Dynamics of Machines

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Module-13 Lecture-2
Vibration of continuous systems

Another very common type of element which finds its usage in mechanical systems, machines or shaft, we will take up now, the oscillation of a very common type of element. We will consider, of course, the perfectly circular shafts of some length and uniform cross-section.

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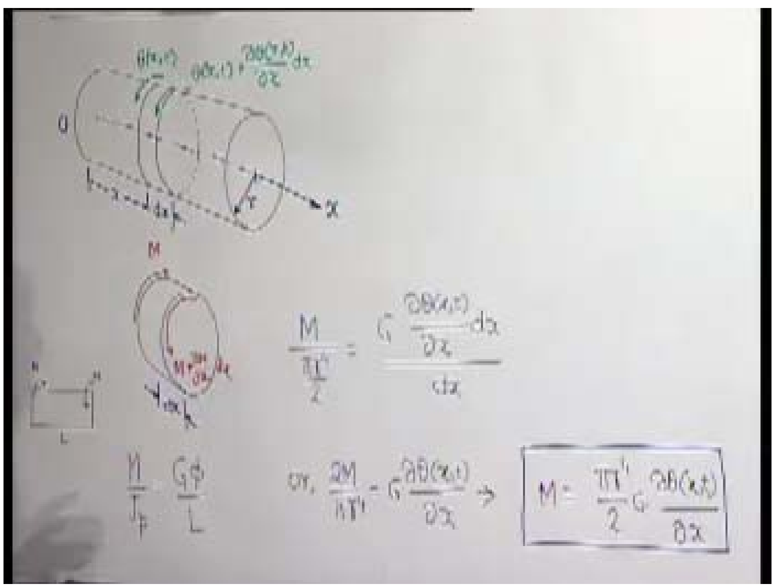

This is the uniform circular shaft in question and the mode of oscillation, as I have already explained in my previous presentation that it will be about an axis which is called the longitudinal axis x whose origin is here and that is o. We again consider an element at a distance x from the origin whose length or thickness is dx. If we imagine this element of infinitesimally small thickness, the displacements will be that if you take the plane which is at a distance x its rotation. Now, the displacement will not be linear displacement, but angular rotation. Let this be theta which will be a function of x and time. As we found in case of a bar it was u, that is, the longitudinal displacement. It was also a function of its position and also the instant we are considering. Obviously, the rotation of the other one will be the amount of rotation of this plus the slight difference. So, these are the displacements of the two ends of the element. Now, let us draw the elements separately. This is the element.

Now, if you consider the motion of this element and find out the equation of motion which is a valid for all such elements at all position. Now, here, as we know that the total moment which will be resisting the rotation of this, from this side, we say M. The moment which will be trying to… (Refer Slide Time: 04:16). So, there will be two moments, we are seeing, one on this phase, because of the twist like this, this phase will be subjected to resisting moment and this phase will be subjected to moment in the positive direction like theta. Therefore, we can say, this is a shaft of length dx and we have not mentioned the size; let us consider radius to be r. Then the twist which this shaft is subjected to it may be element, but nevertheless it is a shaft will satisfy our standard relation; Moment on a shaft by the polar second moment of area is equal to clear modular G, amount of twist psi and the length is L. This is the standard relation we know for the torsional of a shaft. Length is L.

If we now compare this with this standard formula, the moment on this space is M and this place is also M, just slightly a different value; so, M will be M. Second moment of area of a circle with radius r is half or pi, t is the clear modular. What is the amount of twist? Now, this end has moved by theta and this end has moved by theta plus something. So, the amount of twist of this elemental shaft is nothing but that extra something, otherwise this rigid body rotation is like theta and then an extra amount. So, this is the amount of twist, so that will take the place of pi (Refer Slide Time: 07:09 to 08:04). The length of the element of shaft here is nothing but dx, or we get 2M by pi r to the power 4 equal to this, or from this we get M is equal to pi r to the power 4 by 2 into G[.]. So, this is what we find; the expression of moment in terms of the displacement and the material properties, and the size of the shaft that is r.

