Dynamics of Machines

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Module-13 Lecture-1

Vibration of Continuous Systems: Longitudinal Vibration of Prismatic Bars

Till now, we have been considering vibration of idealized system in the form of lump parameter bodies or systems where the inertia, the restoration, dissipation are all segregated and considered to be located in individual bodies. However in real life, the bodies are all continuous elastic bodies and I think lastly what we would like to do is to develop or to discuss something about the vibration of such continuous systems or elastic body.

The first thing, we have to keep in mind when we discuss the vibration of continuous system or elastic bodies, such bodies constitute infinite number of materials points or particles and the complete description of the configuration of the elastic body under consideration at any instant of time will require infinite number of coordinates. Therefore, you may also say that, such continuous systems or elastic bodies possess infinite degrees of freedom. So in one sense, we may also consider elastic bodies to be one extreme situation of multi-degree freedom only when that the multi becomes infinite. So, the real life system they constitute predominantly of these kinds of elastic bodies in the form of a plate or a beam or a column or a shaft or a bar whatever or in a rigid body. What we will do is we will try to consider in this limited time and scope, only few idealized bodies in the form of uniform bars, uniform shafts and uniform beams, we will avoid the discussion of plates and other kinds of higher level objects or complicated bodies. Our objective will be predominantly to discuss the ways and means by which such systems can be analyzed and to get some results of some simple cases.

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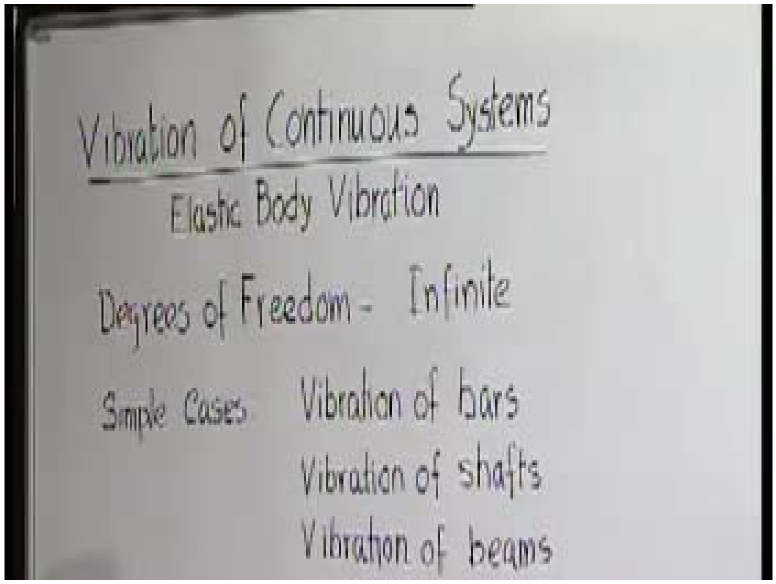

So simple objects or simple geometry will be… (Refer Slide Time: 03:30). What do we mean by a bar object like this of any cross-section which can be anything. Only thing it has to be uniform and the body as to be prismatic that means, it remains uniform everywhere. So, if you consider a particular plane here, this plane will oscillate like this and this motion can be described as a function of t, position of this plane in a coordinate system that means, this distance is x and time because, the position of the plane in its longitudinal vibration is continuously changing with time. So, the displacement of the plane from its equilibrium position which is shown in this line will be a function of x that is where this plane is located and at what time or what instant you are considering. Say for example: this maybe the plane here, may have a vibration which is somewhat different; maybe it vibrates with this magnitude.

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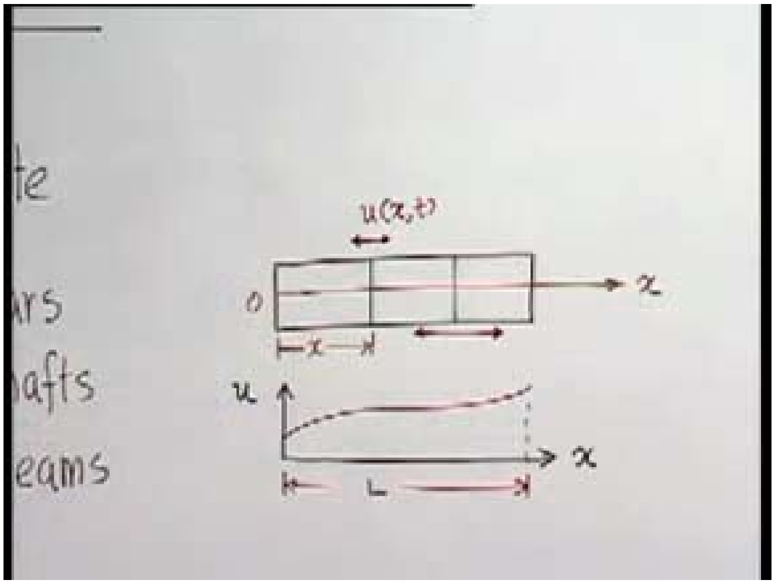

