Dynamics of Machines

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Module - 2 Lecture - 1

Space Motion of Rigid Bodies

In the previous lectures, we have studied dynamics of machines in which we determine the unknown joint reactions and forces and moments, which must act on the system when the motion is prescribed. However, we noticed that the situation we have considered was not completely general. What we studied or the systems which we studied were moving in a particular manner so that, each and every component is having motions in either the same plane or parallel planes. In other words, we considered only those types of situations where all the points or particles constituting the body of the system were moving in parallel planes.

This is a very special and convenient situation and we had no difficulty in determining the forces and moments. But, a real machine or in a real situation the condition may be different. What I mean to say is that all the particles constituting a system may not move in parallel planes. What we will see in the coming few lecture is that, when you consider space motion instead of plane motion of rigid bodies some new concepts merge. So, before we take off the dynamics of such systems or bodies first let us consider and investigate the motion of a rigid body in space or space motion of rigid bodies.

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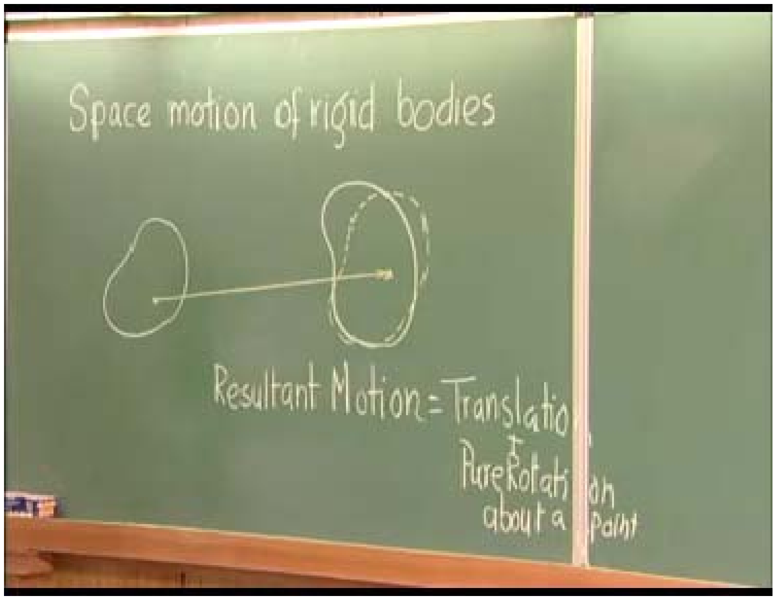

If we consider a rigid body in this initial position and we consider the same in its final position. The resulting total motion can be split into two motions: one is translation of one of the points of the rigid body, so that by this motion the rigid body came to this position that is a pure translation where all the particles of the body execute identical displacement.

Once it reaches this position then, keeping this reference point chosen by us, we provide a rotation to the body, so that it comes to the desired position. Thus, any general motion of a rigid body can be split into a pure translation plus a pure rotation about a point. Therefore, I think it is clear that we should try to investigate the motion of a rigid body by splitting it into two components: one, which is a pure translation of the body and then, a rotation of the body about that point. Now, translation of a body is a very simple situation we know that each and every particle of the body, they execute identical motion. So the dynamics of such a motion is very simple, what we can do? We can treat the whole object as a particle concentrated at a point which we are considering or at any point and it will be identical to the case of a particle motion. What really needs to be done here now to get into the actual problems of space motion that how to analyze or how to handle the dynamics of a rigid body about a point.

This kind of motion, we call spheric motion. Why we call this motion a spheric motion is simple to understand that, if you take this kind of a body and one point is fixed, we call it O. Then any particle of that rigid body will move on this surface of the sphere with this as the radius and O as the center, because this distance has to remain fixed by the definition of a rigid body. So, all particles constituting the rigid body they move on spheres or concentric spheres with O as their centers, but the radius of the various spheres depend on the distance of the point of the particle from O. That is why such a motion is called a spheric motion.

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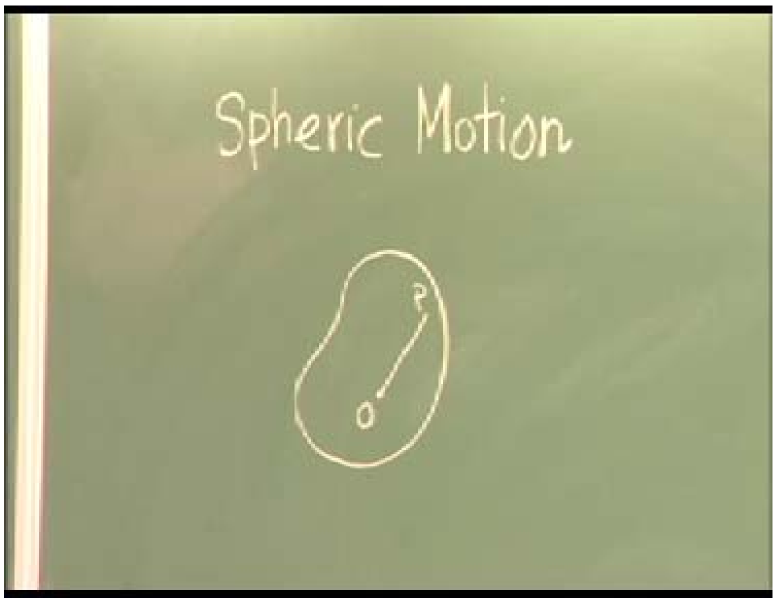

Thus, any general motion of a rigid body is a combination of a pure translation and a spheric motion. Now, before we proceed further, let us consider the question of rotation. What do we mean by rotation? I think you know from your kinematics class is nothing but the change of orientation. That means if we identify a reference line on the body and we notice its original inclination, then rotation will be given by the change of that line from its initial to final position.

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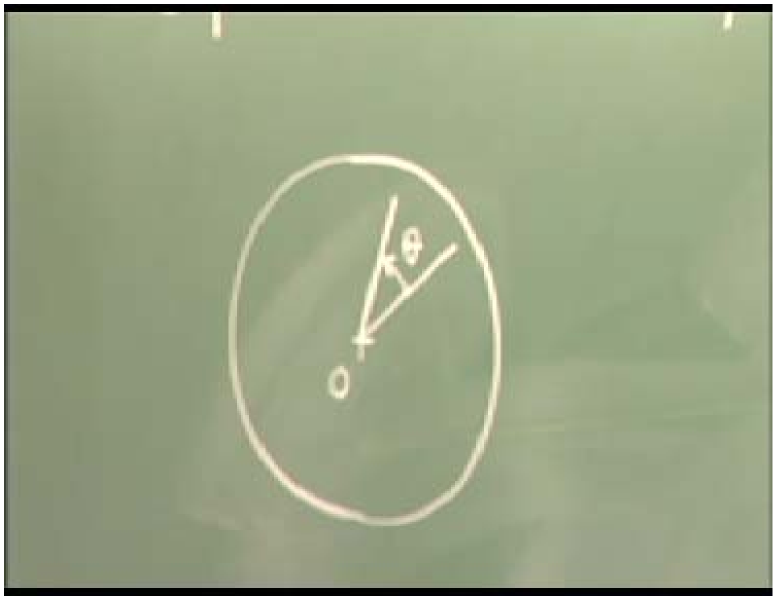