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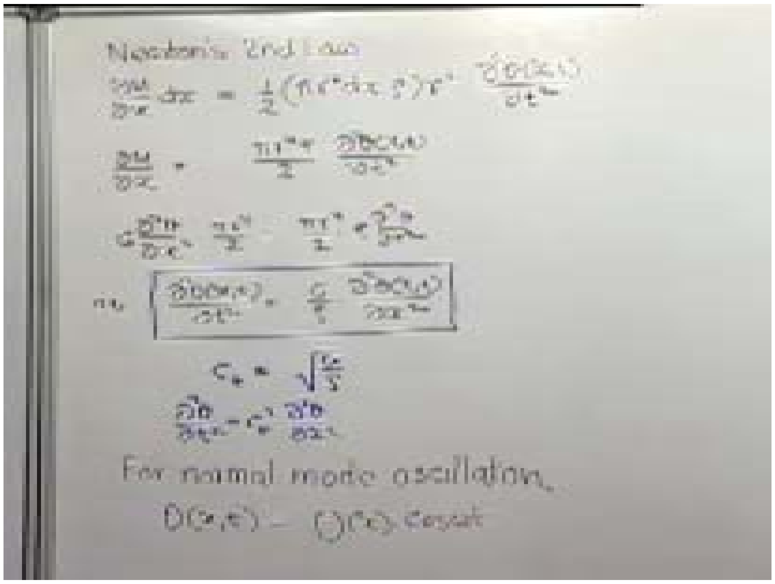

Next, again, we find out the dynamics of this element; what is the total moment acting on this. Total moment acting on this is resisting moment M and a forward moment M plus something. Therefore, (Refer Slide Time: 08:49) this is the total moment acting on the element in the forward direction that is theta[.]. Therefore, this must be equal to Newton's second law. This total moment must be equal to moment of inertia of this element into the angular acceleration. Moment of inertia will be how much? For this, we know it will be half mass of this element, that is, volume of the element is this (Refer Slide Time: 09:34) into rho; that is, the mass of the element into r square; this is polar moment of inertia of the element into angular acceleration.

Now, what will be angular acceleration? The amount of rotation is given by theta. So, del theta del t will be velocity and del square theta and del t square will be the angular acceleration (Refer Slide Time: 10:02 to 11:22). From this, we get dx and dx cancelled, so, pi r to the power 4 rho by 2 into this. Now, M is this. Using M and substituting it here, we get, del M del x will be equal to G del square theta by del x square into pi r to the power 4 by 2 is equal to pi r to the power 4 by 2 rho or finally, we get this equation . We get an equation which is identical in form with the equation of motion for longitudinal oscillation of a bar. This is the equation of motion which is valid or that is the relationship between the angular rotation, time, material properties, location and so on.

If you remember previous one, in case of longitudinal oscillation it was exactly same, only here it was G by rho. So, the velocity of propagation of torsional wave is given and the equation is (Refer Slide Time: 12:18 to 13:18). So, this is the velocity of propagation of a torsional wave. Now, since the form of the equation is exactly same as the previous one, for normal mode oscillation, let us consider, all the elements will execute harmonic oscillation of same frequency. Therefore, theta x can be written as a product function of x only into a harmonic function of time, it can be cosine omega t or sin omega t.

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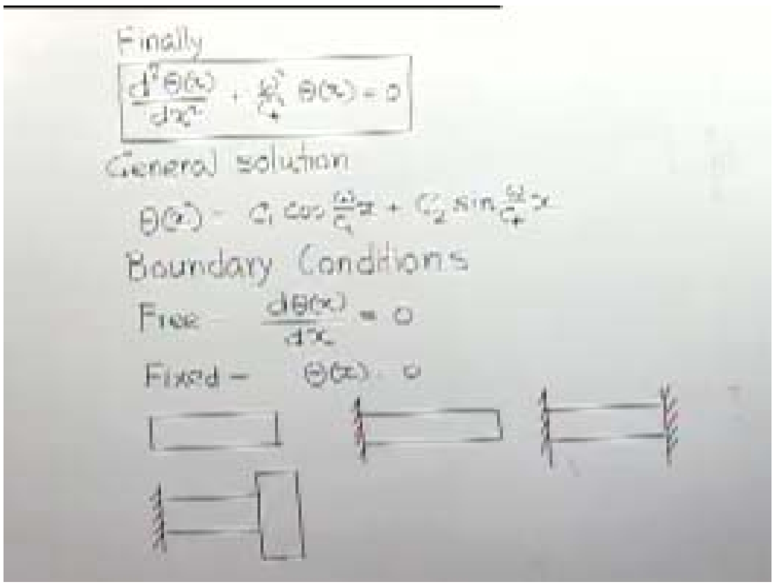