So, when you plot, the amount of deformation or magnitude of deformation you will find it may be function of the total length of the body. So therefore, what is happening that this bar all particles of course you have to consider only one-dimensional motion that means, all particles in a plane x is same, also will have identical motion. Therefore, the motion is purely a function of x coordinate and of course, the instant when its location is considered. So therefore, what is happening that the amplitude of motion of every plane depends where this plain is located. This kind of vibration that means, the motion or the displacement of the particles on the plane, are along the length of the object like this. This kind of vibration or oscillation is called longitudinal oscillation. So, this is the kind of motion, we will have in case of longitudinal oscillation of a body uniform oscillation.

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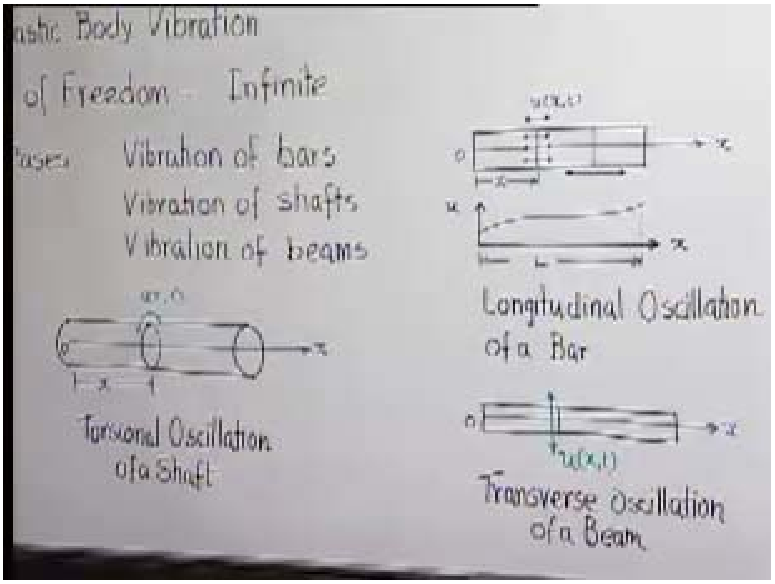

Next possibility is uniform shaft and so now any plane here which is at a distance x from the origin. Here, each particle motion will not be in this direction but in a direction that means which this circular plane will only rotate about its center which is here and this kind of motion, it will be purely angular motion about its equilibrium position and we may consider this to be theta which will be again a function of x and instant when we are considering. So, here the whole thing is moving in a direction which is perpendicular to this longitudinal axis and that is called torsional motion or angular rotation. Here also you will find that the amplitude of torsional oscillation of each plane depends on the location where it is located and amplitude will be a function of that and actual value of theta at any instant will also be (Refer Slide Time: 09:12). This kind of motion is called torsional oscillation of a shaft and finally if we have a body like this which keeps the coordinate system in this. Now, if we take up an element in this or a plane in this then each particle here all the particles oscillate in a direction which is perpendicular to this in the transverse direction. So, again now you see here the motion was along this x here, it is perpendicular to that in the transverse direction and this particular mode of the system is considered to be beam mode when the load etcetera all the movements are in the lateral transverse direction. Again here the magnitude of transverse oscillation amplitude depends on the position of this point and the actual value of the displacement depends also on the time and this kind of oscillation is called transverse oscillation of a beam. So, these are three types of vibration which we will discuss and what we will discuss here will be always a free vibration that means our objective will be predominantly to figure out or find out the natural frequencies and mode shape whenever possible. Secondly, we have to also keep in mind few things that means, this bar we are considering in this case is uniform and prismatic. The shaft we are considering is uniform straight and also circular and in case of beam it is again a beam with uniform cross-section. So therefore, the uniform bar, uniform shaft and uniform beam we will investigate. We will also keep in mind that all these systems there will be certain amount of damping involved but, we will ignore the damping. Finally, all the cases, we will consider the magnitude of vibration is small compared to the dimensions of the object. So, if always, we will consider small objects. So therefore, we start our discussion with the first case that means longitudinal vibration of a uniform bar. We have noted the point uniform, small oscillation, free oscillation and undamped.