In a plane rotation case or cases where the body moves in a plane, the matter is simple. We call that if this is the fixed point and this is the line, if the line takes this position, then, this is the amount of rotation. It is very simple, you are doing it from your school days. The question is that this rotation what kind of a quantity is this? Like a displacement, you all know, it has a magnitude and a direction and we represent displacement by vector quantities. Similarly, a linear velocity- It has a direction and magnitude, we treat such quantities as vector quantities. First, let us see whether it is possible for us to treat a finite rotation like this theta as a vector quantity or not. Now, in case of plane motion, such rotation can be treated as a quantity by giving a convention that if it is in the anti-clockwise direction, we consider being positive or negative depending on one's choice and the other direction is of opposite polarity. Therefore, we can always say a rotation is either positive or negative and its magnitude is the angle of rotation, so no problem, but in space the situation is very different. So, what we will do now? Let us see whether the finite rotation satisfies certain conditions which must be satisfied by vector quantities.

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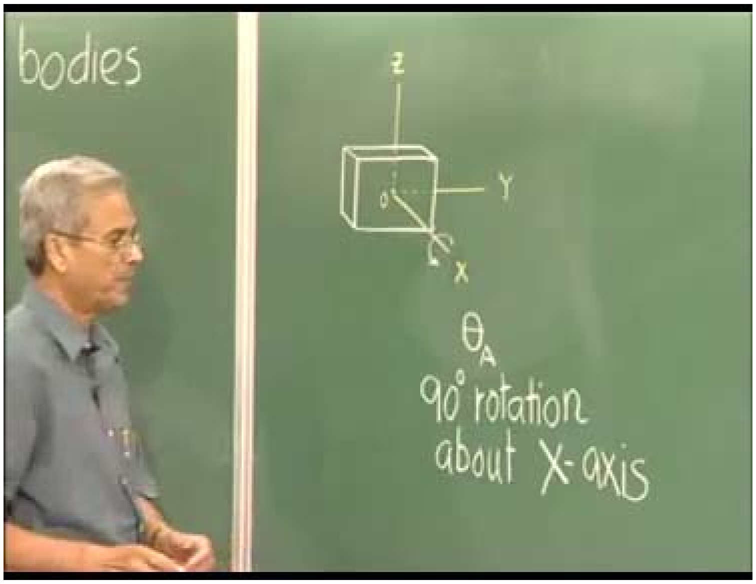

We start with a simple problem. Let us consider a rectangular block. It is initially in this position. Just to mark, let us consider this as the X-coordinate, this as the Y-coordinate and this as the Z-coordinate and this is the fixed point. Let us define, rather theta_A as a 90 degree rotation about the X-axis. So, if we apply theta_A to this body what will be the position or resulting position of the body? We have to rotate it by 90 degrees in this direction. Now positive and negative, we can give some kind of nomenclature or convention that if this is the positive direction of X, then a positive rotation is given to a right-handed rule. We will move in the forward direction or positive direction of X.

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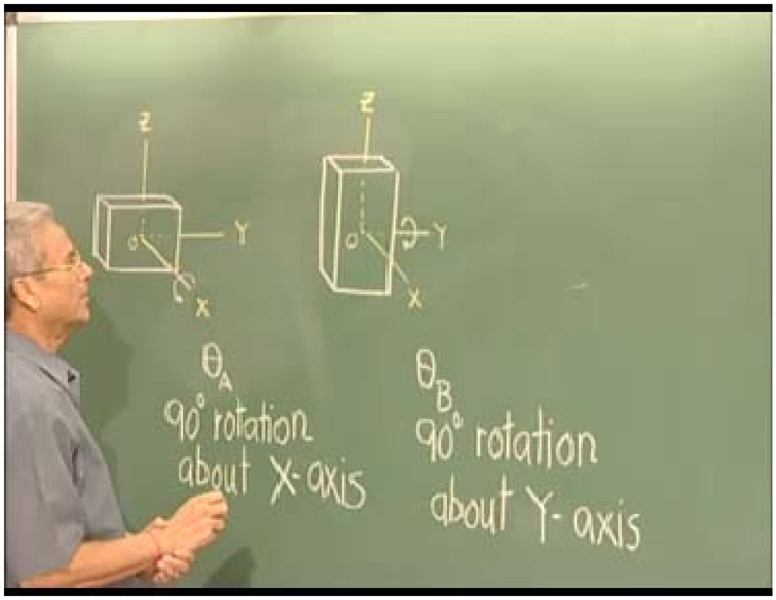

With that, we know that our body will come to this position where this is the X, these are the fixed coordinates. So this is the position after 90 degree rotation about X, which you call as theta_A, we get a finite position. Suppose, we define another angle theta_B, that is a 90 degree rotation about Y-axis. So, if you add this angle theta_B, where will it go? It is a rotation about this 90 degree so it will come to a location, we get this resulting position by theta_A adding to this theta_B.

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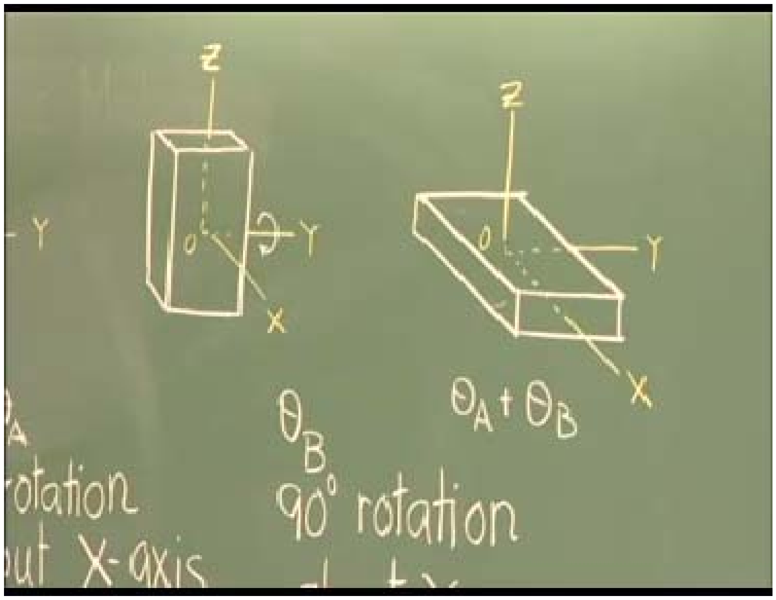

Now, let us reverse the order. What do you do now? We start with the initial position. Say, this is the body in the original position X, Y.