If we substitute this here, we get (Refer Slide Time: 13:45 to 14:43). So, when you substitute this theta here and cosine omega t gets cancelled, this is the equation. General solution will be this; as before, omega can be found out from the boundary conditions of the shaft. So, the solution should be very similar, that means both ends free or one end free and one end fixed or both ends fixed, we will get similar kinds of things. So, for example, here also, we will find boundary conditions can be of two types; Free means no moment acting there and as we know moment is proportional to theta dx, forget about the cosine omega t and if that has to be 0 all the time. It means it is same as Refer Slide Time: 16:00) and for fixed ends will be equal to 0, for all kinds.

In case something else is attached, a disk, then the inertia force of that disk is the total moment acting at that end. Similarly, we can proceed to solve various kinds of solutions and we will get exactly similar expressions only. Here, C will be replaced by C_t whose value is equal to square root of G by rho. In case of longitudinal propagation, it was G by rho square root. We will not repeat that; rather, I think we would like to solve an example. So, one can easily solve various types of conditions; free free, fixed free, fixed fixed and a shaft carrying a disk at the end. The mode shapes also will be similar to what we got in case of longitudinal vibration of bar for the same condition, same form of solution; only X will be replaced by C. C will be replaced by $C_{t.}$

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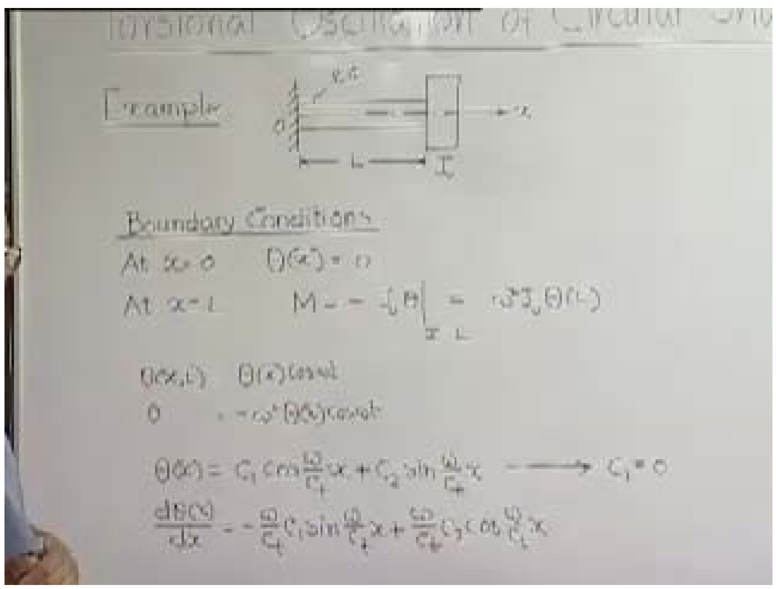

Instead of solving this type of problems which will be identical in form with the problem which we solve for longitudinal oscillation of bars, let us consider a case, where we have to deal with the other kind of problems. That means, a shaft carrying a disk at the end, the polar moment of inertia of the disk is J_0 and the length of the shaft is L. Now, what will be the boundary conditions? We have to find out the natural frequency of disk. Here, the material density is rho and the modular is G. The boundary conditions, we find that we consider this x and this as origin. At x is equal to 0, theta x equal to 0, because it is a fixed end. At x equal to L, the moment which is acting on this will be equal to the inertia loading. Now, inertia moment, acting across, acting on this will be minus J_0 into theta two dot at x equal to L.

Now, if we know that theta which is a function of x and t written for natural mode of oscillation like this, then theta two dot is nothing but minus omega square (Refer Slide Time: 20:52). So, we can substitute it there and find out the value. The moment which is acting here, also will be dynamic moment, moment will be magnitude of the moment, because it is also a harmonic function of time into cosine omega t, so magnitude of the moment at this free end when you are dealing only with magnitude, then it will minus J_0 into minus omega square theta x. So you can say, when you are dealing with only the magnitude is equal to omega square J_0 theta of L. We have to keep in mind that here the magnitude we have to find. We have already seen that theta of x we get as C_1 cosine omega by C_t into x plus C_2 sin omega by C_t into x; this is the form of the solution

So we know (Refer Slide Time: 22:22 to 23:16) we differentiate once with respect to x and we get d theta x by dx. Now, if we use the boundary conditions, theta x is equal to 0, at x is equal to 0. If we put x is equal to 0, this term is 0, and here it is only $C_{1,}$ so, this is 0. That means, this will give us this. Therefore, we will have theta x as simply C_2 sin omega by C_t into x is 0.

