Dynamics of Machines

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Module - 12 Lecture - 5

Free Vibration of Multiple Degrees of Freedom System; Approximate Methods

We have seen that determination of the natural frequencies and the natural modes of multi-degree freedom systems are quite computationally intense, particularly when the number of degrees of freedom increases. In many situations, the designers will be happy to have some approximate idea about the fundamental frequency, that is, the lowest natural frequency.

In this lecture, we would like to present techniques of quickly and approximately finding out the lowest natural frequency. There are quite a few methods and we would like to discuss two methods: one method, which will give us the lower bound, that means, the actual natural frequency of the first mode will be higher than the value we find; another method by which we will get the upper bound, that means the value of the actual natural frequency will be lower than the approximate value. Therefore, if we apply both the methods it will be possible to get quickly a band and the actual natural frequency will be in that band or that range and that can sometimes be very useful in the designing stage. The first method which we will discuss is called Dunkerley’s Approach.

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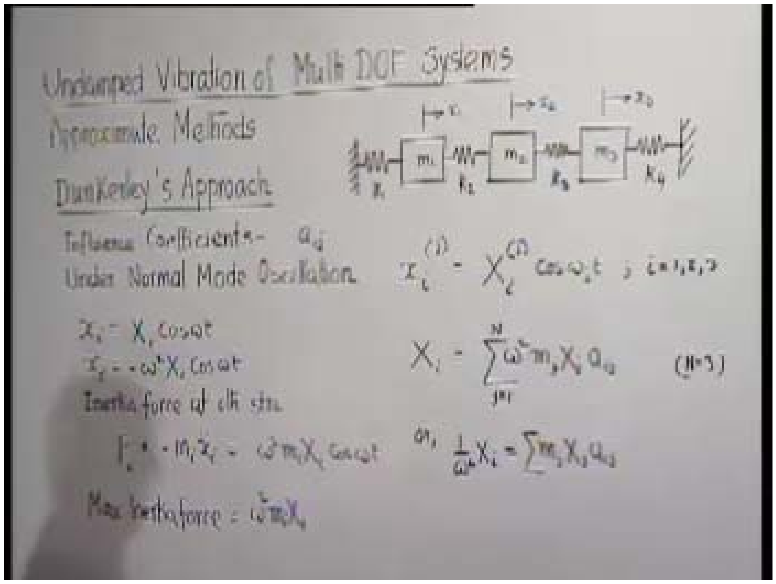

We will take up a particular case as an example, but whatever we discuss and derive will be quite general as you will realize. Let us take this 3 degree freedom system. For this particular system, for example, we can find out the influence coefficients. Now one thing is perhaps clear to you - that there will be two kinds of equations while finding the flexibility matrix and using the influence coefficients upwards. The cases which we solve like this, masses are there, but the other end is free. Those cases are relatively simpler for finding out the flexibility matrix because they are statically determinate. We can just apply the load anywhere, immediately we know the load under which each individual is being subjected. But a case like this when this end is also connected it becomes statically indeterminate. Still, it is possible to find out the deflection by applying unit load, only thing that it is slightly more involved, and therefore, I think one has to be careful.

The influence coefficients are a_{11} or a_{ij}. Now, we know that under normal mode oscillation, each coordinate or displacement can be written as x_i and say the mode is jth mode is equal to X_i jth mode cosine omega$_j$ t, which means, all i varying from 1, 2, 3 for the three stations and j is a particular natural mode. So, in this case, we will be mostly concerned with the fundamental frequency in the approximate method. We may consider this to be 1 and now onwards for the sake of simplicity, we will not put this as first, but we will understand that what we are writing is represented as first natural frequency or the first mode of oscillation. So, we can write that x_i is equal to capital X_i cosine omega t. Omega, obviously, is then the first natural frequency because that is our concern; x two dot_i will be minus omega square X_i omega t and the inertia force at each station is F_i is equal to, [this i we should not substitute, so we call it F; F] will be minus m_i x_i two dot at the station I, which will be nothing but omega square m_i X_i cosine omega t. Now, we also know that all the stations, all the masses reached their extreme position either way at the same instant when their velocities become 0 and of course acceleration becomes maximum.