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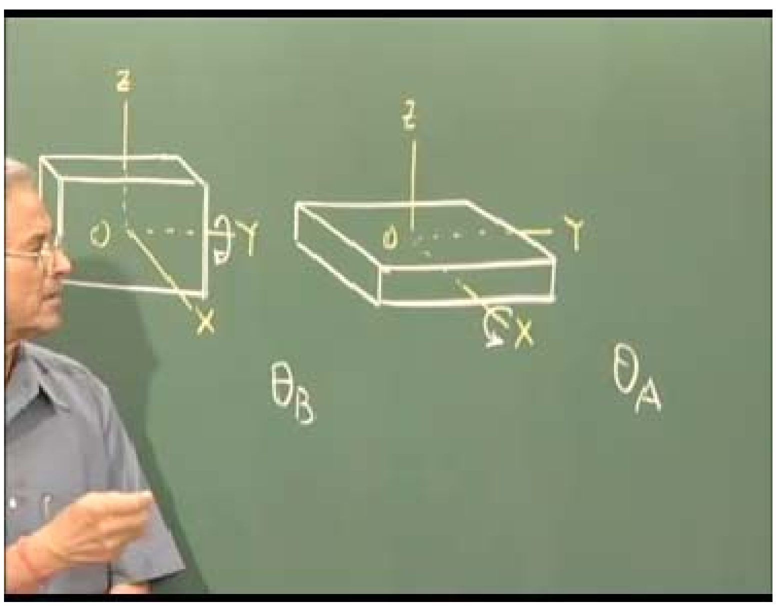

Now, with theta$_B$, let us first apply theta$_B$ on this, that is a 90 degree rotation about Y-axis. So theta$_B$ will lead to what you can see. This is a rotation of 90 degrees about the Y-axis and we come here. We apply theta$_A$, which is a 90 degree rotation about the x-axis.

So if we apply a 90 degree rotation about X and this is therefore the situation we get through rotations theta$_B$ plus theta$_A$.

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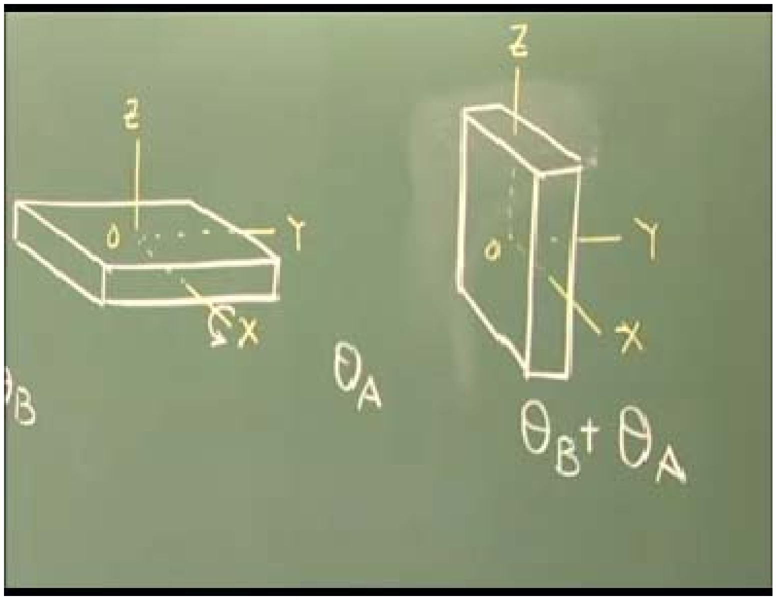

If you compare the results it is very clear that theta$_A$ plus theta$_B$ does not yield the same result as theta$_A$ plus theta$_B$. But you know for vector quantities the addition should be independent of the order in which the quantities are being added. Thus, it proves very clearly that finite rotations cannot be treated as vector quantities. So if you cannot treat rotation as a vector quantity it becomes a very difficult situation. Luckily, there is one advantage when you consider rotations which are very small in magnitude like this.

Let us now consider cases with a very small rotation. Let us consider a stick which is free to rotate about this point O. We give the fixed co-ordinates system as before like this, this is the X-axis, this is the Y-axis and this is the z axis. Now, let us give very small delta quantity rotations, delta theta$_A$ and delta theta$_B$ to this body. What happens? Let, delta theta$_A$ be a rotation about the X-axis, but angle is very small, where it will go? Now, you can see that it will come somewhere here where this angle we consider as delta theta$_A$. Next, we give delta theta$_B$ that is a very small rotation about the Y-axis that is we have to give a small amount of rotation about this. In this condition, we all understand that this whole thing will describe a surface of a cone with Y-axis as the central axis.

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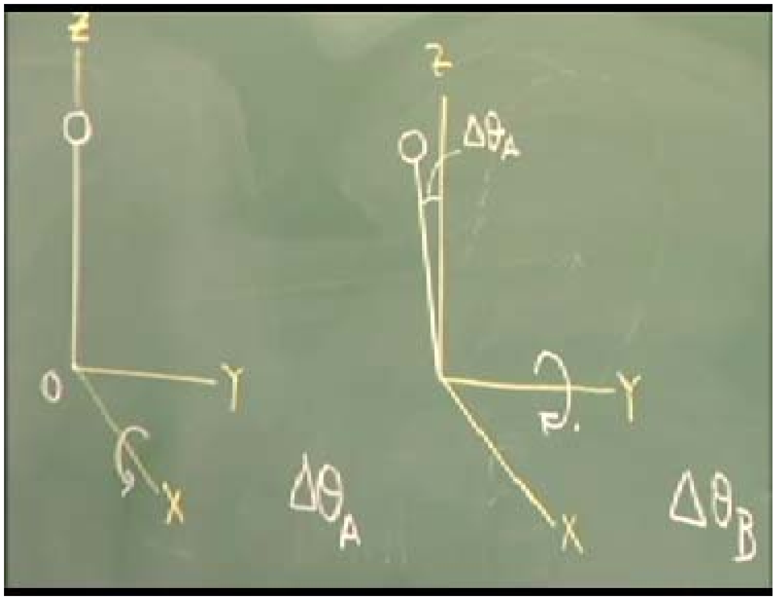

It has come here, now it will go somewhere here. That is, after this we will get the position somewhere here. If we reverse the order, first give a small delta theta$_B$ rotation about the Y-axis then you will find that this point, the head comes from here to here. Then we rotate about X-axis, brings it here. In earlier case, what happened? First, we rotated about X-axis it came here and then we rotated about Y-axis it came here.