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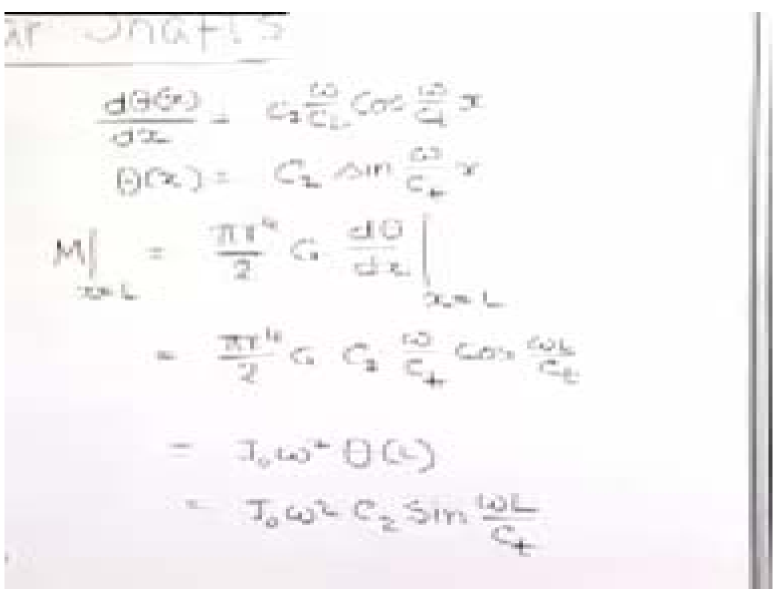

Let us find out moment at x is equal to L. How do you find out moment? Magnitude of moment at x equal to L will be equal to (Refer Slide Time: 24:07 to 25:10). We have already seen the expression for moment was pi r to the power 4 by 2 G d theta by dx. Therefore, moment at x is equal to L will be the same thing when we calculate x. So, therefore, this is nothing but, now C_1 is 0, so, d theta x by dx is equal to C_2 into this. So, if you put x is equal to L here we get this expression. This we have found out already; this is equal to J_0 omega square theta L. Now, theta L will be nothing but J_0 omega square. This theta L will be nothing but C_2 sine omega L by C_t. Therefore; we get an equation in this form.

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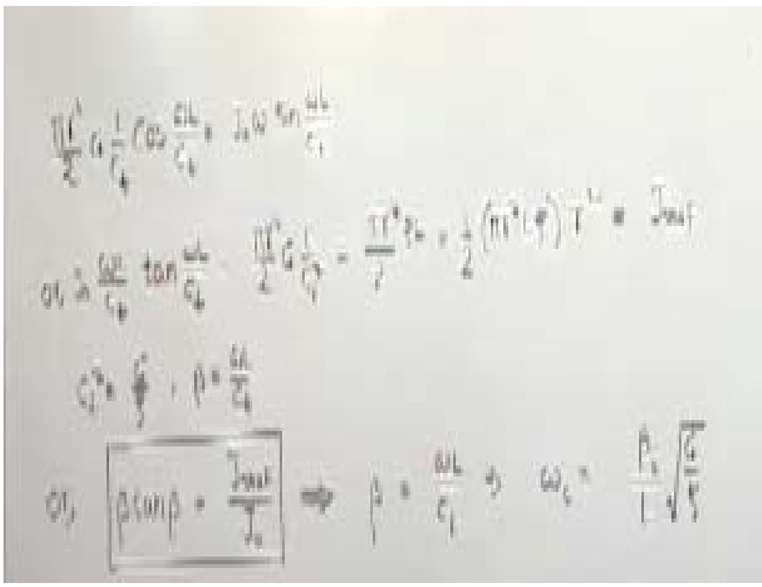