So, maximum inertia force equal to omega square m_i X_i. At this instant, the deflection at the ith location will be due to all these inertia forces, because, there is no other external force acting. Therefore, it will be omega square m_j X_j a_{ij}, this is the maximum inertia force at this instant at the jth station and the deflection at the ith station due to this force at jth station is multiplied by a_{ij}. This we sum up from j equal to 1 to N; this N is the number of degrees of freedom. In this particular case, N is equal to 3 or this we have done (Refer Slide Time 09:19). The equation will be this (Refer Slide Time 09:40)..

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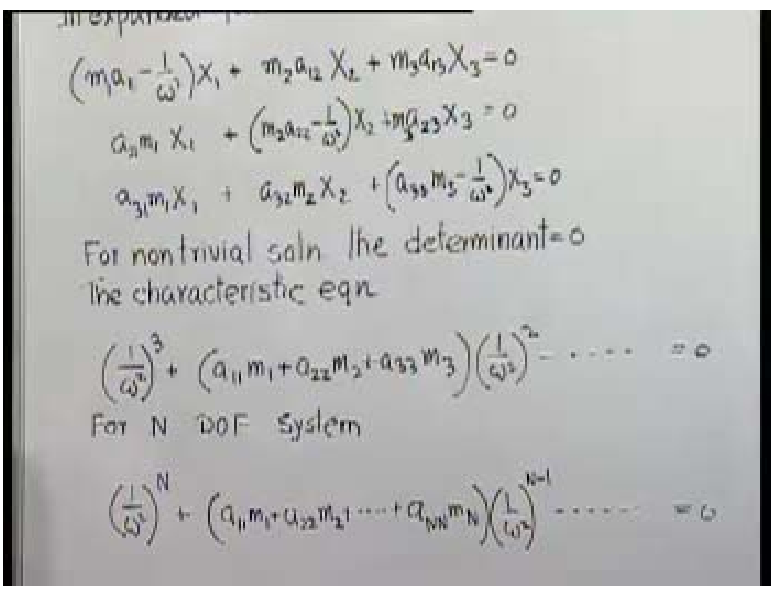

So, for three cases N is equal to 3, the equation in expanded form m_1 a_{11} minus 1 by omega square (Refer Slide Time: 10:39). It is the same set of equations which we have derived earlier and you all understand now that a non-trivial solution will be possible with determinant being 0. So, for non-trivial solution the determinant will be 0, this we have done.

So now, if you write the determinant the characteristic equation becomes like this (Refer Slide Time: 12:27 min). It becomes 1 by omega square power 3 plus a_{11} (Refer Slide Time: 12:33); you need not go to the higher modes. So, for the general case, N degree of freedom system, this will become 1 by omega square power N (Refer Slide Time: 13:30); we have not derived it, we have just written the calculation, but one can easily do the whole calculation for N degree of freedom system case and the characteristic equation for the first two terms will be like this. Solutions of this equation are the natural frequency of the system. So, if the solutions, (Refer Slide Time: 14:18) which represent the natural frequencies be omega$_{n1}$, that is the first natural frequency; omega$_{n2}$, that is, the second natural frequency and so on; omega$_{n3}$ is the third natural frequency in this particular case; in general, it will be then these are the routes of … that means, the 1 by omega$_{n}$ square, 1 by omega$_{n2}$ square, 1 by omega$_{nn}$ square these will be the routes of this characteristic equations.

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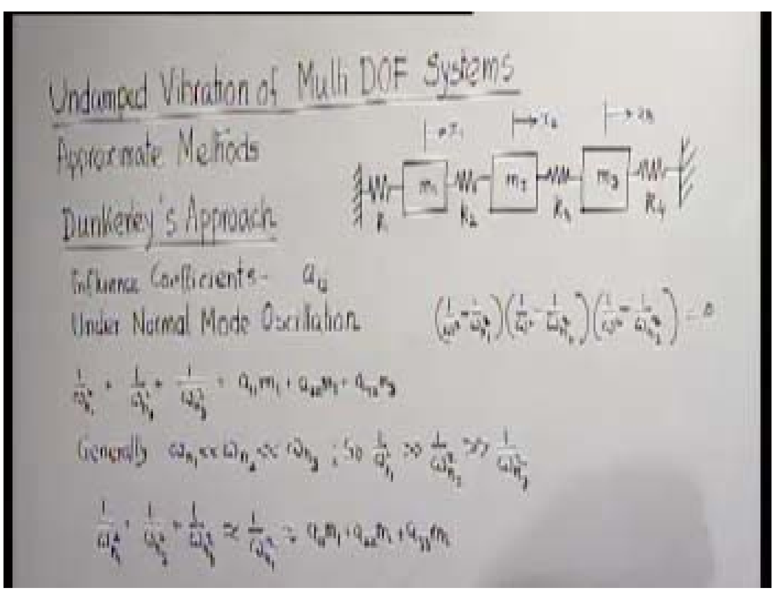