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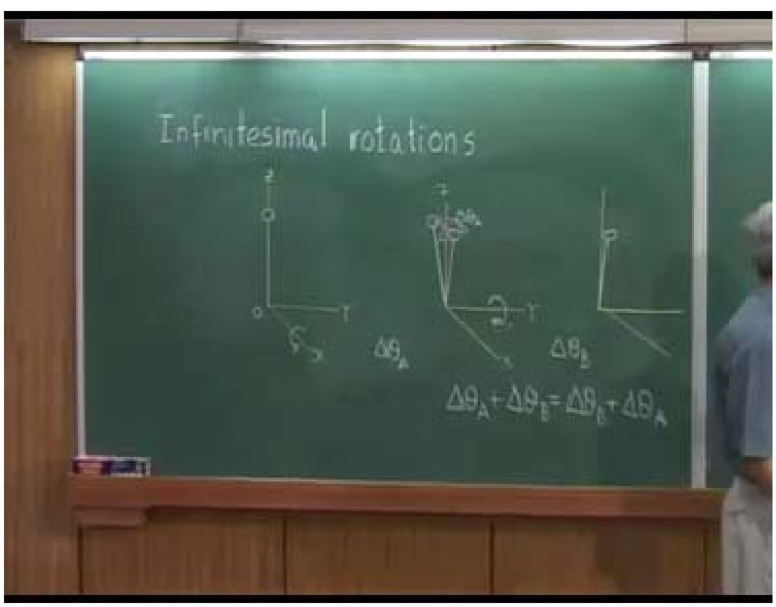

So you see, when these rotation angles are of infinitesimal magnitude the final result will be the same. Thus we can say, for infinitesimal rotations delta theta$_A$ plus delta theta$_B$ is equal to delta theta$_B$ plus delta theta$_A$.

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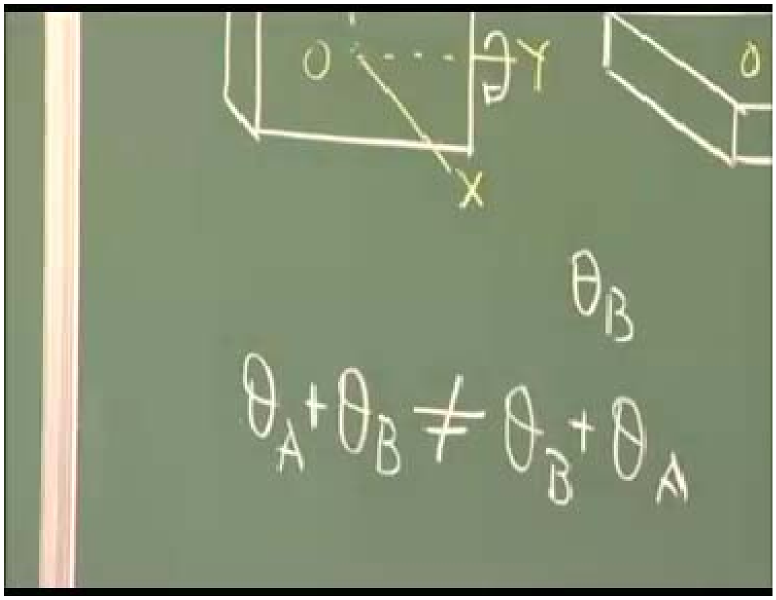

On the other hand, here, what we noticed that theta$_A$ plus theta$_B$ was not equal to theta$_B$ plus theta$_A$ in space motion. In plane motion of course it satisfies. So there is an advantage to us, if we treat infinitesimal quantity of rotation, we can find that they can be treated as vectors. This is a big advantage, why? Because when you define angular velocity, angular velocity omega is defined as the limit deltat tending to 0, delta theta by deltat, we all know that. If delta theta can be represented by a vector then omega also is a vector. So, angular velocity is a vector quantity, their additions will satisfy the vector addition.

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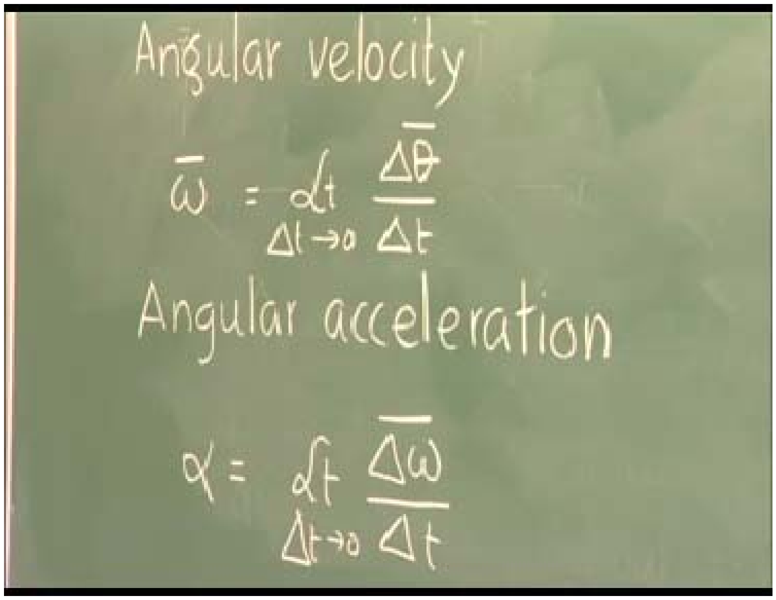

Similarly, if we consider angular acceleration, how it is defined? Angular acceleration is defined as…, but angular velocity change, is definitely a vectorial quantity, because angular velocity is a vectorial quantity. Thus, angular acceleration is also a vectorial quantity. What we have seen so far is that finite amount of angular rotations in space cannot be treated as vectorial quantities. On the other hand, infinitesimal rotation space can be treated as vectorial quantities and this satisfies the vector addition rule. Since angular velocity is defined through infinitesimal angular rotations which are vectors, so angular velocity is a vectorial quantity so is the case with angular acceleration.

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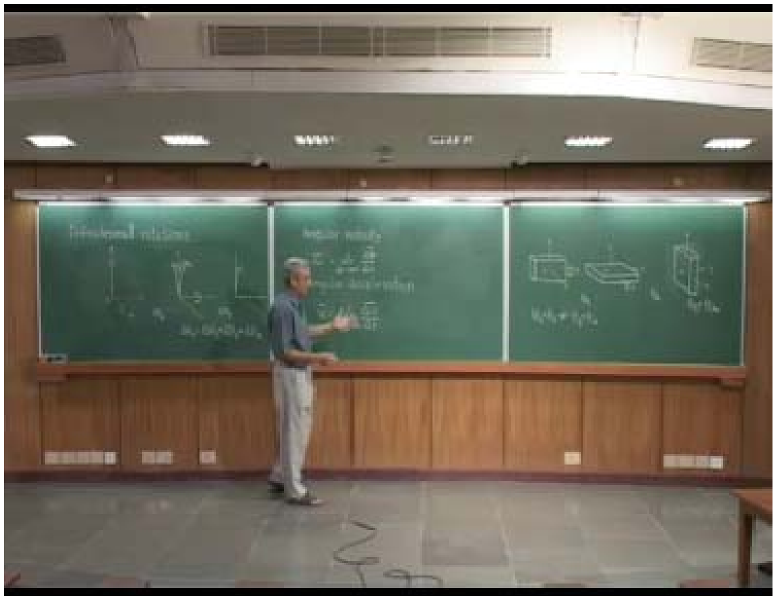

However, unlike normal displacement force, velocity, acceleration for linear motions, here there is a slight difference. In case of a linear displacement, we can say it is a displacement s vector, absolutely no ambiguities there, similarly for velocity or acceleration. Now for angular velocity, I say that it is an angular velocity of omega about this axis. How do we give it a vector motion? There we have to follow a convention, generally, what is followed is that right-hand screw rule. If you take a right-handed screw rule and rotate it along the direction of omega, the moment of this screw will be forward direction.