I think C_2 will get cancelled and it can be written in this form, or we will get omega L by C_t (Refer Slide Time: 27:13 to 29:32). C_t, we have already found out that C_t was C_t square is equal to G by rho. If you use it here, we get this. How much is this? This will be 1 by 2 pi r square into L into r square. Now, this is pi r square is the cross sectional area and L is the total length of the shaft. This is nothing but the total volume of the shaft into rho. This is the total mass of the circular shaft and 1 by 2 into mass of the shaft into r square is nothing but second moment of area of the shaft. If we treat the shaft as a rotor then polar moment of area will be this. Therefore, the equation finally we get, beta into tan beta J sin t by J_0. Thus, we get a transcendental equation and this transcendental equation can be solved only by computational techniques or in the earlier days it is used to be solved by using standard table instrument, one standard book of instrument.

What we will do here, we will give the solutions, first three solutions. First three solutions, of course in this state will require the two values. Here, we do not have and we have not described. If we want to keep it in this form, then it is alright. But for solving a numerical problem, it will be essential for us to give the numerical values of the length, the values of the moment of inertia etc.

Then, this can come and given to be beta. Once you know beta, beta is equal to (Refer Slide Time: 31:02), we know beta is equal to omega L by C_t. Therefore, you will get infinite number of values of beta. Therefore, from this, we will get particular value of $omega_i$ is nothing but $beta_i$ by L into square root of G by rho. Substituting the value of beta here, what we get from that transcendental equation will give us the corresponding frequency, and of course, i will be 1, 2, 3 and so on. This is just an example.

So, there can be another problem in which both sides of a shaft carries two discs; one with J_1 moment of inertia, other side will be J_2. The shaft dimensions for given materialistic types, we can find out again, the equation for which that means, a transcendental equation we will get and again solving this, we will get ultimately the natural frequency. Thus, we find that torsional vibration of shaft are very similar in its form and general form of solution also and equation also, as in the case of longitudinal vibration of mass.

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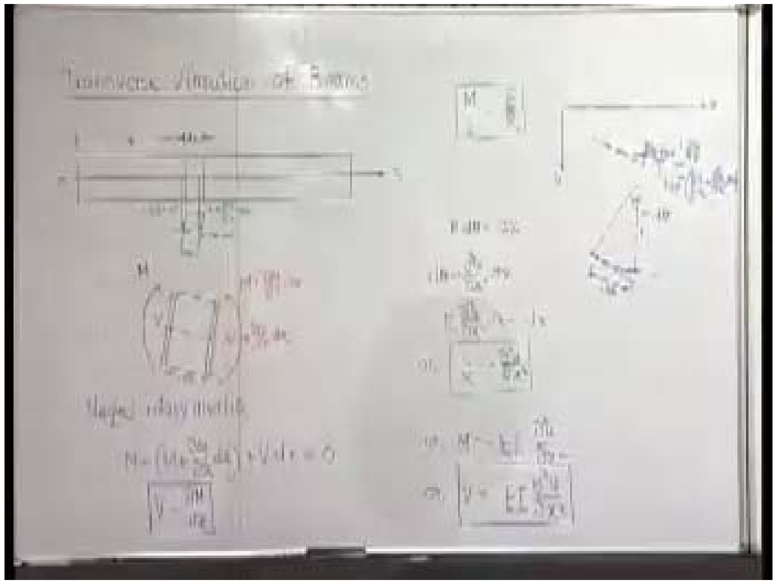


The next type of commonly used elements we find in machine with be beam. As I have mentioned, in case of beam, the direction of moment of the particles of the machine object will be perpendicular to the longitudinal axis of the element.

Next, we take up vibration, that kind of vibration is almost common in machines and structures. Therefore, we should pick up cracking more details. It is true that longitudinal vibration of bars etc., are important but in reality what happens, we get most of the time lateral vibration of the structure moment.

We come now to transverse vibration of beam and such problem, again as mentioned earlier, we will take up uniform beam with a straight and we represent this beam by this, its longitudinal axis is x, with origin at one end as before, now the displacement of the member or the element will be in the transverse direction. Let us consider one particular element. As we have been doing for the other cases, we will derive the equation of motion valid for this element. So, this element comes at sometime here. So that the space which is at a distance x from the origin is displaced by an amount u and the space which is at a distance x plus dx is displaced little bit more than the previous one. Obviously, it will be u plus del u by del x into dx, where dx is the distance between the two ends of this element.

