens approximate methods when you are talking about. If x is not known, we assume from mode shape. If we assume mode shape then and substituted there we get reasonably accurate values of omega. Only thing, it is better to satisfy as many conditions on x, as actually it has to satisfy on x.

Therefore, this satisfies as many boundary conditions as possible. If you can satisfy all the boundary conditions, you will get best possible result. So for example, if you take a simplest possible case, to see the duty of the method, let us take case of a bar with one end fixed, the other end free, whose solution we already know. Let us consider X a very simple form is say, some constant into x. so only thing what we satisfy. There are two conditions that, this is the fixed end, that means, there is no displacement at x equal to 0 and this is free, that is, strain is 0; that means, dx is 0. Now, here we find that at x is equal to 0, X is 0 that is condition.

But the other end, we are not satisfying the condition dx_a by dx is constant C and it is not equal to 0 at x is equal to L. So, we are satisfying only one of the two boundary conditions. But if you try to figure out the natural frequency, let us find out the fundamental frequency. (Refer Slide Time: 54:08 to 55:18 min)So, first the numerator, which is a uniform case, so A into E will go outside and dx a dx will be simply C Square into dx. So, that numerator becomes A E Csquare L. The denominator again, A and rho both are uniform, as a special case. So, this is equal to a rho C square. So, it is A rho C square L cube by 3. So, using this we will get omega square approximately equal to this by this.

So, what we get 3E by L square rho or omega is approximately equal to 1.67 something I do not know L square root of E by rho. If you look into the exact value what we found, we have solved it already is equal to pi by 2L square root of E by rho, which is equal to 3.14 it should be 1.5.

So, you see it is reasonably close. The difference is only 0.1 may be just 10 percent not with such a crude approximation; only one boundary condition, if you make it slightly better so that is 0, this strain is 0 and the value is 0, which is not very difficult to get. Then, we will get very accurate results, which we will see in the subsequent classes. So, you can see that even a very crude approximation about the X, we get a reasonably draft estimate about the natural frequency, which in many cases may be enough ao far as design is concerned.