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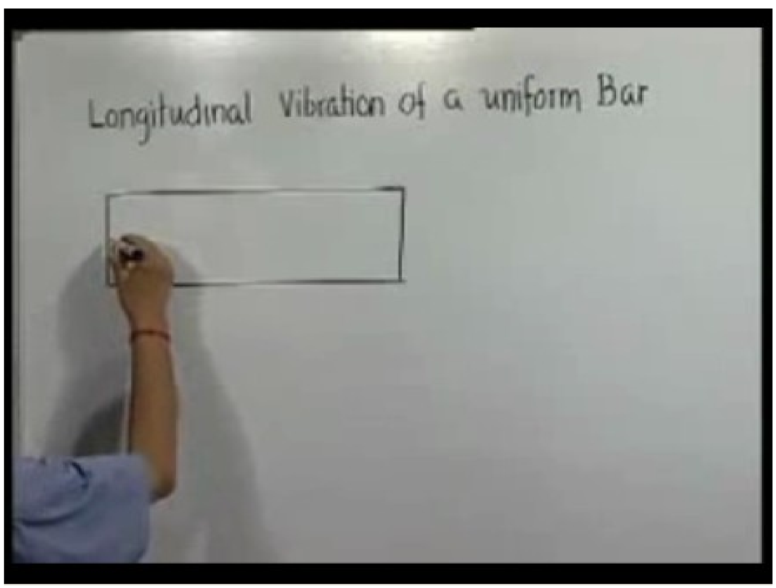

So, first case is the longitudinal vibration of a uniform prismatic bar. This is our uniform prismatic bar. Whatever may be the cross-section it is uniform and we now fix of a coordinate system attached to the body with one end as the origin. Then, we consider the technique of solving this kind of problems to be used is to identify an element at any general location x and find out what will be the dynamics of the element which is valid at any instant at any location. Once you find out the generalized equation of motion of a generalized element of the body that represents the motion characteristics of the whole system. So, we identify an element. Remember this element is infinitively small this is (Refer Slide Time: 15:21) P this point is Q; this point is R and S. The location of this plane, left end of the plane is at x and thickness of the element is dx is infinitively small, this is the equilibrium position. Now, at instant of time t each plane is oscillating. So, at instant of time t, let us consider this to be the location and similarly R prime S prime be the plane RS at time t and P prime Q prime is the plane PQ at time t. This is Q at x at time t. So, displacement of the other plane will be u into x t plus, now if the displacement u is a function of x then displacement at x plus dx will be displacement at x. Suppose here, we are finding the displacement here and displacement here dx, so it will be displacement here plus the extra amount of displacement which is nothing but the rate at which the displacement changes with x into that means that rate into this will be equal. So, in mathematical term what we write is du by dx into dx, so displacement at location x is ux, u so displacement at location x plus x is u plus rate at which the displacement changes with x into dx. So, the element now in this general condition, (Refer Slide Time: 18:28) this is the element at time t how much is now the length of the element?

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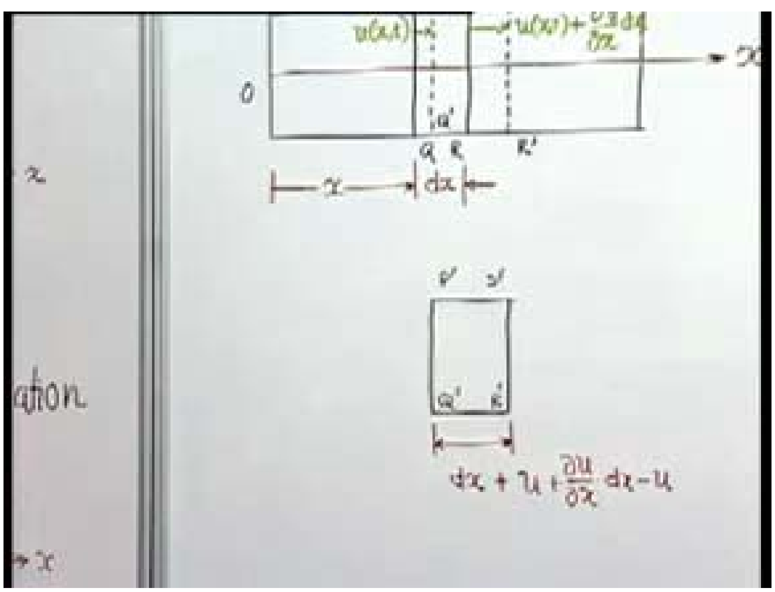

This length of the element is original dx of the element plus how much it has gone on this side that is: u plus del u by del x into dx that has gone this side minus this much. So, that is equal to original length of the element: x plus del u by del x into dx. So, originally the element had a length dx, now the element has length this much, so the strength of this element e is equal to the final length minus original length which is this one divided by the original length dx. So, strain in the element at instant of time t is give by this one. You all know that obviously length will be subjected to tensile stress, it is one dimensional phase let the phase at this be sigma x at time t then, phase here will be whatever phase is acting on this side plus the rate at which this phase is changing its location at any continuous function, if you know the value at x then value at x plus dx is generally this. So, we have done it for displacement, we can do it for phase, we can do it for anything. So therefore, the overall stress this is subjected to is sigma x on this side and sigma x plus an infinitesimally small quantity on this side.