Now, if we consider a free body diagram of this element. The two ends, now when a beam bends like this, the moment acting here will be M and bending moment not twisting moment in the other will be same as M plus little difference from this (Refer Slide Time: 35:04). Then, we also know that during bending, this phase will be subjected to a shear force, which is say V; the other will be also subjected to V plus del V by del x into dx. Now, all this quantities V, M, u they are all continuous functions of x and of course at each point there is a value depending on that x and subject to multiplied by a harmonic function of time. So that everything fluctuates harmonically with time.

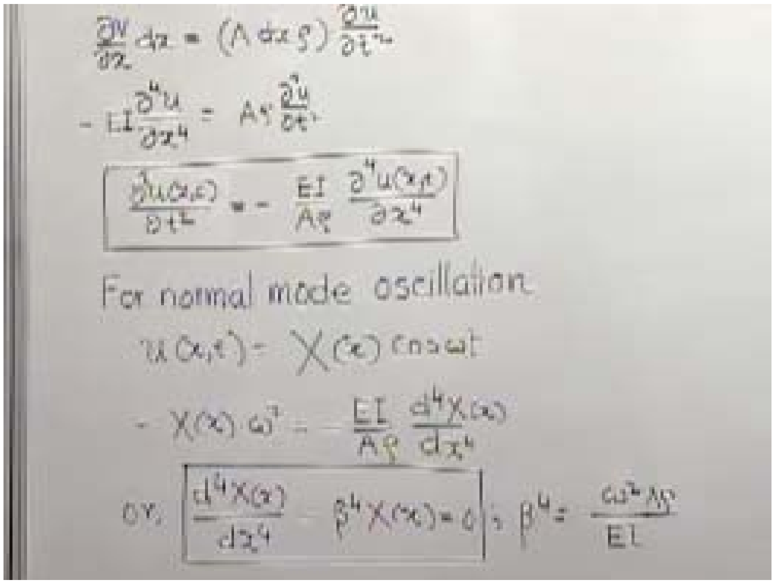
Now, let us now consider the relationships between various components. First of all, you see that the total moment we neglect rotary inertia. It means that in this direction rotation, first of all it is small vibration, so this u is very small. It hardly moves too much away from the original location and the slope etc., are all very small; slope of the bending. Therefore, the angular motion of the element is ignored. If this rotary inertia rotary motion is ignored, then the total moment which is subjected to must be 0. So, what is the total moment?

If we take the total moment in the clockwise direction it will be M minus anti-clockwise direction which is M plus (Refer Slide Time: 37:02). That is, the moment in the anti-clockwise direction plus there is a clockwise moment because of the two Vs. This is slightly different from V, but nevertheless the total moment produced by two parallel and opposite forces will be V into dx. This behaves like a couple with arm length dx. This must be equal to 0, because there is no rotary inertia and this gives us the shear force is equal to del M by del x. That is the relationship we get.

Now, still we have to get the relationship of this quantity in terms of the deflection. So, we know that for bending theory, beam equation says bending moment by second moment of area I is equal to E by R, where M is the bending moment; I is the second moment of area of the cross-section is equal to E is the of the module of elasticity of the material and R is the radius of curvature. Now, if we see the radius of curvature in terms of displacement, how do we find out? (Refer Slide Time: 39:15) At one point, here for example, the slope is, if this displacement is tan inverse del u by del x in this direction it is u. So, del u by del x is nothing but the slope of the tangent to the point to that curve, which is nothing but the central line. So, at this point, we can find out the slope and the slope will be tan inverse. Because you know this tan angle is equal to del u del x. If you go to this point (Refer Slide Time: 40:20) this angle will be equal to tan inverse, what will be the slope? It is del u del x and here it is plus del 2 u by del x square into dx, the slight difference.