This is actually a vector where the line of action is the axis of rotation of the angular velocity, the arrow head is decided by the right-hand screw rule, the length of that vector is the magnitude of omega. So representation of an angular velocity through a vector has to follow this convention, same is the case with angular acceleration. We have to follow a convention like that as you have done in angular velocity, the line of action of the velocity or the angular acceleration vector will be the axis. These arrow head will be decided by the right-hand screw rule and length will be decided by the magnitude. These are very basic things about rotation in space and we have already seen something very different from the planar situation where you found that the rotations cannot be treated as vector quantities if they are finite magnitude. On the other hand, angular velocity and angular acceleration can be treated as vectors.

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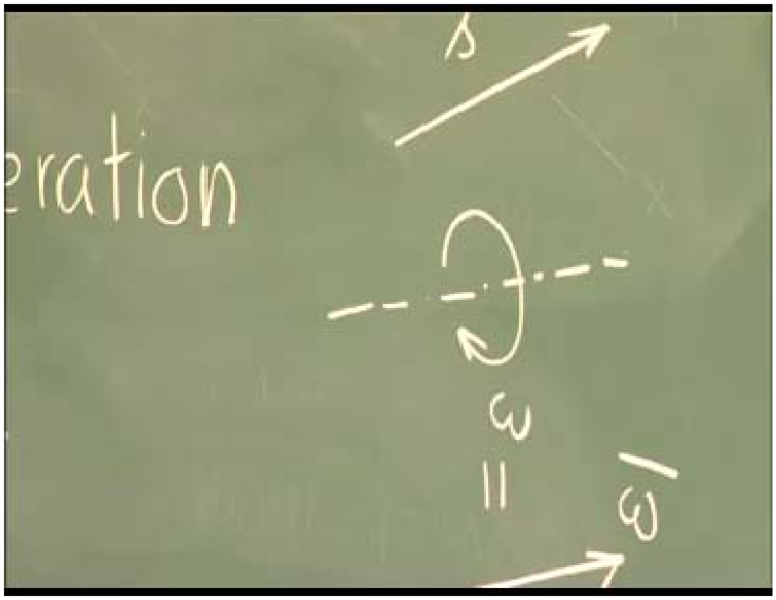

Now, let us consider the case of a rigid body in spheric motion. First, let us try the situation, rigid bodies in spheric motion. If we say a rigid body is moving or rotating about a fixed point, there can be many examples actually even in our common life. For example, it is out of fashion but in earlier days spinning top used to be a very popular toy for the children.

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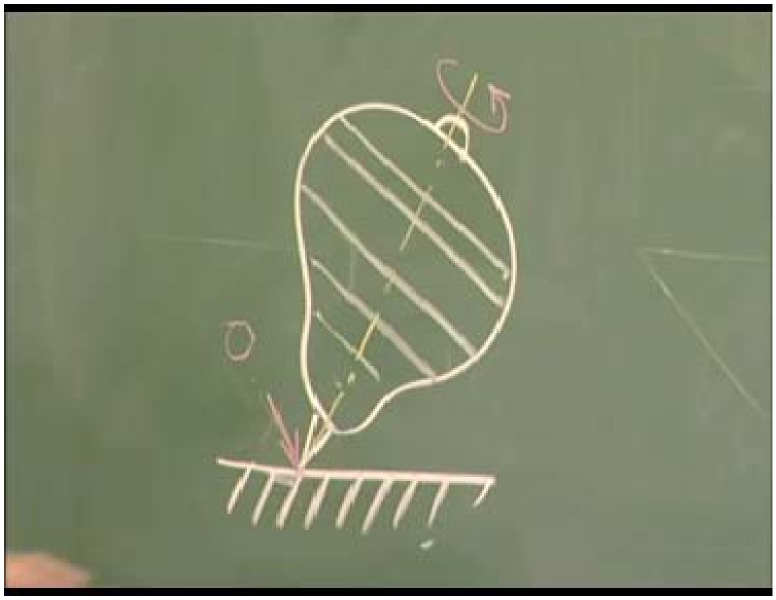

This is a top which rotates about its axis at a very high speed. We know the axis itself can also slowly move, but all the time this point O, which is a particle on the whole top. It is at the end, keep up this, this point is fixed. So each and every particle of this top is moving on surface of spheres with O as the center. Another similar situation, we can create. Let us consider a roller or a cylindrical body which is free to rotate about a central axis like this and the axis itself again is connected to a vertical rod which is hinged. Now, if we give a motion like this. That means, this is being given a rotation with angular velocity $omega_2$ and this is being given an angular velocity $omega_1$, then what kind of motion is this? It may not look apparently that it is a spheric motion, but in reality it is. You just imagine that this body is extended up to this, just imagine, hypothetical. Then, this particle on this body is neither going to have a motion in any direction as you can see. So, it can be treated as a fixed point. Why? Because this axis of rotation and this axis of rotation are intersecting here and so neither of the rotations can produce any motion of a point because it lies on both the axis.

Therefore, this is also a case of spheric motion. There is a very important concept which you must consider. Instantaneously, it can be shown or you will see that the body can be considered to be rotating about an axis passing through that fixed point at any instant.

The axis or this instantaneous axis changes continuously its location, but at an instant there is an axis about which this body rotates.

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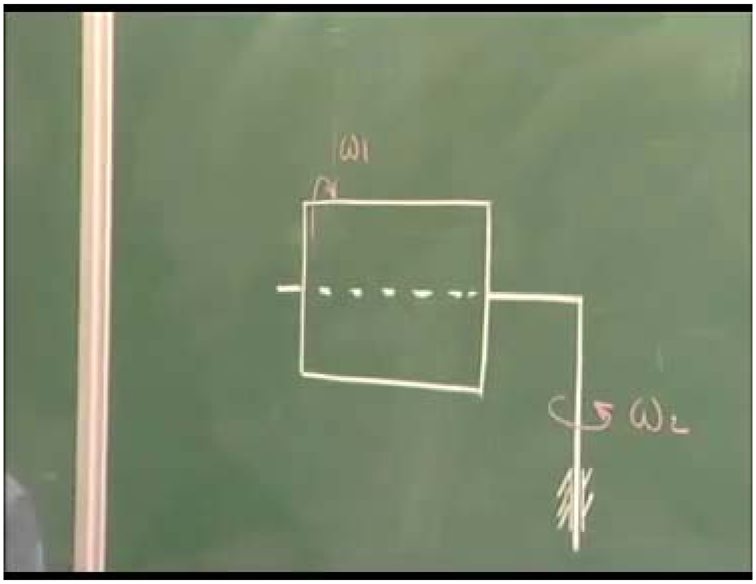

So, if this is the cylinder and this frame containing this cylinder is given an angular velocity, $omega_2$ and this body is given an angular velocity, $omega_1$. What is the instantaneous axis about which this body can be considered to be rotating at this instant?