We already know that sum of the routes; we can write them like this: 1 by omega$_{n1}$ square plus 1 by omega$_{n2}$ square plus for this one it will be omega$_{n3}$ square will be equal to this (Refer Slide Time 15:55). This is the known result of algebra, but one can easily do this by this technique. That means this equation for 3 degrees of freedom then it becomes 1 by omega (Refer Slide Time 16:30). Now, this is an equation of the same order and if we expand it, the routes of this equation are very obviously this, this and this. Therefore, this equation is same as the characteristic equation for 3 degrees of freedom system; that means that equation will be same as this (Refer Slide Time: 17:10 min). When you compare these two equations, you will find that first term will be the same and this second term, whose power is 1 by omega square to the power 2, its coefficient will be this; that you can see easily. From that the coefficient will be this and that equation coefficient will be this (Refer Slide Time: 17:31 min). [Therefore, the… will be…].

Now, comes the main logic of Dunkerley's equation. Generally, omega$_{n1}$ much less compared to omega$_{n2}$ which is again much less compared to omega$_{n3}$; generally, they are gradually increasing. So, 1 by omega$_{n1}$ square is much more than 1 by omega$_{n2}$ square. So, this is the largest term when compared to these. These are approximately equal to this (Refer Slide Time: 18:41); that means, these two terms can be ignored since they are much smaller than this. This, of course, can be written like this (Refer Slide Time: 18:55).

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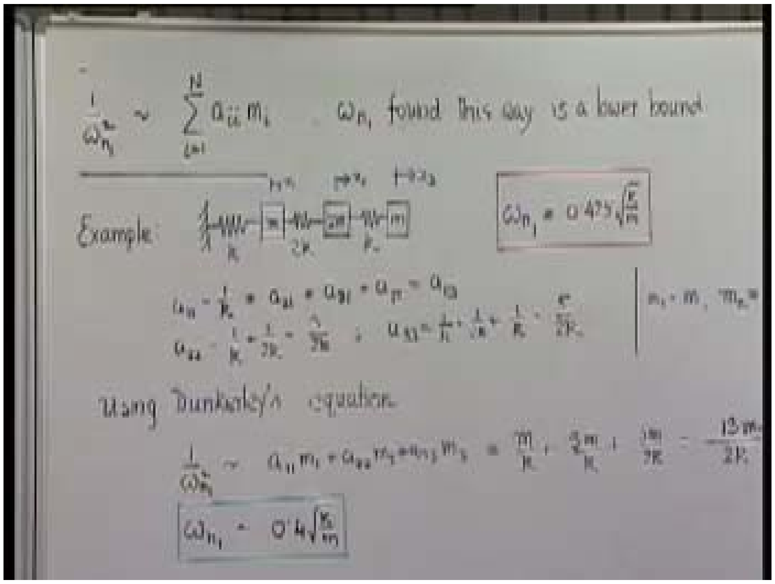

For a general case, Dunkerley's equation says that the fundamental natural frequency, that means, 1 by omega$_{n1}$ square is approximately equal to sigma$_1$ a_{ii} m_i I equal to 1. From this, we can calculate the approximate value of fundamental natural frequency. Now this value is going to be what, upper bound or lower bound? We can see here that, of course, this 1 by omega$_{n1}$ square is less than this, because, two terms we have ignored; we have assumed this whole thing to be 1 by omega$_{n1}$ square. So the real omega$_1$ by omega$_{n1}$ square, actual value will be less than this. When you take the inverse, the actual omega$_{n1}$ square will be more than the value of omega$_n$ squared what we get from here; that means it is nothing but a lower bound (Refer Slide Time: 20:45). We can say that the real natural frequency for the first mode is never going to be less than this. It will be always more than that and approximately equal to this.