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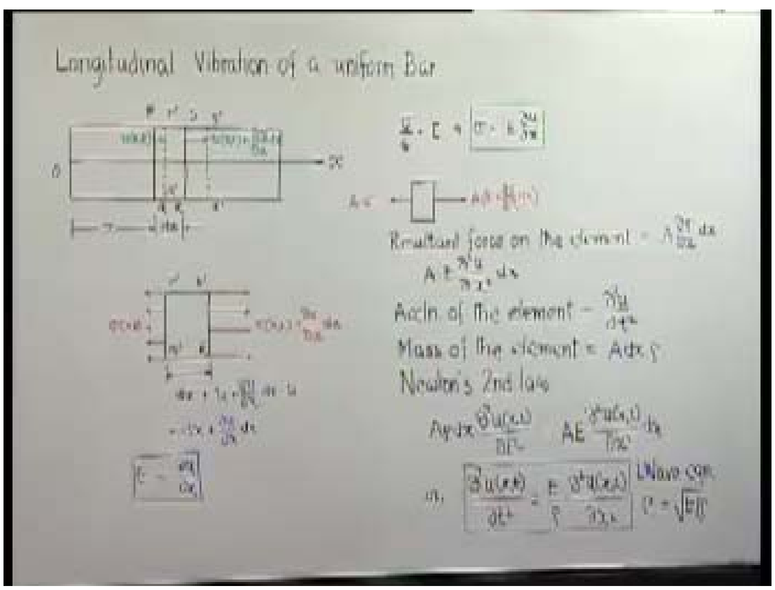

You may consider that, it is predominantly subjected to a stress of sigma x. If this strain is sigma x and this stress is epsilon then, what is the relation? You know that stress by strain is equal to E so this will give us that sigma will be equal to E into del u del x. So, if you now consider a free body diagram of the element in any instant of time, what is the force acting on this side cross-sectional area A into sigma and total force in this side cross-sectional area A into sigma plus del sigma by del x into dx. So, there is a resultant force in the positive x direction that is equal to force in this direction minus force in the opposite direction for what it remains is A into (Refer Slide Time: 23:17). Now, we have this relation, we have now expressed trace in terms of displacement quantities and materials quantities.

If you substitute it here, this is equal to A into E into d square u by del x square into del x using this relation substituting sigma here. This is the resultant force acting on the element. Motion characteristics of kinematics of the element it would be this end of the element is having an instantaneous position u what will be the instantaneous velocity of this end; it will be del u del t. Remember now it will be a function of time which is important to find out the kinematics. So, if the displacement is u, so the velocity will be del u del t and acceleration will be del 2 u del t square. This side also will be same plus an infinitesimally extra change because of this infinitesimally small distance dx. Overall you may consider that, the acceleration of the element is del 2 u del t square plus higher order infinitesimally term that term we ignored because it is higher order.

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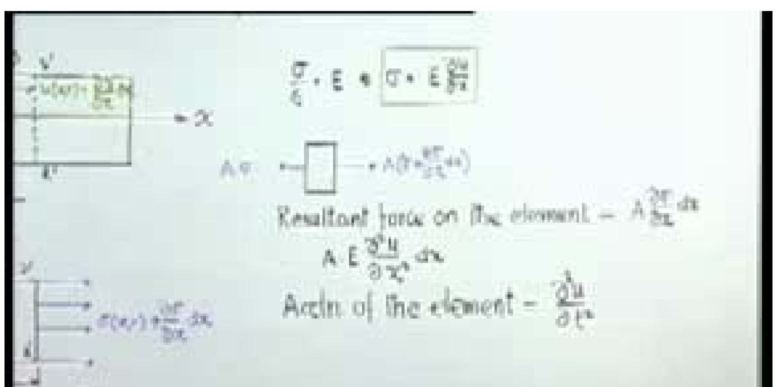

Now let us apply Newton's law. Newton's second law says this mass of the element which is in volume of the element that is cross-sectional area A into length this is the volume of the element and multiplied by the rho that is density of the material that gives the mass of the element. So, mass of the element into acceleration of the element must be equal to resulting force so Newton's second law when applied to this, we get mass of the element A rho into dx into acceleration of the element. This is the acceleration of the element must be equal to total force in the element which is A into E (Refer Slide Time: 26:52). So, finally what we get or (27:10) (Refer Slide Time: 27:40) this is the equation of motion of a general element anywhere in the bar. So therefore, since this is valid for the whole beam so this is the equation of motion of the bar. Now of course, we have to see how we consider the solution of this. This is known as the longitudinal wave equation where, the velocity of longitudinal wave is C and this is nothing but (Refer Slide Time: 28:24) so it is in the standard wave equation form del 2 u del t square is equal to C square into del 2 u del t.