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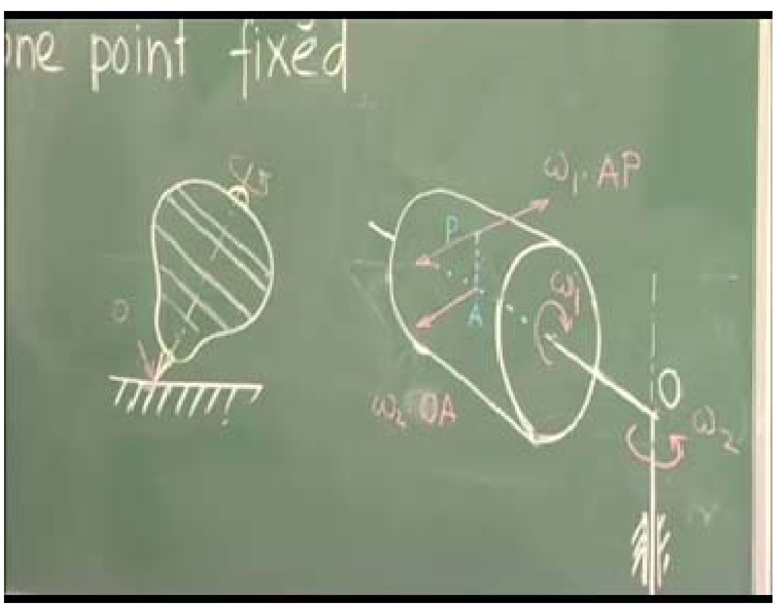

So to do that, let us consider a situation where this is drum or a cylinder made of some completely transparent plastic and this transparent plastic drum has some black particles. When it rotates you will find, say one particle here. Let us represent this particle by this. This particle is at a height or if we slightly change its location and also color to avoid confusion with this. Let us consider this particle P and at a distance R from the axis of rotation. Let us consider this point as A. Now at this point A on this axis has a velocity because of this rotation and this velocity is in the horizontal plane. How much is this? It will be ω_2 into OA. So, you have seen that this point in this axis will have a velocity because of the axis of rotation about a vertical axis is ω_2 into OA. Now, if you want to find out the velocity of this point without any rotation of this, we know that since its distance from the vertical axis of rotation is same, this also will be same as ω_2 into ω_A. Now superimposed on that is the rotation ω_1. Because of this rotation ω_1, this particle P will have a velocity with respect to A in this direction and its magnitude is ω_1 into AP. There can be a situation if the location of point P is such that these two velocities are equal and opposite.

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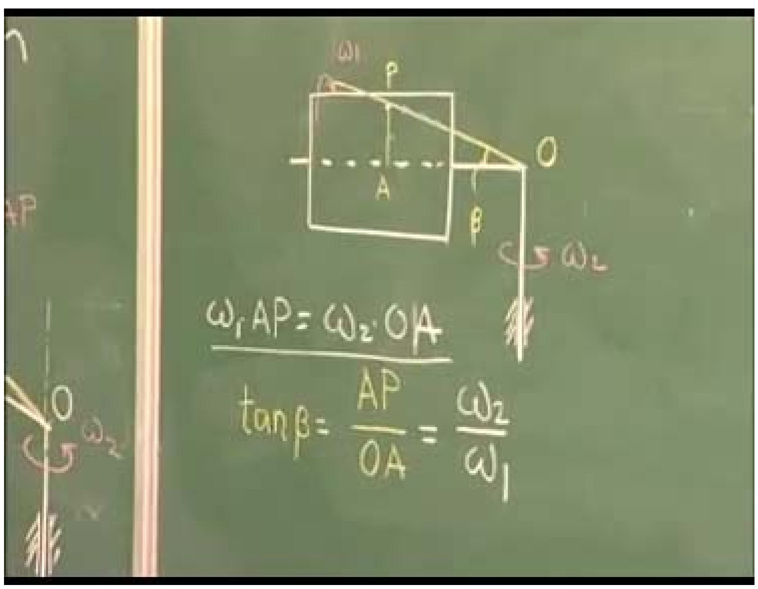

Then what will happen? At this instant, this particle will have 0 velocities. When these velocities are large this particle will be stationary and I think if you use some light to focus on it this particle will be visible at this instant. Remember, the whole thing we are doing is at an instant. Similarly, there will be many other particles which will also satisfy the same kind of situation or condition up to this. So, what we will get here is a line. If I draw it here it will be easier to follow. This line on which all the particles will be temporarily at this instant motionless will be visible and this line defines the instantaneous axis of rotation of the rigid body. So even though we keep one point fixed, a general motion of a rigid body is such that, at any instant, it can be considered to be rotating instantaneous about an axis which, of course continuously changes its position. For example, now you want to see this axis of rotation, what determines this angle? It is very easy because if you consider this point as P and this as A, then this is a right angle triangle and tan beta is given by AP by OA and from the condition here you see AP by OA is nothing but omega_2 by omega_1, this angle depends on the ratio of the two angular velocities. So if omega_1 and omega_2 remains constant, this angle beta will remain constant.

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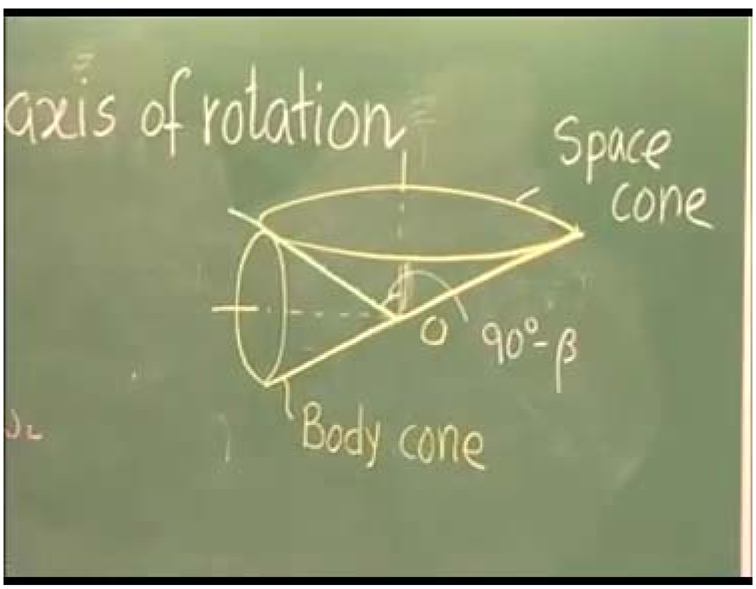