We can apply this to our one example we have taken. Let us see that how we get that. Let us take the same problem which we have already solved; then you can compare the results. This is the problem which you have solved and its first natural frequency is this. If you remember it is equal to this; we found out by matrix equation method and which if it is done accurately, it will give an exact answer. Let us apply this: this is x_1, this is x_2 and this is x_3 (Refer Slide Time: 22:35). So what will be a_{11} we have already found out earlier, but let us find again. a_{11} is the deflection of session one when unit load is applied to session one itself which is nothing but the stretch of this. (Refer Slide Time: 22:55) This will also be the deflection at session two due to a unit force at one, also deflection at … because this shift just like a rigid body. Again, we know that deflection at here due to unit load here and here are going to be same. That is why, actually the flexibility matrix [here]. a_{22} is equal to stretch of this string plus stretch of this string that will be the deflection here if a unit load is applied at station two. This is 1 by k plus 1 by 2k and 3 by 2k equals a_{33} because we will not require others; we will require only a_{11}, a_{22}, and a_{33}. This will be 1 by k plus 1 by 2k plus stretch of this string 1 by k equal to 5 by 2k.

So by Dunkerley's approach (Refer Slide Time: 24:19) which is this - 1 by omega$_{n1}$ square is approximately equal to a_{11} m_1 plus a_{22} m_2 plus a_{33} m_3. We also know that m_1 equal to m, m_2 is equal to 2m and m_3 equal to 3m. So this will be equal to a_{11} is 1 by k into m which is simply 1 by k; a_{22} is 3 by 2k and this is 2m, it will be simply 3 m by k plus a_{33} is 5 by 2k and m 3 is m, it will be 5m by 2k and this is equal to (Refer Slide Time 25:54). If we find out omega$_{n1}$ from this we will get square root of 1 by 6.5 (Refer Slide Time 26:40). When you compare this result, the actual value is here. We find that this is slightly lower than the actual value, but we find it is quite close; it is 0.4 here and it is 0.425 here; that means the difference is in the second and that too not much. You can calculate the percentage error and also that point is revealed here that it is a lower bound. That means the real value is above. You can see that this is a very quick method of estimating a very approximate value of fundamental frequencies of a system, because, calculating the influence coefficient, it is generally very simple if the case is statically indeterminate like this. Then also it can be done; only it will involve little bit more analysis, not very complicated. Mass matrix is a diagonal matrix and finding the mass is very simple; just by observation and calculation of this term is also a straightforward calculation. Thus, we can solve or get the approximate value very quickly. Here, the advantage may not be very clear, but if you have a 10 degree freedom system, you will find that this is a very quick method compared to the earlier one.

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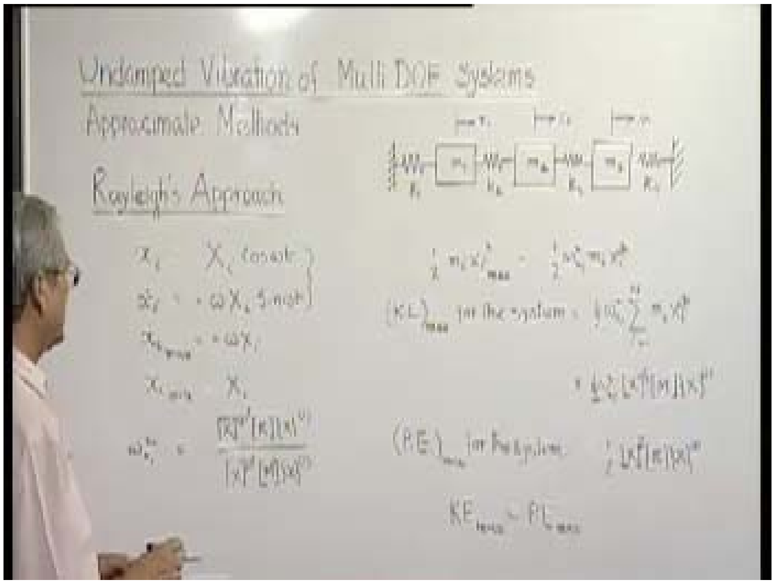