So now let us consider longitudinal vibration but as I mentioned that, we are considering three vibrations and also we have to keep in mind, we are considering natural mode oscillation because, our objective is to find out the natural frequency. So, for a natural mode oscillation all the particles must move or oscillate with the same frequency omega. As we have done in case of multi degree freedom system here also it is the same thing that each and every plane is vibrating with the same frequency omega. Depending on their location they are either in phase or opposite phase. So, therefore each one with vibration can be represented by its amplitude multiplied by cosine omega (29:28). So, therefore this displacement or natural mode oscillation with a frequency omega or displacement which are harmonic functions of time will have same frequency. So, therefore we can write that, displacement u (x, t) can be represented by an amplitude of vibration which is a function of x only into cosine which means that, the amplitude of motion is dependent on x which is a function of x only and sample remains same it does not change its time.

So, only thing what is changing with time is which is represented by the harmonic function of time but the amplitude does not change. That means what we are doing is we are representing the displacement function which is dependent on position and time. As a product of two functions with one of them is a function of x only other one is a function of time. With this understanding which is valid everywhere for multi degree freedom, so it will be exactly valid in case of infinite degree of freedom system also.

Now if we substitute this in the equation what we get? Now the first time we are differentiating with time only it's a partial differentiation, so X remains outside minus and second derivative of this becomes omega square is equal to c square and that is also partial differentiation but with x. It will be differentiation of this function X with x into now since it is a function of X only differentiation with respect to x need not be partial or we write this equation in this form (32:50). So therefore, we get an equation for the amplitude of every particle or every plane. Solution of this, we know solving what we know, we did not do it again, we know that X (x) will be equal to C_1 cosine omega by c x plus C_2 sine omega by c x.

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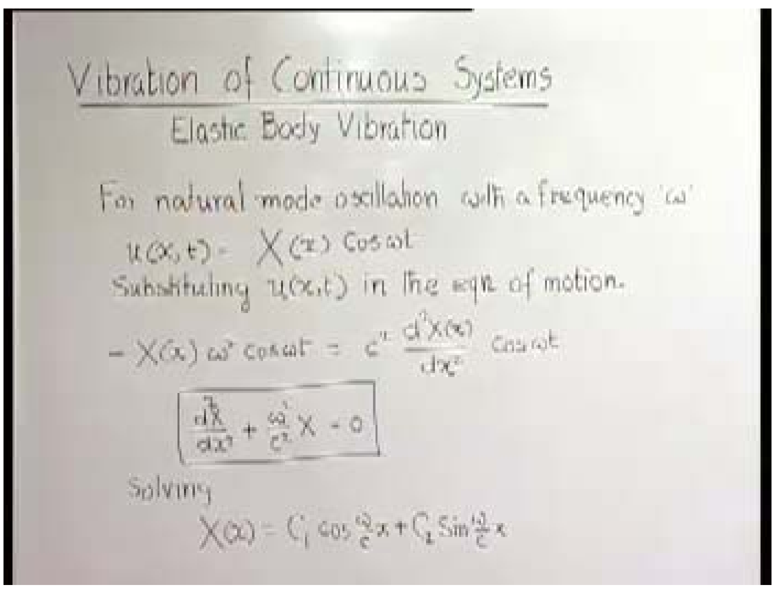

This is the general solution of the second order; we have already solved it number of times. Here this C_1 and C_2these two unknown quantities and omega, we will see depend on the longitudinal vibrations. So for finding out these quantities we have to next take up each and every different case. Therefore, let us consider the boundary condition. Now boundary conditions can be specified if the boundary is fixed. It means X equal to zero that means the displacement of that particular point which is fixed is zero because that is what the definition of (Refer Slide Time: 34:58). So, even if I multiply by cosine omega t at all times it remains free. What is the meaning of free? Free means there is no force acting on it, if there is no force acting on it means no space is acting on it that means sigma is zero. If sigma is zero then it means del u del x is zero and del u del x is zero at all time and if we use u as a product of X into cosine omega t, it will be del dx, dx is zero for all times that means multiplied by cosine omega theta, so this condition has to be satisfied.