In such a situation angle remains constant, so what happens to instantaneous center or instantaneous access of rotation? You see, it describes a cone and the vertex or semi-vertex angle of this cone is how much? It is nothing but 90 degree minus beta. This particular cone, which is described by the instantaneous axis of rotation during the motion of the rigid body, is called the space cone. Why? Because this cone is described in space and remains stationary or fixed in space. Suppose, an observer is sitting within this body or drum or this cylinder, what he or she will observe so far as this instant near success of protection is concerned? To the observer it will appear because the observer is rotating like this. The observer will find, with respect to the body, the axis is rotating in the opposite direction. So in the body this instantaneous axis will describe another cone like this and you can easily tell that since the semi-vertex angle of the space cone is this, the semi-vertex angle of the cone described by the angle of rotation within the body is this, these two will sum to 90 degrees. Now, this cone is the cone described by the instantaneous axis of rotation with respect to the body, that is why, we call it a body cone.

Now, the motion of the rigid body will be equivalent to the motion which I will be just describing that you attach this space cone or the body cone to the body and create a space cone and keep it fixed in space. Then, if you roll the body cone over the space cone the motion will be exactly the same as what you were getting in this way. Why it is so? In this case, if we ask that where the instantaneous axis of rotation is, it is very easy to find out that because it is the touching line between the two cones is the instantaneous axis of rotation. Why? Because body cone is attached to the body and these particles at this instant on the body has zero velocity because the space cone has zero velocity and it is a pure rolling action. So the points on the body cone, at this instant, along the line or along this line of contact between the two has zero velocity and so this is the instantaneous axis of rotation. If this is the instantaneous axis of rotation we can also tell that this is also the instantaneous direction of the resultant angular velocity of the body, why? If a body is rotating then obviously, the particles which define the instantaneous axis of rotation have zero velocity.

Therefore, the angular velocity of a body has to be along the same. If a rigid body’s instantaneous axis of rotation is this, we all know that all particles on the body along this line has zero velocity which is the instantaneous axis of rotation. Obviously, we define this as the direction of angular velocity as just described little earlier in today's lecture. So, this line along which the two cone touch is the instantaneous axis of rotation and also the direction of the resultant angular velocity of the body, which is nothing but the vectorial sum of the two angular velocities.

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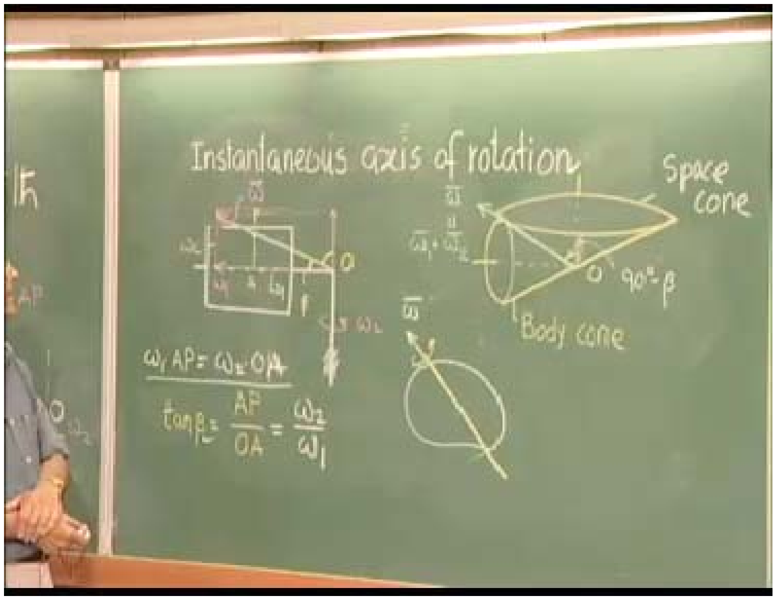

How do we get it? It is very simple to show that if $omega_2$ can be shown like this and $omega_1$ can be shown like this because it is rotating like this. So, the vector sum will produce the resultant angular velocity and it will be along this because this length is the magnitude of $omega_2$ and this length is the magnitude of $omega_1$. Now $omega_2$ by $omega_1$ is tan beta. Therefore, we find that motion of a rigid body with one point fixed in general case is such that it can be considered to be a rotation about an instantaneous axis passing through the fixed point. Now, these instantaneous axis of rotation moves in space producing a space cone which is fixed in space so far as this relative motion of these instantaneous axis with respect to the body itself is concerned. Since the body is rotating like this and this angle remains fixed, obviously an observer within the body will see this axis to rotate in the opposite direction discovering another cone with beta as the semi-vertex axis. Thus, the resultant motion of the body can be generated again in an equivalent way which is a rolling of the body cone over the fixed space.

Next, I think comes the question of the angular velocity. We have found out that angular velocity of a rigid body can be described as a vector along the instantaneous axis of rotation, its direction will be decided by the right-hand screw convention which you have followed and its length will be the magnitude of the velocity. What about acceleration? This is a very complex case and rather, unusual case. You will see that to understand this we need to understand the rate of change of vector which is moving in space. So, first let us take up the case of the rate of change of a vector.

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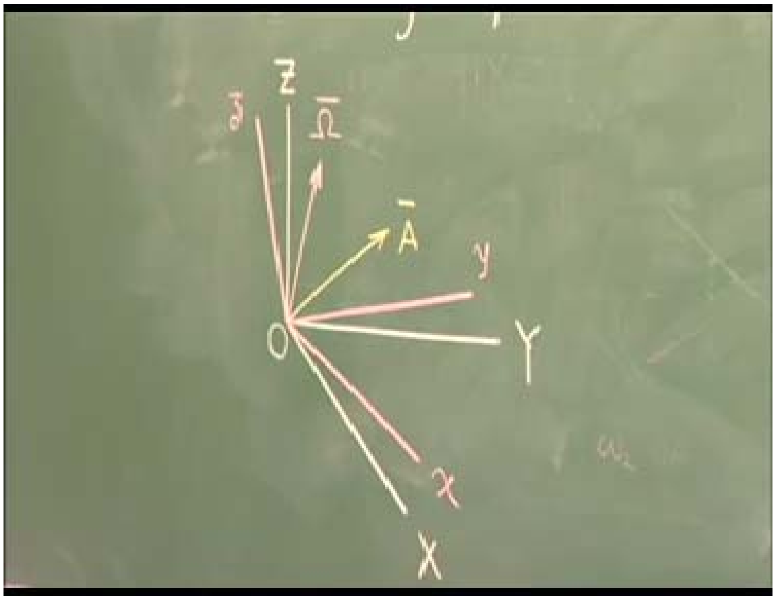

So, before taking up the matter of angular acceleration, these are all topics of basic dynamics. I am sure you have done but even for the sake of it, you are using again to determine angular acceleration, we will repeat it. If we take a fixed coordinates system, X, Y and Z and another coordinate system x, y, z, which has the same origin as the fixed coordinates system, but which has a rotation with respect to the fixed axis. What I mean to say is pink coordinate system is rotating with this point fixed and white coordinate system is fixed in space, the angular velocity of rotation of this moving coordinate system is omega.