Now let us take up the other method. This is an extremely important approach and we will refer to it at a later stage also, but understand the case here. Now, under normal mode oscillation condition, we have seen that all masses are oscillating with the same frequency and almost they are either same phase or exactly opposite phase. It means that each mass attains its maximum speed at the same instant of time which is the maximum velocity of it. If the normal mode can be represented like this (Refer Slide Time 29:44), then x_i dot is velocity omega X_i sine omega t and this maximum value of this t dot of each one is x dot_{imax} is omega X_i is this, because maximum value of sine omega theta will be only 1.

Similarly, all these masses attain their extreme position which represents the extreme stretching of all the elastic members. Or if there is gravity then the extreme position in the gravitational potential energy as well. Therefore, we know that extreme positions are all equal and they happen at the same instant and the values are nothing but the capital X_i. So, that is also quite straightforward, because, the maximum value of the kinetic energy is 1.What will be maximum kinetic energy of the whole system? The maximum kinetic energy of ith mass will be half m_i into velocity is going to be half omega square m_i X_i square. So, for all the total kinetic energy of the system will be just sum of all this (Refer Slide Time: 31:45), it has N degrees of freedom or N masses. This can be in matrix form.

It can be written like this (Refer Slide Time: 32:17). This is a straightforward demonstration of this and I am avoiding it to find out that this is nothing but this (Refer Slide Time: 32:43).

Similarly, maximum potential energy of the system, each one will be stiffness matrix. If stiffness matrix be k (Refer Slide Time: 33:25), then this can be also easily seen through a little bit analysis that total potential energy maximum for this stiffness. We know that for a conservative system where there is no dispersion and now when the kinetic energy is maximum; the potential energy is 0. Similarly, when the potential energy is maximum; the kinetic energy is 0. Therefore, both of these two must be equal to the total mechanical energy of the system and so they must be same; that we have used even in the solution of single degree freedom.

Remember, all these things that we are doing are not mentioned during the case of normal mode oscillation, then only this is valid; then only they all will be attaining 0 velocity and maximum velocity at the same instant. Though we are not showing any i here as in particular mode, this is only valid for natural mode oscillation. So if it is natural mode we can show like this (Refer Slide Time: 34:45). Therefore, for the first mode $omega_{n1}$ square can be expressed as this (Refer Slide Time: 35:30) because then it will be natural mode technique. Now, Rayleigh's principle says that if we assume column matrix X as the first natural mode and evaluate this quantity we will get a quantity which we call as Rayleigh's quotient.

Rayleigh's quotient R is for an assumed mode say, X transpose k X which need not be exactly same as the first mode and divide this by this quantity (Refer Slide Time: 37:01). Then R tends to $omega_{n1}$ as X approaches first mode. The most important thing is it will approach from the higher side and when the assumed x becomes identical with the actual first mode then R becomes equal to $omega_{n1}$. So for an assumed shape X, whatever value of the Rayleigh's quotient we get, it will be always slightly higher than the real natural frequency. Of course, I must say R is not [38:15]. So, determination of fundamental frequency using Rayleigh's principle will always give us an upper bound.

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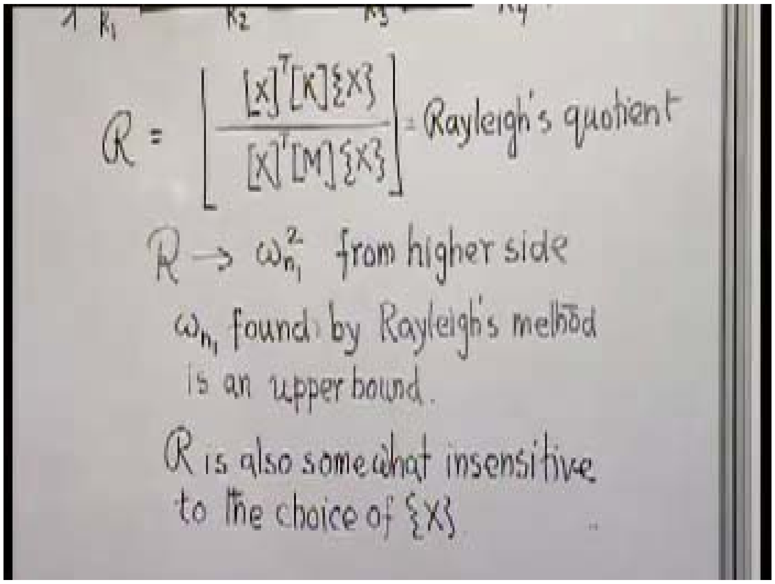