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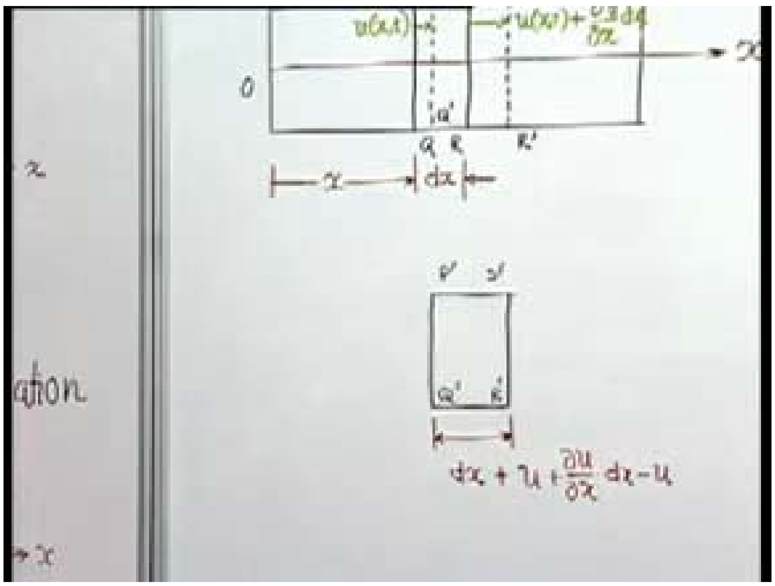

That means it will be space free all the time and X will have to be zero if it is fixed (36:06) these are the two things on geometric boundary conditions. Sometimes there can be other kinds of boundary condition maybe you attach a heavy load here. So, what happens is the whole thing is oscillating along with that heavy load then this end will not be force free. It will be subjected to D Alembert’s principle subjected to the inertia force of the block which is attached to the (Refer Slide Time: 36:29) so that has to be calculated and such problems we will take up as examples. We have to keep in mind that the primarily the conditions which can be subjected to or can be imposed on this beam (36:42) but either is equal to zero or dx is equal to zero. So, let us take up cases one by one. So, now first case is free-free bar which means a bar of length L both ends are free the material property l, modulus of elasticity is E and density is rho. So, solve the longitudinal vibration problem and find out the natural frequency. Remember this all the time that for any free vibration problem it is the natural frequency which can be determined but, the amplitudes cannot be determined it depends on the initial condition how you start and our objective is (Refer Slide Time: 38:04). So therefore, if we take you know that vibration will be given by u (x, t) is nothing but the function of x into cosine omega t for natural mode oscillation. For X we have X(x)= C_1 cos omega/c x + C_2 sin omega/c x, dx where c is equal to square root of E by rho we remember from physics for the longitudinal wave equation this C is the velocity of propagation of the new wave form. If we apply what are the boundary conditions? Boundary condition if it is at x equal to zero we know it is free that is dX by dx zero at x equal to L again dX by dx so both ends are free.

If we apply x equal to zero dX by dx is equal to minus omega by c, C_1 and this is zero and that is zero means C_1 equal to zero because dX by dx at x equal to zero is omega by c into C_1, omega is not zero c cannot be zero capital C_1 as zero. The second boundary condition if we apply what we find is omega by c, C_2 sine omega L by c that is the second boundary condition x is equal to l. C_2 cannot be zero if both C_1 and C_2 are zero then there is no vibration frankly speaking, x is equal to zero if x is equal to zero there is no vibration that is not possible. Omega is not zero, C is not zero the only possibility for which a vibration may exist but, these conditions are satisfied is sine omega L /c =0 that tells us that sine omega L /c has to be 0 or that tells us that omega by L by t has to be integer multiple of y.

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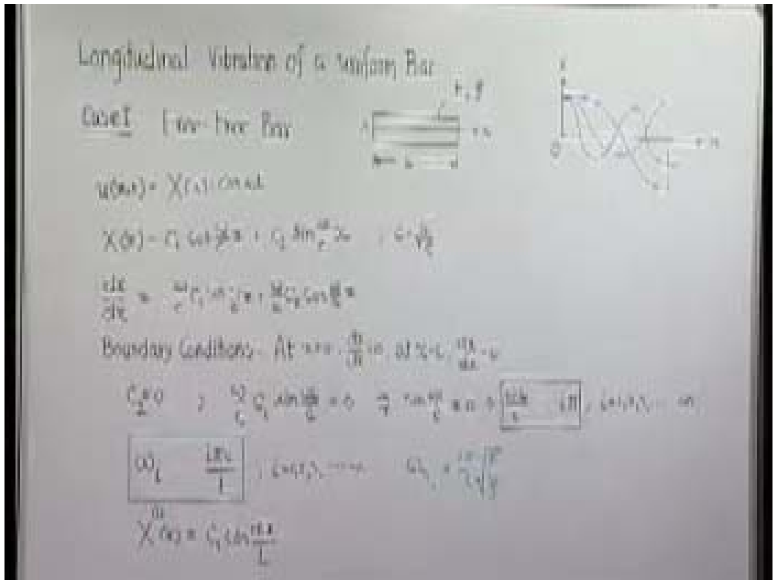