Now, suppose we have a vector A. What our objective is to find out the rate of change of this vector with time as seen from the fixed coordinates system or from fixed space. To do that what we will do? We will split the problem into two parts: one is we will consider a situation where A is fixed in the pink system that means, an observer sitting in the moving coordinate system will find A to be fixed, no rate of change with respect to the… The quantity is defined by dA by $dt_{x\ y\ z}$. Now, that observer whatever rate of change he finds is the rate of change with respect to the moving coordinate system. Suppose, if you take a coordinate vector A which is not changing with respect to x, y and z what the another observer sitting in the fixed coordinate system X, Y and Z will see? Now there you know that this A vector will also have the same rotation as the coordinate system x, y and z given by omega and it will describe a cone with this as the vertex and capital omega as the axis. So in such situation you already know that from kinematics course, the change of this point will be omega cross A. This will be the rate of change of the vector seen by an observer if this vector A is fixed with respect to x, y and z.

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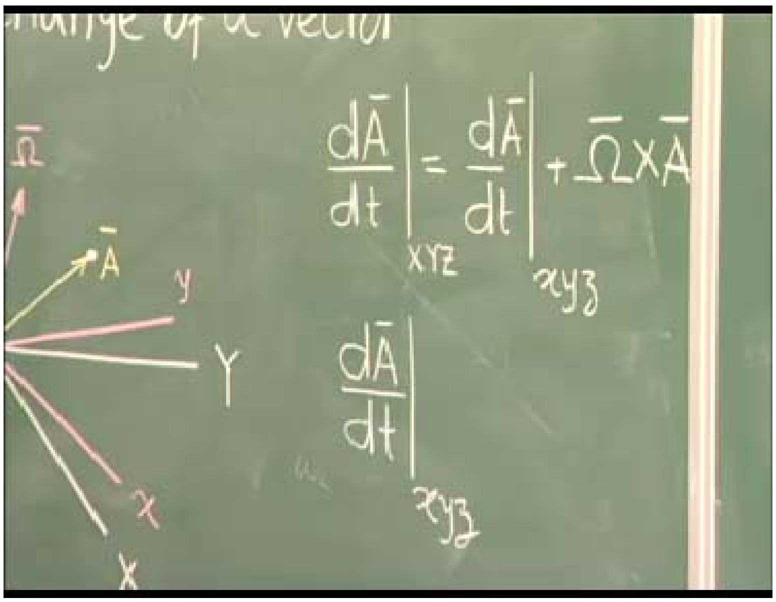

In general case, when A is changing with respect to the moving coordinate system, also to get the total rate of change with respect to a fixed coordinate system we have to add the rate of change with respect to the moving coordinate system. This quantities or this analysis is again a repetition. It has been done in details in kinematics course. So rate of change of a vector with respect to fixed coordinate system is rate of change of the same vector as observed by an observer sitting in a moving coordinate system rotating at an angular velocity omega plus omega cross A. So, the example which we have seen or we have done so far is the space cone or body cone.

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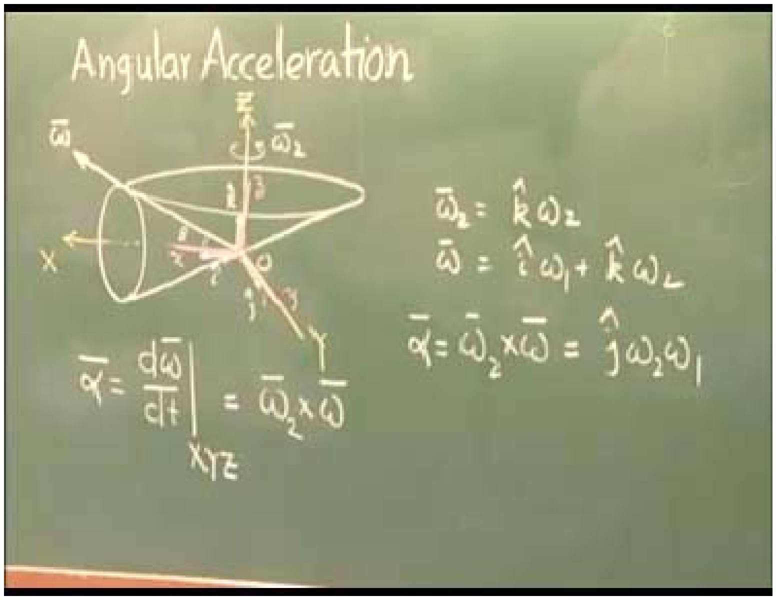

This is the space cone, this is the angular velocity vector at an instant what will be the angular acceleration? Let us find it out now. Here we can treat the problem like this. Let us treat this as a fixed coordinate system and we also define the space cone axis which coincides at this instant, but they have a finite velocity with respect to the fixed coordinate system X, Y and Z. What is the angular velocity of this pink system x, y and z obviously, omega$_2$. That is the angular velocity of the moving coordinate system. At the same time we find that this angular velocity omega does not change with respect to the moving coordinate system, it remains with a same angle beta all along, its magnitude is also same all along because none of the magnitudes of the velocities omega$_1$ omega$_2$ are changing. So, magnitude of omega also remain constant.

So, what will be the angular acceleration? It will be nothing but domega by dt as seen by a fixed observer, obviously and that is equal to domega by dt with respect to x, y and z is 0. So it will be simply omega$_2$ cross omega. Following this capital omega is actually omega$_2$ and vector A is the angular velocity vector. Now, I think further what can be done if we give i, j, k as the unit vector directions then, we will find along the fixed x, y and z, so that i, j, k are constant. So, omega$_2$ is actually k omega$_2$ and omega is equal to i omega$_1$ plus komega$_2$, that is the omega vector is summation of omega$_1$, omega$_2$ and angular acceleration alpha will be which is nothing but, according to the vector algebra rules.

Therefore, we find that the motion of a rigid body with one point fixed has certain interesting features. One is that it can be considered at an instant to rotate about an instantaneous axis of rotation which passes through point O. We also find that the motion of this rigid body, which is a complex motion with only one point fixed, can be generated by an equivalent way in which a body cone connected to the body or connected to the object under consideration, rigidly rolls over a space cone, the motion will be exactly identical. Once we have done that it is purely a kinematics, what we have done of spheric motion. In next class, we have to see how natural properties of a rigid body in space motion can be defined and then we proceed further to find out or to analyze the dynamics of such systems involved in space motion of rigid bodies.