There is another very important point to be noted here is that the value of Rayleigh's quotient as it is nearer to omega$_{n1}$ is somewhat insensitive to the choice of X, that means, even if the choice of X is somewhat different or quite different from the first mode, the value of this quotient will be somewhat nearer to this. That means a large error in X will not be reflected as a large error in this natural frequency. Therefore R is also (Refer Slide Time 39:54). Now, let us solve the same problem using Rayleigh's.

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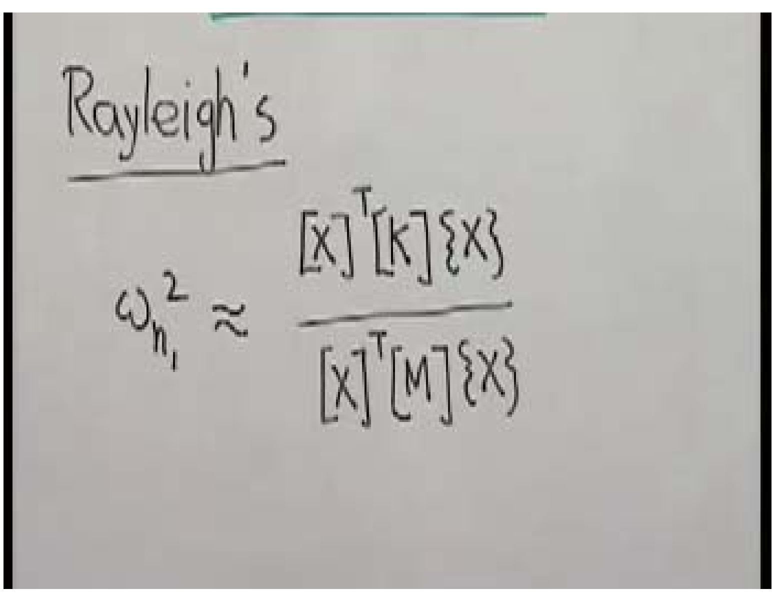

So Rayleigh's technique says [40:36] (Refer Slide Time: 40:48). So, let us find out what we get for solving the same case. As you can see, we will need the k matrix and the m matrix; m matrix is quite straightforward.

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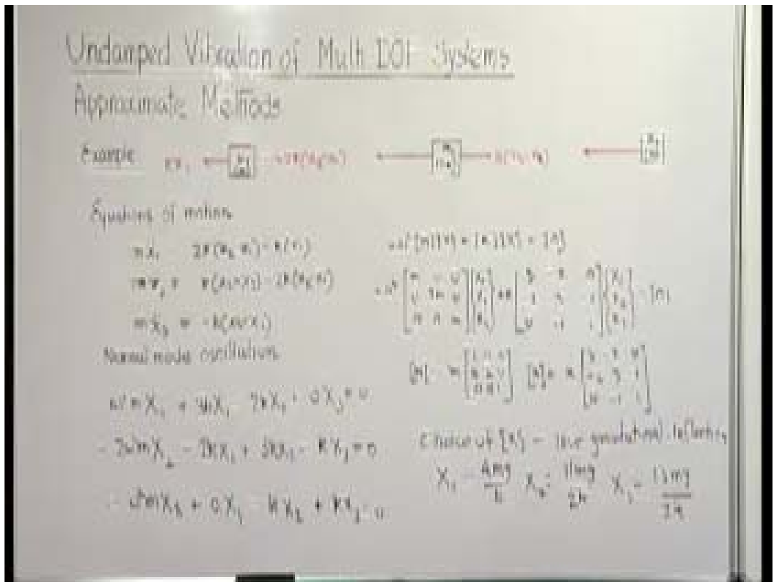