y is equal to 1, 2, 3 etc. So therefore, we find out that this will be possible that is by the equation etc if and only if the natural frequencies are according to i pi e by l. So, there are infinite number of frequencies for which this condition is satisfied and that is called the natural frequencies. These are the natural frequencies, c of course is nothing but square root of (Refer Slide Time: 42:57). So, you can write separately natural frequency of the ith mode is equal to i pi by l square root of t by rho. There is infinite number of natural frequencies because it has infinite number of degrees of freedom but, the first natural frequency is pi by L into square root of t by rho. The mode if you want to find out mode means x in the mode shape. So mode shape if you want to find out first mode as a function of x will be: mode is nothing but the pattern of deformation that we have seen in case of finite lump bodies it used to be ratio between the finite amplitude. For an elastic body, it will be the continuous function x as a function of (Refer Slide Time: 44:02). So this will be C_2 sine omega by c is i pi into x by L, so this will be the ith mode. So therefore, if we want to plot the mode shape…, (Refer Slide Time: 44:34). Now remember the displacement is not in that direction it is only I am graphically plotting, displacement is also in this direction of x. So, one as to be careful of not being confused that the actual displacement in this mode. The first mode says it will be sine pi x pi l.

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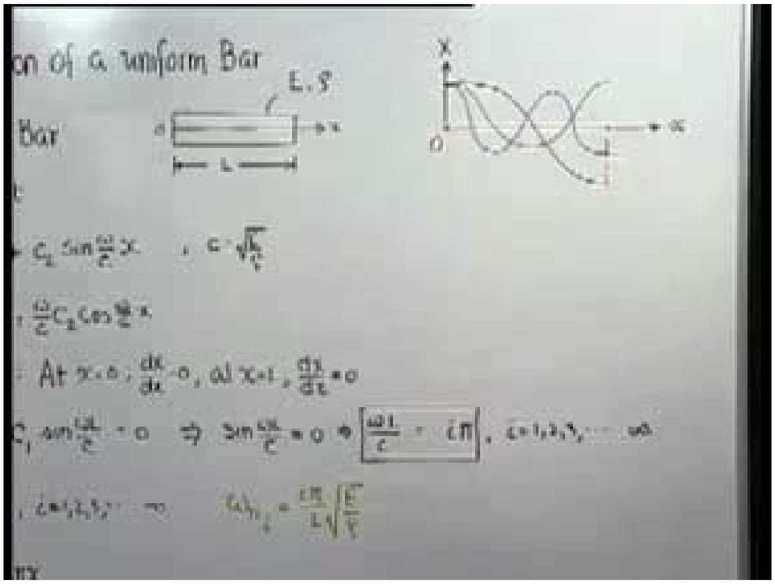

So, C_2 will be zero and the mode shape will be, C_2 is zero, C_1 cosine i pi. Now so we start from here this will be the curve. Second mode will be this; (Refer Slide Time: 46:33) third mode will be this. So therefore, we will find that first, second, third mode in all the cases the boundary condition have to be satisfied and is given by this one. But as we will see later that, mode shape cannot be found out analytically in most of the cases (Refer Slide Time: 46:57) we will see that later. If the object or the beam or bar is fixed at one end and free at the other end then what will be the every second (Refer Slide Time: 47:12). Case two: at one end it is fixed to a wall and other end is free, so boundary conditions will be at x equal to zero, X will be equal to zero at x equal to L. Now we know that X (x) is C_1 cosine omega/c x + C_2 sin omega/c x and we already know this (Refer Slide Time: 48:19) we should not make the same mistake. When we use the boundary condition for x is equal to zero this is equal to zero, this will give us C_1 equal to zero. Now for x equal to L we have to use this that means this is zero when you substitute x=L (Refer Slide Time: 49:04). Now C_1 is already zero. So therefore, this will be here omega by c into cosine omega by c L equal to zero. Now the same logic we cannot have C three equal to zero then there is no vibration.