Now, only thing we have to keep in mind that when we are taking this (Refer Slide Time: 40:55to 42:00) if we draw two perpendicular to a two different locations, this angle will be the difference between the slope at this point and this point. All amount of rotation of this because this is also perpendicular to this; this is also perpendicular to this. So, this is nothing but the difference in inflation of these two tangents. That is given by simply this one. Since, this is all small, so you call it delta theta. Of course, the two ends of the two elements are dx. So, we know that R into delta theta equal to dx; delta theta we can replace by d. Now, in terms of u, what we get here? Since these angles are all very small, we can always write tan theta is equal to theta. Therefore, the difference between this and this will be d theta will be del 2 u by del x square into dx. Only thing we have to keep in mind that, as x is increasing; slope is decreasing. So, this is definitely a negative quantity, because if you go to increasing x, del u del x decreases. Therefore, del 2 u and del theta square must be a negative quantity. Now, that you have to keep in mind. Finally, we have to shift that here.

Therefore, you will get minus R del 2 u del x square dx equal to dx or 1 by R equal to minus del 2 u del x square . So, you can write in this equation, using this is a standard equation and now you have got another relationship. Using this and this, we get M is equal to minus EI del 2 u by del x square. This is the positive definition of M. Therefore, to maintain this sine equality and this is always negative. That is why for positive M, del 2 u del x. Therefore, V will be del M del x. So, it will be minus EI del 3 u by del x cube.



Next, we come to again as usual to Newton's second law. So, total force acting on this in the transverse direction is V plus del V by dx minus V. So, resultant force is only del V by del x into dx. That will be (Refer Slide Time: 45:39 to 48:51) and this must be equal to the acceleration of the element in the transverse direction multiplied by its mass. Mass will be A into dx is the volume into rho that is the mass and del 2 u by del t square is the acceleration. So, using this del V del x into dx we get this expression. This is the equation of motion for the element. It depends on modulus of velocity, second moment of area I, A of cross-section and the density like this. So, for normal mode oscillation, we can take obviously as before that this is composed of function of X. When you substitute this here, we get minus x into omega square, cosine omega t gets cancelled from both sides, a final form you can write like this (Refer Slide Time: 48:51).

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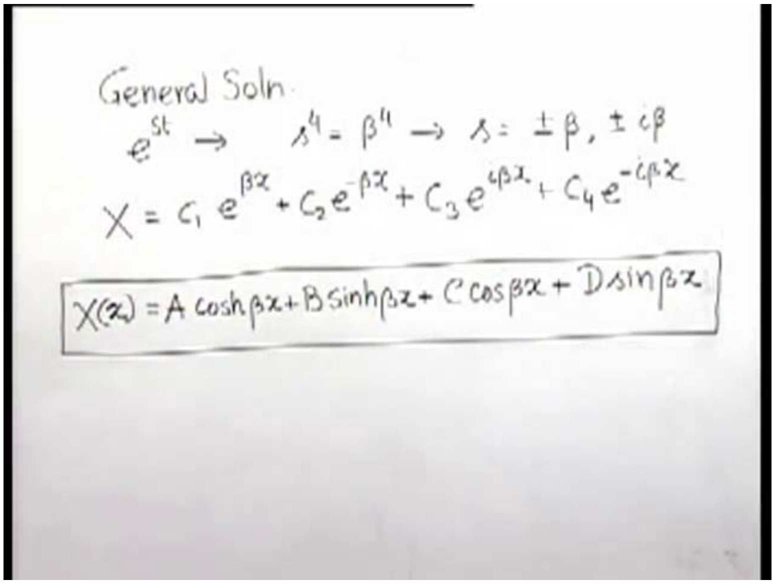

The general solution, what we can get, if you take e to the power st form solution. Then, that gives us the characteristic equation as x to the power 4 equal to beta to the power 4. So, this gives us s equal to either plus minus beta or plus minus i beta. There are four rules; plus beta, minus beta, plus ibeta and minus ibeta. Therefore, the solution will be of the form C_1 e to the power beta x plus C_2 e to the power minus beta x plus C_3 e to the power i beta x plus C_4 e to the power minus i beta x. Therefore, the solution will have both hyperbolic term and harmonic term. This will be lead to either hyperbolic function; this will be lead to harmonic function. Or we can write like this, A cosine hyperbolic beta x plus B (Refer Slide Time: 51:11) and the A B C D etc., including beta will be determined by the boundary conditions. In case of beam, boundary conditions will be four in number two ends; each end you have to tell either deflection or the slope. There will be two conditions or in the case where it is force is free, then force will be 0 and bending moment will be 0. We will take up this in the next presentation along with the solution of some problem.