Let us solve the same example. To find out the k matrix, best way for us will be to derive the equations of motion. So, for mass one force acting are 2k into x_2 minus x_1, for mass one it is m; for $mass_2$ it is 2m; this force will be same as this and for $mass_3$ it is again the same. The equations of motion can be written. For the three masses it will be m x_1 two dot equal to 2k into x_2 minus x_1 minus k into x_1; 2m x two dot. The second mass is k into x_3 minus x_1 minus 2k into x_2 minus x_1 and for the third mass it is this (Refer Slide Time: 44:33). So, now we rewrite them in a manner and for normal mode oscillation. [So that we can….]. (Refer Slide Time 44:53) omega square m X_1 plus 3k X_1 minus 2k X_2 plus 0 X_3 is 0. This one will be 2 omega square m (Refer Slide Time 45:51), here, this X_1 term will be plus k X_1 plus X_2 term will be 2k X_2 and k X_3 (Refer Slide Time 46:35) and third one will be minus omega square m X_3, (Refer Slide Time 46:54). There will be no X_1 term; so 0 $X_{1,}$ for X_2 it will be minus k X_2 and for X_3 plus k X_3 and so we are not interested.

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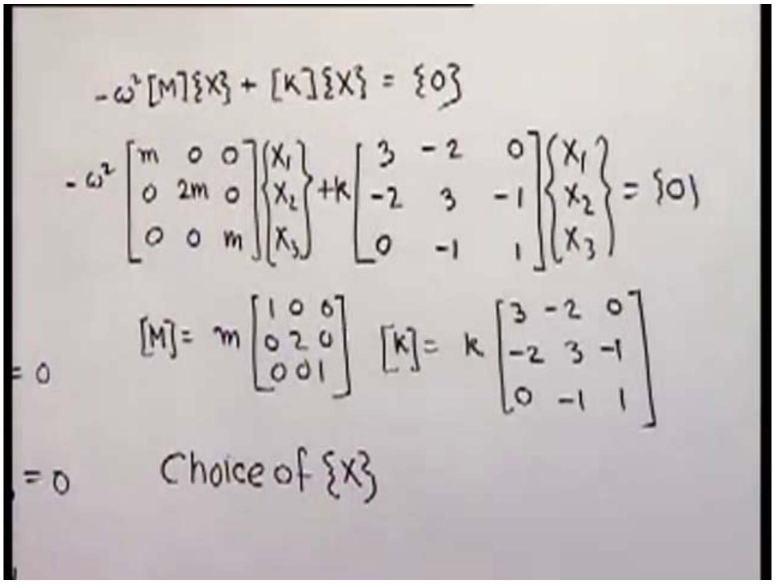

This becomes minus omega square m X plus k X equal to 0. This is the equation in matrix form. So, obviously, this is nothing but m 0 0; 0 2m 0; 0 0 m into X_1 X_2 X_3 plus k matrix: 3, minus 2, 0; minus 2, 3, minus 1; 0, minus 1, 1; m matrix is this (Refer Slide Time 49:36) and k matrix is this (Refer Slide Time 49:47).

Now, from the application of this, the choice of x is true that we will assume some value of x. If we make the choice somewhat logical then obviously the results are going to be better; otherwise, if we take some extremely wild choice deliberately, we will get something. Here, when it moves in the first mode we will find the inertial load here is proportional to m; here it is proportional to 2m, but not exactly because the amplitudes are different and amplitude here is more. Again here it will be. Therefore, everywhere you will find the force acting is somewhat related to the mass and the deflection is also related to the stiffness. So, one quick way of having a reasonable [51:04] is…. Let us see what is the phase of each position or displacement at each position due to a gravitational pull; that means if we hang it what will be the deflection of these in the static equilibrium position? That is one very common technique used (Refer Slide Time: 51:27).