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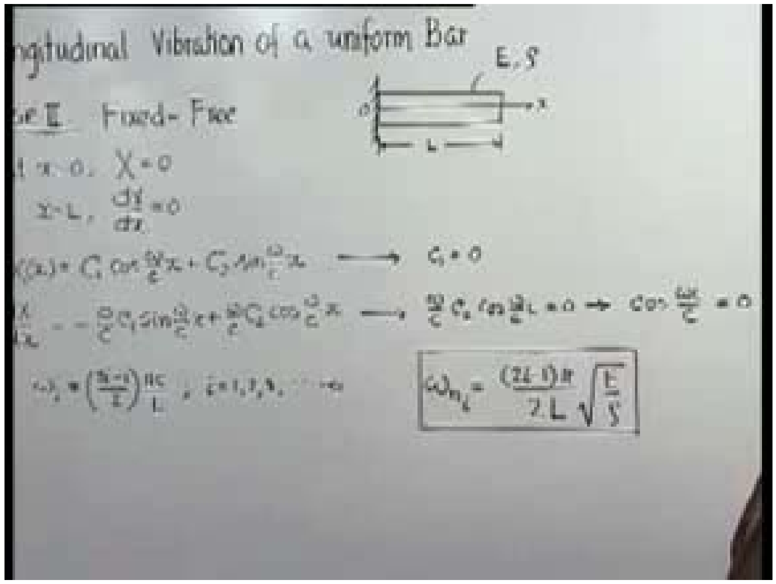

Omega cannot be zero, C cannot be zero only possible situation which is permitted is cosine omega L and we can say that, omega$_i$ in this case, this will be zero that means a integer multiple of omega L by c will have to be integer multiple of pi by 2. It will be two i minus one by i where i varies from 1, 2, 3 so on. So, i is equal to 1 which is i C by two L that means omega L by C will be pi by two for pi by i equal to two, it will be 3 pi by two and so on. So; the natural frequency of the ith mode with one end fixed and one other end free given by two i minus 1 by 2 into pi by L square root of E, so that would be case for natural frequency and the mode shape if you want to plot will be given by C_1 is zero it is C_2 sine (omega by c is) pi t, (Refer Slide Time: 52:08). So, x is equal to zero, it will be always zero and x is equal to L will be again quantity which is integer multiple of pi by two that means either plus one or minus one. The first mode will be something like this; second mode will be something like this; third mode and so on. Again I remind you that this kind of very analytical closed form solutions (Refer Slide Time: 53:06) for the mode shape is possible only with limited values. (Refer Slide Time: 52:11). So, I think you can see and you can find out the natural frequencies in the other types of the boundary condition that means, you can make both end fixed.

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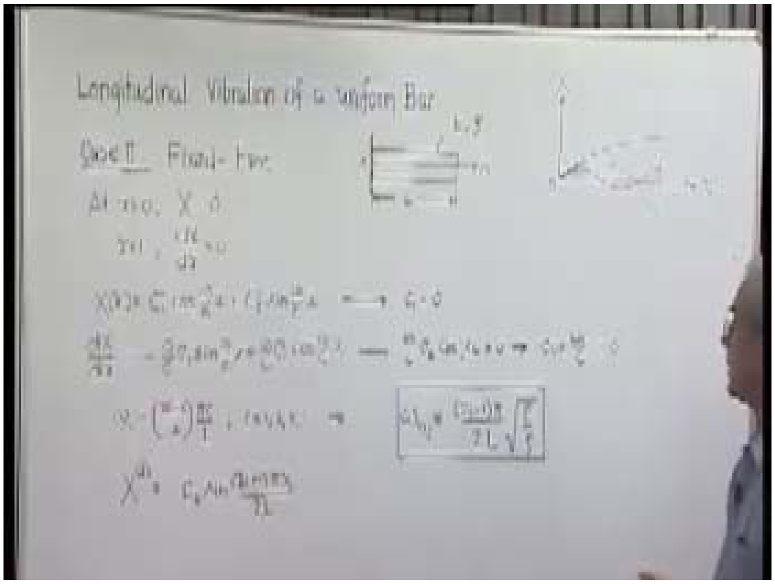

One thing to be noticed here is that, in first natural frequency of this one end fixed one end free is how much (53:38) is fixed free and both ends free first natural frequency was (53:55). It can be physically justified that why it is just half in case of fixed free. The reason is this you could imagine a free case like this (Refer Slide Time: 54:18) if you take a free-free beam, remember that, if we take a uniform beam of length two L, its natural frequency will be same. Why, because the midpoint of this, if it is a free vibration midpoint which is representing this center of mass of the beam cannot shift position. So therefore, it is obvious that for free oscillation the midpoint of the uniform bar will remain fixed. So therefore, whether you fix it like this or you make a bar of length 2L and allow it to vibrate freely this point will remain fixed. (Refer Slide Time: 55:06). So, the motion of this half and this are same. Therefore, the natural frequency of this is nothing but the natural frequency that is first mode of a bar of length two L and so therefore you can see it is this (Refer Slide Time: 55:24). So, that is the physical justification (Refer Slide Time: 55:30). So we will take up other types of simple bodies like shaft in the next presentation.