Gravitational deflection that means if it hangs what will be the X_1? X_1 will be something like mg, 2mg and mg; I will not do the detailed calculations. So here it will be 4mg by k;

X_2 will be 11mg by k; X_3 will be 13mg by k. So we can take.. it will be 1 1.38 1.3…; it is in that ratio 4 11 13. If divided by 4, first one will be 1, second one will be 1.38 in this ratio. Let us use this as first mode which is just due to gravitational problem; using this we get… I will not calculate the whole thing. R is going to be k by m and 1.35 in the top and 7.47 in the numerator. This will be equal to .181 and R is supposed to be square of the natural frequency. So first natural frequency we get approximately equal to square root of this becomes 0.42 (Refer Slide Time: 54:38). Now, this is extremely revealing, because, even using an approximate method to use the first mode by just using the gravitational pull and the subsequent information, the natural frequency we get is very close, rather, almost same, but there may be difference in the higher order third, fourth and fifth. You can see… Now again to demonstrate, as I mentioned, this value accuracy is not very sensitively dependant on the correctness of the choice. If I now say let us have quick method of finding out this. Even for finding the gravitational pull and doing it you have to do some calculation.

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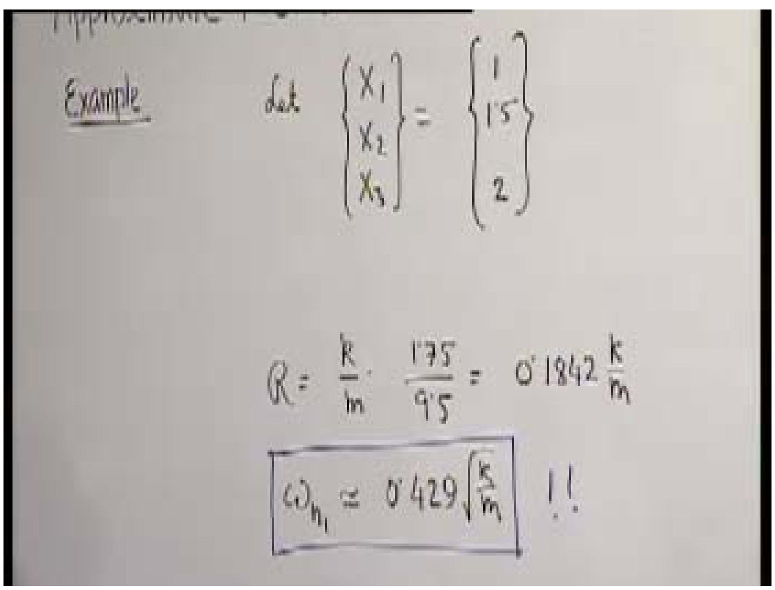

Let us say that X_1 X_2 X_3 be 1. Then this one being the same force we apply everywhere for example, then this will stretch by another half amount of this. So, the next one will be 1 plus half of it, so it will be 1.5. The other one, again, we approximately say another half will go; so therefore this. This is a very crude way of [56:27] as you can see. If we use this, we will get Rayleigh's quotient as k by m into 1.75 by 9.5 equal to 0.1842 k by m and this will result in approximate value of this (Refer Slide Time: 56:52). So surprise - that even if we make such a crude way, we assume the mode shape you can see it is only in the third place. This is the actual merit of Rayleigh's principle; otherwise, this expression is not the real thing, but it is actually insensitivity to the errors in the choice of X in the result. That is the main merit of this Rayleigh's principle and as you said that you will always get it. Here it was very close. You did not see the difference, but here you can find we have made some errors in the choice, more errors compared to the previous one. This is also a static deflection case; this is not the real mode.

We have found out the real mode. In the previous lecture, you can compare and find out what was the real mode. Even if you make this, you will get some value which is always more than the real mode; that is why, we get always an upper bound. The proof that why it is in upper bound shows that truth can be found in [58:20 – 58:30]. Thus, for a designer, the quick way of estimating the first natural frequency even for a complicated system can be done without going to computer and running a whole program. Just by this (58:45) which gives some ideas. So therefore, if we say that this is the Dunkerley's method and this is the Rayleigh's method, then we know that it must be between this and this (59:00). We narrow the gap and the designer can satisfy whether this is in dangerous zone or not, that means whether there is some excitation from outside is possible (59:11).